From far-out mystery to a useful tool

The new filter definition provided by the LEAN Community

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A little of History

Bourbaki congress at Dieulefit in 1938 from Wikipedia.

Henri Cartan in 1968 from Wikipedia.

Problem formalizing limits

$$\lim_{x\to x_0} f(x) = y_0 \quad \lim_{x\to x_0^+} f(x) = y_0 \quad \lim_{x\to x_0} f(x) = \infty$$

$$\lim_{x\to x_0^-} f(x) = -\infty \quad \lim_{x\to \infty} f(x) = y_0 \quad \lim_{x\to -\infty} f(x) = \infty$$

Commonly we have 15 definitions of limits of one dimension $\mathbb{R}$-functions.

Theorem on uniqueness of limits

Let $A \subseteq \mathbb{R}$ a subset of the reals and $f : A \to \mathbb{R}$ a function, then giving a point $x_0 \in A$ if the limit of the function f at this point exists then is unique.

In order to prove this theorem we will actually need 15 proves.

The first definition

Filter definition (Henri Cartan, General Topology)

A filter on a set X is a set $\mathcal{F}$ of subsets which has the following properties

(F_I) Every subset of X which contains a set of $\mathcal{F}$ belongs to $\mathcal{F}$.

(F_{II}) Every finite intersection of sets of $\mathcal{F}$ belongs to $\mathcal{F}$.

(F_{III}) The empty set is not in $\mathcal{F}$.

The usual definition

Filter Definition

Let X be a set, a filter is a subsets family of the power set $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the properties

(i) The universal set is in the filter $X \in \mathcal{F}$.

(ii) If $A \in \mathcal{F}$, then for all $B \in \mathcal{P}(X)$ satisfying $A \subseteq B$, we have $B \in \mathcal{F}$.

(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. (Closed for finite intersections)

(iv) The emptyset is not in the filter $\varnothing \notin \mathcal{F}$.

Filter Examples

Principal Filter

Let X a set and $A \subseteq X$ a non-empty subset. The principal filter of the subset A, which will be detonated as $P(A)$ is defined as the collection of subsets $\{B \subseteq X \mid A \subseteq A\}$.

Fréchet Filter

Let X a non-finite set, the Fréchet Filter is defined as the collection of cofinite subsets $\{A \subseteq X \mid A^c \text{ is finite}\}$.

Neighbourhoods Filter

Let (X, Z) be a topological space and $x \in X$ a point, the Neighbourhood Filter, which we will denotated as $\mathcal{N}(x)$, is defined as the collection of neighbourhoods of the x point (e.q. the dot is in the interior set).

Partial Order

Filter Partial Order

We say that a filter $\mathcal{F}$ is finer than a filter $\mathcal{V}$ if, as sets, $\mathcal{V} \subseteq \mathcal{F}$ and, from now on, it will be denoted as $\mathcal{F} \leq \mathcal{V}$.

This order might be anti-intuitive with the instinct we have from the Set Theory. A useful trick could be establishing a relation between the inclusion form sets and the principal filters.

$$A_1 \subseteq A_2 \subseteq \cdots \subseteq A_n \Longrightarrow P(A_1) \leq P(A_2) \leq \cdots \leq P(A_n)$$

Limit definition

Filter convergence

Giving a topological space (X, τ), a filter $\mathcal{F}$ of X we will say that $\mathcal{F}$ converges to $x \in X$ if $\mathcal{F} \leq \mathcal{N}(x)$ and, will be detonated as $\mathcal{F} \to x$.

Limit of a function at a point

Giving two topological spaces (X, τ_X) and (Y, τ_Y) an application $f : X \to Y$ between them. The limit of the function f at the point $x_0 \in X$ is $y_0 \in Y$ if (detonated as $\lim\limits_{x \to x_0} f(x) = y_0$), for all filter $\mathcal{F}$ that converges to x_0, the filter $f_*(\mathcal{F})$ will converge to y_0.

$$\forall \mathcal{F} \text{ X-filter, such that } \mathcal{F} \to x_0 \implies f_*(\mathcal{F}) \to y_0$$

Where f_* is the push forward of the application f.

$$f_*(\mathcal{A}) := \left\{ B \subseteq Y \mid f^{-1}(B) \in \mathcal{A} \right\}$$

Example in $\mathbb{R}^n$

Lemma about limits and Neighbourhoods filters

Giving two topological spaces (X, τ_X) and (Y, τ_Y) an application $f : X \to Y$ between them, then, giving two point $x_0 \in X$ and $y_0 \in Y$, we will have

$$\lim_{x \to x_0} f(x) = y_0 \iff f_*(\mathcal{N}(x_0)) \leq \mathcal{N}(y_0)$$

Let f be a $\mathbb{R}^n$-function and $x_0, y_0 \in \mathbb{R}^n$ two real points, let's see the limit in this case

$$\lim_{x \to x_0} f(x) = y_0 \iff f_*(\mathcal{N}(x_0)) \leq \mathcal{N}(y_0)$$

Using the open sets of the usual topology, it can be concluded

$$\forall \varepsilon \in \mathbb{R}^+, B_\varepsilon(y_0) \in f_*\left(\mathcal{N}(x_0)\right) \iff \forall \varepsilon > 0, \exists \delta > 0 \text{ s.t. } f\left(B_\delta(x_0)\right) \subseteq B_\varepsilon(y_0)$$

This last expression is equivalent to the usual definition of limit

$$\lim_{x \to x_0} f(x) \iff \begin{array}{c} \forall \varepsilon \in \mathbb{R}^+, \exists \delta \in \mathbb{R}^+, \forall x \in \mathbb{R}^n \\ \text{s.t. } |x - x_0| < \delta \text{ then } |f(x) - y_0| < \varepsilon \end{array}$$

Theorem on uniqueness of limits

Let (X, τ) a topological space, then all filters can't converge to two points if and only if the topology is Hausdorff.

Characterisation of continuous functions

Let (X, τ_X) and (Y, τ_Y) two topology spaces and $f : X \to Y$ an application between spaces. The application f is continuous if and only if for all point $x_0 \in X$ $\lim_{x \to x_0} f(x) = f(x_0)$.

Ultrafilter

Let X be a set, an ultrafilter is a filter $\mathcal{F}$ of X satisfying that for all $A \subseteq X$ we have $A \in \mathcal{F}$ or $A^c \in \mathcal{F}$.

Characterisation of compact spaces

A topological space (X, τ_X) is compact if and only if every ultrafilter in X is convergent to some point in X.

Filters arrive to theorem provers

```
/-- A filter `F` on a type `α` is a collection of sets of `α` which contains the whole `α`,
is upwards-closed, and is stable under intersection. We do not forbid this collection to be
all sets of `α`. -/
structure filter (α : Type*) :=
(sets                   : set (set α))
(univ_sets              : set.univ ∈ sets)
(sets_of_superset {x y} : x ∈ sets → x ⊆ y → y ∈ sets)
(inter_sets {x y}       : x ∈ sets → y ∈ sets → x ∩ y ∈ sets)
```

Mathlib filter definition.

Photo of Dr. Patrick Massot from his personal webpage.

The LEAN definition

Filter Definition (LEAN Community)

Let X be a set, a filter is a subsets family of the power set $\mathcal{F} \subseteq \mathcal{P}(X)$ satisfying the properties

(i) The universal set is in the filter $X \in \mathcal{F}$.
(ii) If $A \in \mathcal{F}$, then for all $B \in \mathcal{P}(X)$ satisfying $A \subseteq B$, we have $B \in \mathcal{F}$.
(iii) If $A, B \in \mathcal{F}$, then $A \cap B \in \mathcal{F}$. (Closed for finite intersections)

Giving a filter $\mathcal{F}$ from a set X, it will always satisfy

$$\mathcal{P}(X) = P(\varnothing) \leq \mathcal{F} \leq P(X) = \{X\}$$

Lattice structure

Giving a set X, the set of filters $\mathcal{F}_X$ with the partial order define a lattice structure algebra

$$(\mathcal{F}_X, \sqcup, \sqcap)$$

Characterisation on the join of two filters

Giving two filters $\mathcal{V}, \mathcal{A} \in \mathcal{F}_X$ of a set X the join of the two filters will be described as

$$\mathcal{V} \sqcup \mathcal{A} = \{B \subseteq X \mid B \in \mathcal{V} \text{ and } B \in \mathcal{A}\}$$

Characterisation on the meet of two filters

Giving two filters $\mathcal{V}, \mathcal{A} \in \mathcal{F}_X$ of a set X the meet of the two filters will be described as

$$\mathcal{V} \sqcap \mathcal{A} = \{B \subseteq X \mid \exists V, A \text{ s.t. } V \in \mathcal{V}, A \in \mathcal{A} \text{ and } V \cap A \subseteq B\}$$

Semiring structure

The filter semiring of a set

Giving a set the set of filters $\mathcal{F}$ of a set X the triplet

$$\left(\mathcal{F}_X, \bigsqcup, \sqcap\right)$$

is a semiring structure.

The collection of subsemirings of $(\mathcal{F}_X, \bigsqcup, \sqcap)$ will be denoted as

$$\mathrm{Sub}(\mathcal{F}_X)$$

Definition. Topological subsemiring

We will say that a subsemiring G of $\mathcal{F}_X$ is topological if for all $y \in X$, there exists a filter $\mathcal{F}_y$ such that all subsets $y \in U \subseteq X$, whose principal filter is in G, will be contained in the filter $U \in \mathcal{F}_y$.

The collection of topological subsemirings of the semiring of filters will be denoted, from now on, as

$$\mathrm{Sub}^t(\mathcal{F}_X) = \left\{ G \leq \mathcal{F}_X \mid G \leq^t \mathcal{F}_X \right\}$$

The set of topological spaces of a set X will be denoted as

$$\mathrm{Top}(X) = \{\tau \subseteq \mathcal{P}(X) \mid (X, \tau) \text{ is a topology}\}$$

Correspondence between topologies and $\mathcal{F}_X$-subsemirings

Giving a set X we can define an application between the topological spaces of X and the subsemiirings of the filter semiring associated to X.

$$\begin{aligned}\mathcal{T}_X : \mathrm{Top}(X) &\to \mathrm{Sub}(\mathcal{F}_X)\\ (X,\tau) &\mapsto K^0_{\mathcal{F}_X}\left[\mathcal{N}(y), \forall y \in X\right]\end{aligned}$$

We can also define another application between the subsemiirings and the topological spaces of X.

$$\begin{aligned}\mathcal{S}_X : \mathrm{Sub}(\mathcal{F}_X) &\to \mathrm{Top}(X)\\ G &\mapsto (X,\tau),\ \tau = \{V \subseteq X \mid P(V) \in G\}\end{aligned}$$

Be aware that the equality

$$\mathcal{S}_X \circ \mathcal{T}_X = id$$

will be always satisfied, however, generally, it is not a bijection.

Let's consider the follow relation between subsemirrings

$$G_1 \sim G_2 \text{ iff } \mathcal{S}_X(G_1) = \mathcal{S}_X(G_2)$$

Theorem. Correspondence between topologies and $\mathcal{F}_X$ subsemirings.

Giving a set X, the applications $\mathcal{T}_X$ and $\mathcal{S}_X$ will establish a relation between all the topological spaces of X and the quotient $\mathrm{Sub}(\mathcal{F}_X)/_{\sim}$, where all the elements will have a representation in $\mathrm{Sub}^t(\mathcal{F}_X)$.

$$\mathrm{Top}(X) \rightleftarrows \mathrm{Sub}(\mathcal{F}_X)/_{\sim} \cong \mathrm{Sub}^t(\mathcal{F}_X)/_{\sim}$$

Solving basic propositions

With the relation establish before we can redefine some basic concepts as the closure of a subset.

$$\text{Topological space} \rightleftarrows \text{Subsemiring}$$
$$\{x \in X \mid \forall N \in \mathcal{N}(x), N \cap A \neq \varnothing\} \rightleftarrows \{x \in X \mid \mathcal{F}_x \sqcap P(A) \neq \bot\}$$

Solving basic propositions

Proposition. Union of clousres

Giving a topological space (X, τ) and two subsets $A, B \subseteq X$, it would be satisfied

$$Cl(A \cup B) = Cl(A) \cup Cl(B)$$

Using the correspondence with semirings this proposition can be easily proof

$$y \in Cl(A \cup B) \iff \mathcal{F}_y \sqcap P(A \cup B) \neq \bot \iff$$

$$(\mathcal{F}_y \sqcap P(A)) \sqcup (\mathcal{F}_y \sqcap P(B)) \neq \bot \iff$$

$$\mathcal{F}_y \sqcap P(A) \neq \bot \text{ or } \mathcal{F}_y \sqcap P(B) \neq \bot \iff y \in Cl(A) \text{ or } y \in Cl(B)$$

$$\boxed{Cl(A \cup B) = Cl(A) \cup Cl(B)}$$

$\square$

Solving basic propositions

Proposition. Intersection of closures

Giving a topological space (X, τ) and two subsets $A, B \subseteq X$, it would be satisfied

$$Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)$$

$$y \in Cl(A \cap B) \Longleftrightarrow \mathcal{F}_y \sqcap P(A \cap B) \neq \perp \Longleftrightarrow$$

$$\mathcal{F}_y \sqcap P(A) \sqcap P(B) \neq \perp \Longrightarrow$$

$$\mathcal{F}_y \sqcap P(A) \neq \perp \text{ and } \mathcal{F}_y \sqcap P(B) \neq \perp \Longleftrightarrow y \in Cl(A) \text{ and } y \in Cl(B)$$

$$\boxed{Cl(A \cap B) \subseteq Cl(A) \cap Cl(B)}$$

$\square$

Inducted topology

Remember that a topology inducted by an application $f : X \to Y$ is the more finer topology (in Y) such that the application f is continuous.

Proposition. The inductive topological subsemiring

Giving an application $f : X \to Y$ and a topological subsemirring $G \in \mathrm{Sub}^t(\mathcal{F}_X)$, the inductive topology will be related with the subsemiring

$$K^0_{\mathcal{F}_Y}\left[f_*(\mathcal{F}_x),\ \forall x \in X\right]$$

In this case, we are asking for the continuous at every point instead for the global continuity.

T_2 and T_1 spaces

We can also find the equivalent definition of Hausdorff Spaces (T_2) in the filters subsemirings

$$\begin{array}{ccc} \text{Topological space} & \rightleftarrows & \text{Subsemiring} \\ \begin{array}{c} \forall x \neq y \in X,\ \exists U, V \in \tau \text{ such that} \\ x \in U, y \in V \text{ and } U \cap V = \varnothing \end{array} & \rightleftarrows & \forall x \neq y \in X, \mathcal{F}_x \sqcap \mathcal{F}_y = \bot \end{array}$$

It can also be described a similar relation with Fréchet Spaces (T_1).

$$\begin{array}{ccc} \text{Topological space} & \rightleftarrows & \text{Subsemiring} \\ \begin{array}{c} \forall x \neq y \in X,\ \exists U \in \tau \text{ such that} \\ x \in U \text{ and } y \notin U \end{array} & \rightleftarrows & \forall x \neq y \in X, \mathcal{F}_x \sqcap P(\{y\}) = \bot \end{array}$$

Hausdorff and the subspace topology

Proposition. Hausdorff spaces and their subspaces

Giving a topological subsemiring $G \in \mathrm{Sub}^t(\mathcal{F}_X)$ and a subset $A \subset X$. If G is a Hausdorff subsemiring, then the inductive topological subsemiring, by the projection from X to A, will also be Hausdorff.

Giving two points

$$x \neq y \in A \subseteq X \Longrightarrow \mathcal{F}_x^G \sqcap \mathcal{F}_y^G = \bot$$

In the induced topology we have

$$\mathcal{F}_x \sqcap \mathcal{F}_y \leq f_*(\mathcal{F}_x^G) \sqcap f_*(\mathcal{F}_y^G)$$

$$= \left(P(A) \sqcap \mathcal{F}_x^G\right) \sqcap \left(P(A) \sqcap \mathcal{F}_y^G\right) = P(A) \sqcap \left(\mathcal{F}_x^G \sqcap \mathcal{F}_y^G\right) = \bot$$

□

The closed definition

Filters by Closed Sets

Giving a set X, a filter by closed is a subset of the power set $\mathcal{C} \subseteq \mathcal{P}(X)$ which satisfies

(i) The empty set is in the filter $\varnothing \in \mathcal{C}$

(ii) If a subset A is in the filter and $B \subseteq A$ then B is in $\mathcal{C}$.

(iii) The union of two subsets of the filter is in the filter.

With the invert order relation of the filters by open sets, we will obtain that the set of filters by closed sets $(\mathcal{C}_X)$ is also a lattice. Moreover, it can be prove that $(\mathcal{C}_X, \prod, \sqcup)$ is a semiring.

Theorem. Bijection between closed and open subsemirings

There exists a bijection between the semirings of $\mathcal{F}_X$ and the ones of $\mathcal{C}_X$ induced by isomorphisms between subsemirings.

$$\mathrm{Sub}(\mathcal{F}_X) \rightleftarrows \mathrm{Sub}(\mathcal{C}_X) \text{ and } \mathrm{Sub}^t(\mathcal{F}_X) \rightleftarrows \mathrm{Sub}^t(\mathcal{C}_X)$$

The correspondence with filters by closed sets

As we have done before we define

$$\begin{aligned} \mathcal{T}_X^c : \mathrm{Top}(X) &\to \mathrm{Sub}(\mathcal{C}_X) \\ (X,\tau) &\mapsto K^0_{\mathcal{F}_X}\left[\mathcal{N}_c(y), \forall y \in X\right] \end{aligned}$$

and

$$\begin{aligned} \mathcal{S}_X^c : \mathrm{Sub}(\mathcal{C}_X) &\to \mathrm{Top}(X) \\ G &\mapsto (X,\tau),\ \tau^c = \{V \subseteq X \mid P(V) \in G\} \end{aligned}$$

Theorem. Correspondence between topologies and $\mathcal{C}_X$ subsemirings.

Giving a set X, the applications $\mathcal{T}_X^c$ and $\mathcal{S}_X^c$ will establish a relation between all the topological spaces of X and the quotient ${\mathrm{Sub}(\mathcal{C}_X)}/_{\sim}$, where all the elements will have a representation in $\mathrm{Sub}^t(\mathcal{C}_X)$.

$$\mathrm{Top}(X) \rightleftarrows {\mathrm{Sub}(\mathcal{C}_X)}/_{\sim} \cong {\mathrm{Sub}^t(\mathcal{C}_X)}/_{\sim}$$

The correspondence theorem

Theorem. The big correspondence theorem.

Giving a set X. The applications we have defined before, induce the follow commutative diagram

$$
\begin{tikzcd}
\mathrm{Top}(X) \arrow[r, leftrightarrow] \arrow[dr, leftrightarrow] & \mathrm{Sub}(\mathcal{F}_X)\,/_{\sim} & \cong \arrow[d, leftrightarrow] & \mathrm{Sub}^t(\mathcal{F}_X)\,/_{\sim} \\
 & \mathrm{Sub}(\mathcal{C}_X)\,/_{\sim} & \cong & \mathrm{Sub}^t(\mathcal{C}_X)\,/_{\sim}
\end{tikzcd}
$$

References

- Nicolas Bourbaki, *Topologie Générale*. Chapter I
- LftCM2020 lecture: Topology and filters (Patrick Massot).
- All proofs are publish at carloscaralps.github.io