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Heegner points and periods on products of upper half-planes

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Abstract

Heegner points were first defined by Bryan John Birch in the 1970s. Since their first introduction, Heegner points have been used to partially prove famous conjectures such as the Gauß Conjecture, Hilbert's 12th problem, or the Birch and Swinnerton-Dyer conjecture. One can still see its importance in today's literature with some generalizations, such as Stark-Heegner points, or in its generalizations to plectic points. The definition of Heegner points is a purely algebraic concept, however, using the isomorphism of the modular curve with a concrete $\mathbb{C}$-Riemann surface, one can prove that CM points have an analytic analog, considering concrete periods of the modular form attached to a given elliptic curve. In this document, we start exploring the generalization of such correlation to p-adic CM points over Shimura curves and periods over a $\mathbb{C}_p$-rigid analytic space. We will continue studying the paper by Bertolini, Darmon, and Green, where they find periods over the setting of a product of two upper-half planes imposing certain conditions on the primes involved. We will then generalize the results of this paper to all possible behaviors of the primes associated with these two half-planes. We will finish by generalizing the construction of similar periods on finite products of upper half-planes, and we will prove a generalization of the Heegner Hypothesis for these periods.

Zusammenfassung

Heegner-Punkte wurden erstmals in den 1970er Jahren von Bryan John Birch definiert. Seit ihrer Einführung wurden Heegner-Punkte verwendet, um berühmte Vermutungen wie die Gauß-Vermutung, Hilberts 12. Problem oder die Birch und Swinnerton-Dyer Vermutung teilweise zu beweisen. Ihre Bedeutung ist in der heutigen Literatur noch in einigen Verallgemeinerungen, wie z. B. Stark-Heegner-Punkten, oder in ihren Verallgemeinerungen auf Plectic Punkte erkennbar. Die Definition von Heegner-Punkten ist ein rein algebraisches Konzept. Mithilfe des Isomorphismus der Modulkurve mit einer konkreten $\mathbb{C}$-Riemannsche Fläche kann jedoch gezeigt werden, dass CM Punkte ein analytisches Analogon haben, wenn man konkrete Perioden der Modulform betrachtet, die einer gegebenen elliptischen Kurve zugeordnet sind. In diesem Dokument beginnen wir mit der Untersuchung der Verallgemeinerung einer solchen Korrelation auf p-adische Schwerpunktpunkte über Shimura Kurven und Perioden über einem $\mathbb{C}_p$-rigid-analytischer Raum. Wir untersuchen weiterhin die Arbeit von Bertolini, Darmon und Green, in der sie Perioden über dem Produkt zweier oberer Halbebenen finden, wobei den beteiligten Primzahlen bestimmte Bedingungen auferlegt werden. Anschließend verallgemeinern wir die Ergebnisse dieser Arbeit auf alle möglichen Verhaltensweisen der mit diesen beiden Halbebenen verbundenen Primzahlen. Abschließend verallgemeinern wir die Konstruktion ähnlicher Perioden auf endlichen Produkten oberer Halbebenen und beweisen eine Verallgemeinerung der Heegner-Hypothese für diese Perioden.

Uittreksel

Heegnerpunten werden voor het eerst gedefinieerd door Bryan John Birch in de jaren 70. Sinds zijn eerste introductie zijn Heegnerpunten gebruikt om beroemde vermoedens zoals het vermoeden van Gauß, Hilberts 12e probleem of het vermoeden van Birch en Swinnerton-Dyer gedeeltelijk te bewijzen. Het belang ervan is nog steeds terug te zien in de hedendaagse literatuur met enkele generalisaties, zoals Stark-Heegnerpunten, of in de generalisaties ervan naar plectic punten. De definitie van Heegnerpunten is een puur algebraïsch concept, maar met behulp van de isomorfie van de modulaire kromme met een concreet $\mathbb{C}$-Riemann-oppervlak kan men bewijzen dat CM-punten een analytische analoog hebben, rekening houdend met concrete periodes van de modulaire vorm die verbonden zijn aan een gegeven elliptische kromme. In dit document verkennen we de generalisatie van een dergelijke correlatie naar p-adische CM-punten over Shimura-krommen en periodes over een $\mathbb{C}_p$-rigide analytische ruimte. We zullen het artikel van Bertolini, Darmon en Green verder bestuderen, waarin zij periodes vinden over de setting van een product van twee bovenste halfvlakken die bepaalde voorwaarden stellen aan de betrokken priemgetallen. Vervolgens zullen we de resultaten van dit artikel generaliseren naar alle mogelijke gedragingen van de priemgetallen die geassocieerd zijn met deze twee halfvlakken. We zullen afsluiten met de generalisatie van de constructie van vergelijkbare periodes op eindige producten van bovenste halfvlakken, en we zullen een generalisatie van de Heegner-hypothese voor deze periodes bewijzen.

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"We all feel like the system is too big to change,
but guess what, we are the system and we need to change.
So, how can I help?" Max Goodwin
Al meu germà petit, Pau.

Contents

1 Introduction

We are going to present an example that will underpin the main motivations and methods we are going to generalize in this document. Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. Using the modularity theorem, we have the minimal complex parametrization

$$\Phi : X_0(N) \to E.$$

We shall recall that, using the moduli interpretation of the subspace $Y_0(N)$ of modular curves $X_0(N)$, we obtain the following expression

$$Y_0(N) = \left\{(E, E') \text{ ell. curves over } \mathbb{C} \ : \ E \to E' \ N\text{-isog.}\right\} /\!\cong\ \subseteq X_0(N).$$

Given an order $\mathcal{O}$ of an imaginary quadratic field $K/\mathbb{Q}$, we can use this last expression to define CM points over the modular curve $X_0(N)$ attached to the order $\mathcal{O}$

$$\mathrm{CM}(\mathcal{O}) := \{(E, E') \in X_0(N) |\, \mathrm{End}(E) \cong \mathrm{End}(E') \cong \mathcal{O}\}$$

We assume that $\mathcal{O}$ satisfies the Heegner Hypothesis, so the collection of CM points attached to this order is non-empty, i.e. the ideal $N\mathcal{O}$ factors as $\mathcal{N}\overline{\mathcal{N}}$ cyclic ideals of norm N. Consider the map

$$\delta : \mathrm{Pic}(\mathcal{O}) \to X_0(N)(\mathbb{C})$$

that sends $\mathfrak{a}$ to the pair of elliptic curves $(\mathbb{C}/\mathfrak{a}, \mathbb{C}/\mathcal{N}^{-1}\mathfrak{a})$. We consider the map given by the Artin reciprocity law

$$\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \to \mathrm{Gal}(H/K),$$

where H is the ring class field associated to the order $\mathcal{O}$. The map rec plays an essential role in the following result, which is one of the most important properties of CM points.

Theorem 1.1 (Complex Multiplication theorem)**.** *Given a point $P \in \mathrm{CM}(\mathcal{O})$, we have that $\Phi(P) \in E(H)$. Furthermore, given $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we have the following equation*

$$\Phi(\mathfrak{a}P) = \mathrm{rec}(\mathfrak{a})^{-1}\Phi(P).$$

It is important to remark that CM points can be identified by considering all the δ maps associated to all the possible orientations $\mathcal{N}$ of $\mathcal{O}$ i.e.

$$\mathrm{CM}(\mathcal{O}) = \bigcup_{\mathcal{N} \text{ orientation of } \mathcal{O}} \delta_{\mathcal{N}}(\mathrm{Pic}(\mathcal{O})).$$

The definition of Heegner points we have just given is purely geometrical, which makes the task of producing examples difficult. We are going to relate this definition to concrete integral cycles, in order to do so, we have to start showing the relation between CM points and the following type of embeddings.

Definition 1.2 (Optimal oriented embeddings)**.** *Given an order $\mathcal{O} \subseteq K$, we say that an embedding $\Psi : K \to \mathcal{M}_2(\mathbb{Q})$ is optimal oriented if it satisfies the following conditions*

(i) (Optimal) The embedding is called optimal if it satisfies $\Psi(K) \cap M_0(N) = \Psi(\mathcal{O})$, where $M_0(N)$ is the collection of 2×2 matrices with integer coefficients such that their third entrance is divisible by N.

(ii) (Oriented) We consider an orientation of the Eichler order i.e. a subjective morphism $M_0(N) \to \mathbb{Z}/N\mathbb{Z}$. The embedding is said to be oriented (relative to an orientation $\mathcal{N}$ of $M_0(N)$) if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & M_o(N) \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathcal{N} & \xrightarrow{\sim} & \mathbb{Z}/N\mathbb{Z} \end{array}$$

(iii) If we consider the action generated by the embedding at K, there exists a unique fixed point $\tau_\Psi \in K$ such that

$$\Psi(\lambda) \begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix}.$$

We say that the embedding is oriented at infinity if τ_Φ belongs to $\mathcal{H}$ under a fixed embedding $K \hookrightarrow \mathbb{C}$

Let $\mathrm{Emb}(\mathcal{O}, M_0(N))$ be the collection of optimal oriented embeddings of $\mathcal{O}$ to $M_0(N)$. This collection is equivalent to the Picard group of $\mathcal{O}$ as proven by BDG.

Lemma 1.3. *The assignment $\Psi \to \mathfrak{a}_\Psi$ is a bijection between $\mathrm{Emb}(\mathcal{O}, M_0(N))$ and $\mathrm{Pic}(\mathcal{O})$.*

One should note that this bijection n allows us to define an action in $\mathrm{Emb}(\mathcal{O}, M_0(N))$ by the Picard group, which we will denote as $\Psi^{\mathfrak{a}}$ for all $\Psi \in \mathrm{Emb}(\mathcal{O}, M_0(N))$ and $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$. We fix the common isomorphism $j : \mathcal{H}^*/\Gamma_0(N) \cong X_0(N)$. Let Λ_E be the lattice generated by the periods of a Néron differential ω_E on E, and let

$$\eta : \mathbb{C}/\Lambda_E \to E(\mathbb{C})$$

be the Weierstraß uniformization associated to Λ_E. Up to the Manin constant (which we assume is trivial here), the pull-back $(\Phi \circ j)^*\omega_e$ is equal to $2\pi i f(t)dt$ where f is a normalized weight 2 cusp form attached to E. Given an optimal oriented embedding Ψ, we define the period

$$J_\Psi := \int_{i\infty}^{\tau_\Psi} 2\pi i f(t) dt.$$

Using the result in the Chapter VI.5 of [Sil09], we have that

$$\Phi(\delta(\langle\tau,1\rangle)) = \eta\left(\int_{i\infty}^{\tau}(\Phi\circ j)^{*}\omega_E\right), \tag{1.0.1}$$

which proves that the class $\mathfrak{a}_\Psi$ satisfies $\Phi(\delta(\mathfrak{a}_\Psi)) = \eta(J_\Psi)$. We have found an expression of Heegner points using periods.

Theorem 1.4. *For all $\Psi \in \mathrm{Emb}(\mathcal{O}, M_0(N))$, the point $\eta(J_\Psi)$ belongs to $E(H)$, and for all $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$,*

$$\eta(J_{\Psi^{\mathfrak{b}}}) = \mathrm{rec}(\mathfrak{b})^{-1}\eta(J_\Psi)$$

Proof. The statement is an immediate result of the equation 1.0.1 and the Shimura reciprocity law. □

This theorem enables us with the tools to explicitly compute CM points over modular curves. We are going to exemplify this phenomenon with the concrete setting where $N = 11$. In this case, the associated modular curve has genus 1 (i.e. it is an elliptic curve) and the vector space $S_2(\Gamma_0(11))$ is generated by the unique normalized eigenform

$$f(z) = q\prod(1-q^{11n})^2(1-q^n)^2.$$

Since $X_0(11)$ is an algebraic curve of genus 1, the Shimura construction will be an isomorphism and, therefore, we can identify $X_0(11)$ with the following elliptic curve

$$E_f : y^2 + y = x^3 - x^2 - 10x - 20.$$

We consider the field $K = \mathbb{Q}(\sqrt{-7})$ and its maximal order $\mathcal{O} = \mathbb{Z}\left[\frac{1+\sqrt{-7}}{2}\right]$. It is easy to check that this order satisfies the Heegner Hypothesis i.e. that 11 is split in $\mathcal{O}$. Using the correspondence given by the Lemma 1.3, we get that there's a unique optimal oriented embedding

$$\Psi(\mathbb{Q}+\mathbb{Q}\tau) = \mathbb{Q}\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathbb{Q}\begin{pmatrix}-9/11 & -2/11\\ 1 & 0\end{pmatrix},$$

where $\tau = \frac{-9+\sqrt{-7}}{22}$ is the number in $\mathcal{H}$ that satisfies the conditions given in the proof of the lemma. We use SageMath to compute the approximation of the period associated to J_Ψ (up to the exponential 100)

$$J_\Psi = -0.507683721711822 - 0.405629044516045i.$$

We use the Weiestraß uniformization to get the associated approximated point in the elliptic curve $E(\mathbb{C})$

$$(0.500000000000001 - 1.32287565553229i, -2.00000000000000 - 5.29150262212918i)\,.$$

It is important to point out that the class number of $\mathcal{O}$ is one and, consequently, the associated ring field is K itself. We use the LLL-algorithm and the Theorem 1.4 to find the unique CM point with the approximation values that we have gotten

$$\left(\frac{1-\sqrt{-7}}{2}, -2-2\sqrt{-7}\right).$$

One should note that these computations, although they are slightly different, agree with the results computed by Darmon in section 3.3 of [Dar04]. One should note that the fact that $X_0(11)$ is an elliptic curve implies that the point we have found is actually a CM point and not the image of one, which makes this case specially interesting.

Given an elliptic curve $E/\mathbb{Q}$ of conductor $N = pN^+N^-$, we set X to be the Shimura variety associated to the decomposition (pN^+, N^-). The definition of CM points can be extended in general to points over Shimura curves using the moduli interpretations of such curves i.e., given a closed algebraic field F and a quadratic imaginary order $\mathcal{O}$, we can get the following definition of CM points

$$\mathrm{CM}(\mathcal{O}) := \{(A, A') \in X(F) \ : \ \mathrm{End}(A) \cong \mathrm{End}(A') \cong \mathcal{O}\}$$

where A and A' are abelian varieties over F together with a concrete embedding that will be made explicit in the following sections. One should remark that the endomorphism notion on this definition is in the setting of the concrete category of abelian varieties.

Over the Archimedean complex numbers, one can find equivalences similar to the ones we have explicitly for modular curves. However, over Shimura curves it is also interesting to consider the CM points over p-adic complex numbers. We consider the quaternionic algebra B (split at the prime p) of discriminant N^- over $\mathbb{Q}$ and the Eichler $\mathbb{Z}[1/p]$-order of level N^+. One can define a congruence subgroup using the natural embedding $\iota : B \to M_2(\mathbb{Q}_p)$ as

$$\Gamma := \iota(R_1^\times)$$

where $R_1^\times$ are the units of R which have norm 1. The Riemann surface equivalence will be given by the theory of p-adic uniformization, which will give us the following isomorphism of surfaces

$$X(\mathbb{C}_p) \cong \mathcal{H}_p /_{\Gamma}$$

where $\mathcal{H}_p$ is the p-adic upper half plane which is defined as $\mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$. Using the modularity theorem together with the Jacquet-Langlands correspondence, we can identify the elliptic curve E with a p-adic cusp form of weight 2 with respect to the group Γ, which we will denote as f. Once can associate a multiplicative measure to this cusp form, a choice of f should be made so that the definition of this measure does not depend on a choice of an exponential function, which help us define the following cycle for a given optimal oriented embedding Ψ

$$J_\Psi := \times\!\!\!\!\!\!\int^{\tau_\Psi} f(z)dz \in \mathbb{C}_p^\times \Big/ q_T^{\mathbb{Z}} \cong E(\mathbb{C}_p)$$

where τ_Ψ is the associated fixed point of the action $\Psi(K^\times)$ in $\mathcal{H}_p$. One should note that, in order to define this period, we will need to extend the notion of semi-indefinite integral, which is a non-trivial concept here since the surface $\mathcal{H}_p/\Gamma$ has no cusps and, consequently, we have no notion of infinity.
Let $\mathcal{O} \subseteq K$ be a $\mathbb{Z}[1/p]$-order of conductor prime to N, we assume the following Heegner hypothesis

- all the primes dividing N^- are inert in K,
- all the primes dividing N^+ are split in K,
- p is inert in K.

This hypothesis assure us of the existence of an optimal oriented embedding Ψ with respect to the order $\mathcal{O}$ and, consequently, the period J_Ψ can be considered. We denote H as the ring class field of K associated to the fixed order $\mathcal{O}$. We consider $\sigma_\mathfrak{p}$ the Frobenius element of a prime $\mathfrak{p}$ over p.
In this document, we will see that such a period, which can be naturally identified with a point in $E(\mathbb{C}_p)$, is related to CM points through the complex parametrization existing between X and E, and that therefore, such periods will satisfy the following property.

Theorem 1.5. *For any $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, the point $\eta_p(J_\Psi)$ is a Heegner point in $E(H)$. Furthermore $\eta_p(I_\Psi)$ is the Heegner point $\eta_p(J_\Psi) - \omega\sigma_\mathfrak{p}\eta_p(J_\Psi)$ and for all $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we have*

$$\eta_p(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

A natural question that arises at this point is the following: Could we generally find analytic periods over other settings than a single upper half plane that are related to CM points?
The general answer to this question still remains open, however, the paper of Bertolini-Darmon-Green [BDG07] gives a satisfactory answer for a concrete product of two p-adic upper-half planes.
We consider an elliptic curve $E/\mathbb{Q}$ of conductor $N = pqN^+N^-$ with all the factors prime pairwise, p, q both prime numbers, and N^- with an odd number of prime divisors. Let $B/\mathbb{Q}$ a quaternion algebra (split at p and q) of conductor N^- and an Eichler $\mathbb{Z}[1/pq]$-order of level N^-. Using the embedding $\iota : B \to M_2(\mathbb{Q}_p) \times M_2(\mathbb{Q}_q)$ associated to the algebra B, we define the congruence subgroup

$$\Gamma := \iota(R_1^\times),$$

where $R_1^\times$ are the units of R which have norm 1. One can generalize the concept of Cusp forms of weight 2 to the setting of $\mathcal{T}_p \times \mathcal{H}_q$, where $\mathcal{T}_p$ is the Bruhat-Tits tree of the prime p. We will prove the following isomorphism of cusp forms

$$S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma) \cong S_2^{p-new}(\mathcal{H}_q/\Gamma_q),$$

where Γ_q is the product projection of Γ to $M_2(\mathbb{Q}_q)$. This isomorphism allows us to extend the cusp form f associated to the elliptic curve E to the cusp forms over the setting $(\mathcal{T}_p \times \mathcal{H}_q)/\Gamma$. Similarly to the last case, we can define a measure associated to this cusp form, which at the same time allows us to consider the following double integral

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_1}^{v_2} \omega_f$$

where v_1, v_2 are vertices of the tree $\mathcal{T}$ and τ_1, τ_2 are points on the upper half plane $\mathcal{H}$. One can give a prove the Fubini theorem to this setting by proving that there exists a cusp form $f^\# \in S_2((\mathcal{H}_p \times \mathcal{T}_q)/\Gamma)$ (we are just exchanging the roles of the primes p and q) such that the following equation si satisfied

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_1}^{v_2} \omega_f = \int_{v_1}^{v_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^\#}$$

where the second integral is defined as a product of multiplicative integrals through the different edges of the Bruhat-Tits tree with joining the vertices v_1 and v_2

$$\int_{v_1}^{v_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^\#} = \prod_{\epsilon: v_1 \to v_2} \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{z-\tau_2}{z-\tau_1} d\mu_{f^\#_\epsilon}(z).$$

Following the motivation raised by this last definition, one can define the semi-indefinite double integral of $\omega_{f^\#}$ as

$$\int_{v_1}^{v_2} \times\!\!\!\!\!\!\int^{\tau} \omega_{f^\#} = \prod_{\epsilon: v_1 \to v_2} \times\!\!\!\!\!\!\int^{\tau} f^\#_\epsilon(z) dz.$$

We consider a $\mathbb{Z}[1/pq]$-order quadratic imaginary order $\mathcal{O} \subseteq K$ of conductor prime to N. We will assume the following generalized Heegner Hypothesis

- all the primes dividing N^- are inert in K,
- all the primes dividing N^+ are split in K,
- p is inert in K,
- q split in K.

These hypothesis assure us of the existence of an optimal oriented embedding Ψ with respect to the order $\mathcal{O}$. We consider the fixed point τ_Ψ of the action induced $\Psi(K^+)$ in the upper half plane $\mathcal{H}_p$ and we define the period

$$J_\Psi := \int_x^{\gamma_\Psi x} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_f \in \mathbb{C}_p^\times \Big/ q_T^{\mathbb{Z}} \cong E(\mathbb{C}_p)$$

where γ_Ψ is a generator of $\mathcal{O}_1^\times$ through the embedding Ψ. In the paper [BDG07] the authors prove that these periods can be related to CM points with respect to the maximal $\mathbb{Z}[1/p]$-order $\mathcal{O}_0$ inside $\mathcal{O}$. Let H_0 be the ring class field associated to the order $\mathcal{O}_0$ and $\sigma_\mathfrak{q}$ the element of the Galois group $\mathrm{Gal}(H_0/K)$ corresponding to a prime $\mathfrak{q}$ over q. We define H as the subgroup of H_0 which is fixed by $\sigma_\mathfrak{q}^2$.

Theorem 1.6 (Bertolini, Darmon, Green)**.** *The point $\eta_p(J_\Psi)$ is a global point in $E(H)$ on which the involution $\sigma_\mathfrak{q}$ acts via ω_q. In concrete, $\eta_p(I_\Psi)$ is $\eta_p(J_\Psi) - \omega_p\sigma_\mathfrak{p}\eta_p(J_\Psi)$. Moreover, for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$ the periods satisfy*

$$\eta_p(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

The result is, firstly, shocking because we have started from a purely analytic definition and we have been able to prove that such periods have an algebraic equivalence. In this thesis, we will study for which behaviours of p and q the results of Bertolini-Darmon-Green can be extended, and we will generalize their results to periods defined over finite products of upper half-planes. We will finish this document proving the following generalization of the Heegner Hypothesis.

Theorem 1.7 (Generalization of Heegner Hypothesis)**.** *Let $q_1, ..., q_n$ be a non-repetitive collection of primes strictly dividing N which are not p and consider the ring $T := \mathbb{Z}[1/pq_1 \cdots q_n]$. Given an T-order $\mathcal{O}$ and an optimal oriented embedding associated to $\mathcal{O}$, we consider the values $\eta_p(J_\Psi)$ associated to the period J_Ψ constructed on the rigid analytic space*

$$\mathcal{H}_p \times \mathcal{H}_{q_1} \times \cdots \times \mathcal{H}_{q_n}$$

following the theory developed by Bertolini, Darmon, and Green. These values satisfy the main theorem of complex multiplication if and only if K satisfies the Heegner Hypothesis. In other words, the underlying Shimura curve X_{N^+,pN^-} satisfies

(i) all primes dividing N^+ are split in K,

(ii) all primes dividing pN^- are inert in K.

Furthermore, the values $\eta(J_\Psi)$ will be trivial in all the instances where the strict inequality

$$\mathrm{rk}_\mathbb{Z}(\mathcal{O}^\times) < n$$

is satisfied i.e. at least one of the primes $q_1, ..., q_n$ is inert in K.

2 Preliminary notions

In this section, we will introduce the base concepts that will be used throughout the thesis and their corresponding basic results.

2.1 Elliptic Curves

We aim to introduce the theory related to elliptic curves. The author assumes that the reader might have experience with the basic concepts in this field. In light of this, we will make a small introduction to elliptic curves and then review more carefully the more advance concepts that will be used in this document, especially the ones related to p-adic reduction.

Definition 2.1 (Elliptic Curve)**.** *An elliptic curve* (E, O) *over a field* K *is a smooth projective curve* E *of genus 1 over* K *and a point* $O \in E$.

Following [Sil09, III §3], all elliptic curves can be expressed as cubic equations over $\mathbb{P}^2$ with one point at ∞ (that base point). Therefore, elliptic curves can be defined with the following equation

$$Y^2Z + a_1XYZ + a_3YZ^2 = X^3 + a_2X^2Z + a_4XZ^2 + a_6Z^3,$$

with $a_1, ..., a_6 \in \overline{K}$ and the distinguish point is $O = [0 : 1 : 0]$. Given an elliptic curve E/K, we can define a group operation: Given two points $P, Q \in E$, we consider the third point R in the intersection of the line $\overline{PQ}$ and E. We repeat the same process by considering T the intersection of E and the line $\overline{RO}$. We denote the sum $P \oplus Q$ as the point T.

Proposition 2.2. *The operation defined before for the points of a given elliptic curve* E/K *satisfies the group axioms.*

Proof. One can prove this proposition using the explicit equations given in [Sil09, p. 53]; however, the reader might be interested in the more geometrical proof given in [Ful89]. □

We will explicitly mention the two main results for elliptic curves over the rationals. This first theorem describes the set of rational points for arbitrary elliptic curves.

Theorem 2.3 (Mordell-Weil)**.** *Given an elliptic curve* $E/\mathbb{Q}$*, the set of rational points of this elliptic curve is finitely generated i.e.*

$$E(\mathbb{Q}) \cong E(\mathbb{Q})_{tors} \times \mathbb{Z}^r,$$

where $E(\mathbb{Q})_{tors}$ *is the set of points with finite order (also known as torsion points) and* r *is defined as the algebraic rank of* E.

Proof. The proof can be found in [Sil09, p. VIII]. The author also recommends the more didactic proof given in [Mas18, pp. 32–35], where the underlying computational ideas of this theorem are made explicit. □

The second important result in this setting describes the possibilities of rational torsion points of all elliptic curves.

Theorem 2.4 (Mazur Theorem)**.** *Given an elliptic curve $E/\mathbb{Q}$, its group of torsion points will be of one of the following types*

$$E(\mathbb{Q})_{tors} \cong \begin{cases} \mathbb{Z}/n\mathbb{Z} & 1 \leq n \leq 10, \\ \mathbb{Z}/2n\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} & 1 \leq n \leq 4. \end{cases}$$

Proof. The proof of this theorem can be found in [Maz77] and [Maz78]. □

The elliptic curves over $\mathbb{C}$ can be represented by the points in a complex torus i.e., for every $E/\mathbb{C}$ there exists a two dimensional complex lattice $\omega_1\mathbb{Z} + \omega_2\mathbb{Z} = \Lambda \subseteq \mathbb{C}$ (where ω_1 and ω_2 are $\mathbb{R}$-linearly independent) such that

$$E(\mathbb{C}) \cong \mathbb{C}/\Lambda.$$

The reciprocity of this theorem is also true i.e. given a lattice $\Lambda \subseteq \mathbb{C}$ with the properties given before, then the complex torus $\mathbb{C}/\Lambda$ represents an elliptic curve over $\mathbb{C}$.
Moreover, as commonly done, we define isogenies between elliptic curves as morphisms between the group of points and the associated point of the first elliptic curve is sent to the one of the second.
Given an elliptic curve $E/\mathbb{Q}$ and a prime $p \in \mathbb{N}$, we can consider the reduction of the points of the elliptic curve E to an elliptic curve $\overline{E}/\mathbb{F}_p$. The following classifies elliptic curves by their behavior on their reduction over $\mathbb{F}_p$

- E has good reduction if $\overline{E}$ is non-singular.
- E has additive reduction if $\overline{E}$ has a cusp
- E has split multiplicative reduction if $\overline{E}$ has a node and the tangent lines at such node are in $\mathbb{F}_p$
- E has nonsplit multiplicative reduction if $\overline{E}$ has a node and the tangent lines at such node are not in $\mathbb{F}_p$

For the scope of this document, we will need to use the conductor of a given elliptic curve over the rational numbers $\mathbb{Q}$. This concept can be generalized to different base fields, but this will not be needed in our case. The conductor of an elliptic curve $E/\mathbb{Q}$ is defined as

$$N = \prod_{p|\Delta} p^{f_p(E)},$$

where

$$f_p(E) := \begin{cases} 1 & E \text{ has multiplicative reduction at } p. \\ 2 & E \text{ has additive reduction at } p. \end{cases}$$

Given a prime p of good reduction over E, we define the value a_p as

$$a_p := p + 1 - N_p,$$

where N_p is the number of points in $E(\mathbb{F}_p)$. We can extend this definition to primes of bad reduction as

$$a_p = \begin{cases} 0 & \text{additive reduction,} \\ 1 & \text{split multiplicative reduction,} \\ -1 & \text{non-split multiplicative reduction.} \end{cases}$$

After considering these integers, we can define the L-function of E as the following infinite Euler product

$$L(E,s) = \prod_{p|N} \left(1 - a_p p^{-s} + p^{1-2s}\right)^{-1} \prod_{p\nmid N} \left(1 - a_p p^{-s}\right)^{-1}.$$

We shall remark that this product converges for all $S \in \mathbb{C}$ such that $\Re(S) > 3/2$. The following theorem gives us the common extension of the L-functions for the whole $\mathbb{C}$.

Theorem 2.5. *The function $L(E,s)$ extends to the entire function on $\mathbb{C}$*

$$\Lambda(E,s) = (2\pi)^{-s}\Gamma(S)N^{s/2}L(E,s),$$

where Γ is the common Γ-function. Furthermore, the function Λ satisfies the following functional equation

$$\Lambda(E,s) = \pm\Lambda(E, 2-s).$$

Proof. This Theorem for elliptic curves defined over $\mathbb{Q}$ follows from the Modularity theorem and a similar result for modular forms. Both results will be made explicit in the following pages. □

We aim to generalize the result over $\mathbb{C}$ to the setting of non-Archimedean fields. Let E be an elliptic curve defined over $\mathbb{Q}_p$ with p a prime natural number.

Proposition 2.6 (p-adic uniformization)**.** *Given an elliptic curve $E/\mathbb{Q}_p$ of which comes from an elliptic curve over $\mathbb{Q}$ which has multiplicative reduction over p, there exists an element $q_T \in \mathbb{Z}_p$ such that there exists a p-adic uniformization*

$$\eta_p : \mathbb{C}_p/q^{\mathbb{Z}} \to E(\mathbb{C}_p),$$

Let K be a number field and E an elliptic curve over K. We consider the collection of places of K and we denote them as S_K. We define the height over the abelian variety $E(K)$, for a given point $P \in E(K)\backslash\{O\}$, as

$$h(P) = \frac{1}{2[K:\mathbb{Q}]} \sum_{v\in S_K} [K_v : \mathbb{Q}_v] \max\{-\log|P_x|_v, 0\},$$

where $|\cdot|_v$ is one of the norms associated to v and P_x is the first coordinate of the point P. We need to introduce a well-known norm regarding abelian varieties.

Definition 2.7 (Néron-Tate height)**.** *Given a logarithm height h associated to a symmetric invertible sheaf on an abelian variety, we can define the canonical Néron-Tate height as*

$$\hat{h}(P) = \lim_{n\to\infty} \frac{h(nP)}{n^2}.$$

One should check that this definition satisfies the properties of heights. We denote $\hat{h}$ as the Néron-Tate height associated to the height h defined before. With this height, we can define the inner product for two given points $P, Q \in E(K)$

$$\langle P, Q\rangle = \hat{h}(P+Q) - \hat{h}(P) - \hat{h}(Q).$$

We will leave it to the reader to check the properties of inner products in the space of points of a given elliptic curve.

2.2 Modular forms

Similarly to the last subsection, we would like to focus on non-trivial notions related to modular forms, especially to the construction of the Jacobian associated to $S_2(\Gamma)$ for a modular subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ and the L-functions associated to modular forms.
Given a number a natural number $N \in \mathbb{N}$, we define the principal congruence subgroup of eleven N as

$$\Gamma(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \gamma \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \pmod N\right\} \leq \mathrm{SL}_2(\mathbb{Z}).$$

In general, a congruence subgroup is a a subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ such that Γ satisfies $\Gamma(N) \leq \Gamma$ for some number $N > 0$. The most famous examples of congruence groups, which will be essential in this thesis, are

$$\Gamma_0(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \gamma \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N\right\},$$
$$\Gamma_1(N) := \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}) \ : \ \gamma \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod N\right\}.$$

We define the cusps of a given congruence subgroup Γ as the Γ-orbits in $\mathbb{P}^1(\mathbb{Q})$ i.e. the quotient

$$\Gamma\backslash\mathbb{P}^1(\mathbb{Q}).$$

One should view these points as those that will be on the boundary of the compactification of the space $\mathcal{H}/\Gamma$. After discussing these two concepts, we can give the definition of modular forms with respect to a given congruence subgroup.

Definition 2.8 (Modular Forms)**.** *Given a congruence subgroup $\Gamma \leq \mathrm{SL}_2(\mathbb{Z})$ and an integer k, a modular form of weight k is an holomorphic function $f : \mathcal{H} \to \mathbb{C}$ which is holomorphic at all the cusps of Γ and satisfies*

$$f(\gamma z) = (cy + d)^k j(z) \text{ for all } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma.$$

We will denote the $\mathbb{C}$-vector space of modular forms of weight k as $M_k(\Gamma)$.

We will be especially interested in the concrete subgroup of $M_2(\Gamma)$ of functions that vanish at the infinity cusp. The modular forms that have such behavior will be called cusp forms, and we will denote the $\mathbb{C}$-vector space of cusp forms of weight k of Γ as $S_k(\Gamma)$.
Latter in this document, we will use the j function, which is defined using Eisenstein modular forms

$$j = \frac{1728(240E_4)^3}{(240E_4)^3 - (504E_6)^2}.$$

If the reader is interested in reading a definition of Eisenstein series, the author recommends the fourth chapter of [DS05]. One should notice that j is not a modular form, since it has a pole at the infinity cusp, however, it satisfies the modular equation $j(\gamma z) = j(z)$ for every $\gamma \in \mathrm{SL}_2(\mathbb{Z})$.
Given a modular form $f \in M_k(\Gamma_1(N))$ for given integers k, N, we define the operator associated to a matrix $\alpha \in \mathrm{GL}_2(\mathbb{Q})$ of positive determinant as

$$T_\alpha := \sum_{[\gamma] \in \Gamma \backslash \Gamma \alpha \Gamma} \frac{(\det \gamma)^k}{(cz + d)^k} f(\gamma z) \text{ with } \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We will leave it to the reader to check that this definition does not depend on the choice of the representatives of the quotient. We define the Diamond operator for an integer $d \in (\mathbb{Z}/N\mathbb{Z})^\times$ as

$$\langle d \rangle f := T_\alpha f,$$

where $\alpha \in \Gamma_0(N)$ with its lower right entry equal to d. Given a prime number p, we can also define the Hecke operator associated to this prime as

$$T_p f := \frac{1}{p} T_{\begin{pmatrix} 1 & 0 \\ 0 & p \end{pmatrix}} f.$$

We can extend the definition of Hecke operators to any integer $n \in \mathbb{N}$ by establishing $T_1 = id$ and for $n > 1$ as follows

$$T_{p^r} := T_p T_{p^{r-1}} - p^{k-1} \langle p \rangle T_{p^{r-2}} \qquad \text{for } p \text{ prime and } r > 2,$$
$$T_n := \prod_{p \text{ prime}} T_{p^{e_p}} \qquad \text{for } n = \prod_{p \text{ prime}} p^{e_p}.$$

One should observe that given $n, m \in \mathbb{N}$ such that $\gcd(n, m) = 1$ the following relation will be satisfied

$$T_{nm} = T_n T_m.$$

using the definition of these two operators we can give the definition of the Hecke algebra over the congruent subgroup $\Gamma_1(N)$.

Definition 2.9 (Hecke algebra). *The Hecke algebra, which we denote as $\mathbb{T}$, is a subalgebra of $\mathrm{End}_{\mathbb{C}}(M_k(\Gamma_1(N)))$ generated by the diamon operators $\langle d \rangle$, for $d \in (\mathbb{Z}/N\mathbb{Z})^\times$, and the Hecke operator T_p, with p prime.*

The Hecke algebra is a commutative algebra, and it is possible to get explicit formulas for the actions of the operators in $\mathbb{T}$. To get such formulas and all the associated proofs to these concepts, we recommend reading the fifth chapter of [DS05].

Definition 2.10 (Eigenform). *A modular form $f \in M_k(\Gamma_1(N))$ is an eigenform if it is an eigenvector for all the elements of $\mathbb{T}$ i.e. it is an eigenvector for T_p, with p prime, and $\langle d \rangle$, with $d \in (\mathbb{Z}/N\mathbb{Z})^\times$. Furthermore, we will say that an eigenform is normalized if its first non-constant coefficient in its Fourier expansion at infinity is 1.*

We consider an eigenform $f \in M_k(\Gamma_1(N))$ with Fourier expansion at the infinity cusp

$$f(z) = \sum_{n \in \mathbb{N}} a_n(f) q^n, \text{ with } q = \exp(2\pi i z).$$

If we denote λ_n the associated eigenvalue of T_n, we have the following equation

$$a_n(f) = \lambda a_1(f) \text{ for all } n > 0.$$

In concrete terms, this shows that the coefficients of normalized eigenforms will be the eigenvalues of T_n (with respect to f).
We will finish this subsection by defining the L function associated to a modular form in $M_k(\Gamma_1(N))$ and the common basic results around this concept.

Definition 2.11 (Partial L-function of a modular form). *Given a modular form $f \in S_k(\Gamma_1(N))$ with Fourier expansion at the cusp ∞ given as*

$$f(z) = \sum_{n \in \mathbb{N}} a_n(f) q^n, \text{ with } q = \exp\left(\frac{2\pi i z}{h}\right).$$

We define the partial L-function associated to the modular form f as

$$L(f, s) = \sum_{n \geq 1} a_n(f) n^{-s}$$

which converges for all $s \in \mathbb{C}$ such that $\Re(s)$ big enough.

Given an eigenform $f \in S_k(\Gamma_1(N))$, we denote as a_p and $\chi(d)$ the respective eigenvalues of the hecke operator T_p and the Diemanon operator $\langle d \rangle$. Using this notation, we have the following prime product expression of the partial L function $L(f, s)$

$$L(f,s) = \prod_{p \text{ prime}} (1 - a_p p^{-s} + \chi(p) p^{k-1-2s}) \text{ for } \Re(s) \text{ big enough.}$$

If we define the normalized eigenform $f^*(z) = \overline{f(-\overline{z})}$, one can check that there exists a complex number η_f satisfying

$$\omega_N f = \eta_f f^*,$$

where ω_N is called the Atkin-Lehner operator and it's defined as

$$\omega_N = T_\alpha \text{ with } \alpha = \begin{pmatrix} 0 & -1 \\ N & 0 \end{pmatrix}.$$

We define the completed L-function attached to an eigenform $f \in S_k(\Gamma_1(N))$ as

$$\Lambda(f,s) = N^{s/2} \frac{\Gamma(s)}{(2\pi)^s} L(f,s),$$

where $\Gamma(-)$ is the common Gamma function. The following theorem proves the common properties of completed L-functions.

Theorem 2.12. *Let f be a normalized eigenform of $S_k(\Gamma_1(N))$ and denote f^* as the dual eigenform as defined before. The following two conditions will hold*

(a) The function $\Lambda(f,s)$ can be continued to a monomorphic function on $\mathbb{C}$ with at most poles at $s = 0, k$. Furthermore, if f is a cusp form then the function $\lambda(f,s)$ is holomorphic on all $\mathbb{C}$.

(b) The completed L functions of f and f^* are related by the functional equation

$$\Lambda(f, k-s) = i^k \eta_f N^{-k/2} \Lambda(f^*, s).$$

Proof. One can find the results summarized in this statement by checking subsection 5.9. of [DS05]. □

In particular, this theorem proves the existence of a meromorphic extension of the partial L-function associated to a modular form f.
The relation between L-functions associated to elliptic curves and modular forms will be given in the subsection about the modularity theorem; however, we should introduce the notion of a modular elliptic curve. Given an elliptic curve $E/\mathbb{Q}$ of conductor N, we say that it is modular if there exists a cusp form $f \in S_2(\Gamma_0(N))$ such that

$$L(f,s) = L(E,s).$$

The following proposition will reduce the equation of L-functions given in the last definition to something more manageable.

Proposition 2.13. *Given an elliptic curve $E/\mathbb{Q}$ of conductor N, E is modular if there exists $f \in S_2(\Gamma_0(N))$ such that the following equation is satisfied*

$$\#E(\mathbb{F}_p) = p + 1 - a_p(f)$$

for all prime p of good reduction.

Proof. This result can be immediately deduced using the prime product of both the L function attached to the elliptic curve and the one attached to the modular curves. □

The reader might be questioning how the concept of modular forms can be generalized to a p-adic setting, similarly to what we did with elliptic curves in the last subsection. This generalization is far from an easy process, and it will be a central stone in this thesis. We will learn how to extend the definition of modular forms to non-Archimedean settings in section 4 after getting a clear picture of the p-adic upper half plane.

2.3 Construction of Shimura Curves

In this subsection, we will give the construction of Shimura curves associated to a pair (N^+, N^-), with N^- square-free, following Chapter 3 of [Pet89]. Let B be the quaternion algebra over $\mathbb{Q}$ split at infinity of discriminant N^- and R Eichler order of B of level N^+.

Let $\hat{\mathbb{Z}} = \prod_{\ell \neq \infty} \mathbb{Z}_\ell$ be the profinete competition of $\mathbb{Z}$ and $\hat{\mathbb{Q}} = \hat{\mathbb{Z}} \otimes \mathbb{Q}$ the ring of finite adèles. We define the adèlization of the quaternion algebra and Eichler order as

$$\hat{B} = B \otimes \hat{\mathbb{Q}} \text{ and } \hat{R} = R \otimes \hat{\mathbb{Q}} \subseteq \hat{B}.$$

If the number of prime divisors of N^- is even, we define the associated Shimura curve to the pair (N^+, N^-) as

$$X_{N^+,N^-} = B^\times \backslash Y \times \hat{B}^\times / \hat{R}^\times,$$

where Y is the genus zero curve canonically associated to the algebra B. The definition in this setting is mostly simple, however, when we try to extend such a definition to the case where N^- requires the solution of a moduli problem. We can define the following category attached to the quaternion algebra and the Eichler order R fixed before

$$(N^+, N^-)\text{-Ab.Surf.}/R := \begin{cases} \text{Ob:}(A, \iota) \text{ where A is an Ab. Surf.}/R \text{ and } \iota : R \hookrightarrow \text{End}_r(A) \\ \text{Ar: Maps between Ab. Surf. which preserve embbedings} \end{cases}$$

Using this category, we can define a moduli problem represented by the following functor, which goes from the category of $\mathbb{Q}$-Algebras to the category of sets

$$\begin{aligned} F : \mathbb{Q}\text{-Alg} &\to \text{Set} \\ R &\mapsto \{(\mathcal{A}, \mathcal{A}') \in (N^+, N^-)\text{-Ab.Surf.} | \exists \mathcal{A} \to \mathcal{A}'\} / \cong \end{aligned}$$

The following result assure us of the Scheme representation of this functor, which will immediately give us the definition of Shimura curves.

Theorem 2.14. *The functor $\overline{F}$ is coarsely represented by a regular, connected, two-dimensional scheme $\overline{X}_{N^+,N^-}$ smooth over* $\operatorname{Spce}\mathbb{Z}[1/N]$. *Furthermore, if $N^- \neq 1$, then $\overline{X}_{N^+,N^-}$ is proper over $\mathbb{Z}[1/N]$.*

We define the Shimura curve associated to the data (N^+, N^-) as the moduli space X_{N^+,N^-} which represents the moduli problem we have just defined. When $N^- = 1$ i.e. $B = M_2(\mathbb{Q})$, the solution to the moduli problem, which we will denote as $Y_0(N)$, is non-compact. One should point out that in this case, the moduli problem parametrizes elliptic curves, more specifically, for any closed algebraic field $F/\mathbb{Q}$, we have the following expression

$$Y_0(N)(F) = \{(E, E') \text{ ell. cur.}/F | \exists E \to E' \text{ of deg } N\}/\cong .$$

We denote $X_0(N)$ as the compactification of the curve $Y_0(N)$. The following theorem proves an isomorphism of the $\mathbb{C}$-points of the modular curve and a compact Riemann surface.

Proposition 2.15. *There exists an isomorphism of Riemann surfaces over the field of complex numbers $\mathbb{C}$*

$$X_0(N)(\mathbb{C}) \cong \mathcal{H}^* \big/ \Gamma_0(N),$$

where $\mathcal{H}^* := \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$.

Proof. We will start proving the identification for $Y_0(N)$. We define the map

$$\begin{aligned} \mathcal{H}/\Gamma_0(N) &\to Y_0(N)(\mathbb{C}) \\ \tau &\mapsto (\mathbb{C}/\langle \tau, 1\rangle, \mathbb{C}/\langle \tau, 1/N\rangle). \end{aligned}$$

We consider the natural isogeny $\varphi : \mathbb{C}/\langle \tau, 1\rangle \to \mathbb{C}/\langle \tau, 1/N\rangle$, which sends each element to its class (this is well defined because $\langle \tau, 1\rangle \subseteq \langle \tau, 1/N\rangle$). It is clear that $\ker\varphi \cong \mathbb{Z}/N\mathbb{Z}$, consequently, the map is well defined.
We have injectivity in this map because if we were to have two elements $\tau, \tau' \in \mathcal{H}$ generating the same class of isomorphic $\mathbb{C}$-elliptic curves, then there must exists $\gamma \in \Gamma_0(N)$ such that $\tau = \gamma\tau'$. Surjection comes from the fact that all classes of isomorphic $\mathbb{C}$-elliptic curves can be represented by a torus of the form $\mathbb{C}/\langle \tau, 1\rangle$.
This proves that the map defined before is an isomorphism of Riemann surfaces, considering compactifications, we get the isomorphism that we wanted. □

One can give a similar result for the $\mathbb{C}$-points of the Shimura curves we have defined. We consider the embedding associated to the quaternion algebra B, which exists because B is split at infinity,

$$\iota : B \to M_2(\mathbb{R}).$$

We define the congruent subgroup of $\mathrm{SL}_2(\mathbb{R})$ as $\Gamma_{N^+,N-} := \iota(R)$. This subgroup will give us the following identification of the complex points of the Shimura curves we have defined before.

Proposition 2.16. *The complex points of a given Shimura curve X_{N^+,N^-} can be identified with a Riemann surface using the congruent subgroup Γ_{N^+,N^-} defined before*

$$X_{N^+,N^-}(\mathbb{C}) \cong \mathcal{H} \big/ \Gamma_{N^+,N^-}.$$

Proof. This theorem is proven by using the moduli interpretation of Shimura curves and a similar approach to the modular curves one. □

However interesting this result might be, this thesis will use the focus of the $\mathbb{C}_p$ points of Shimura curves (for some prime $p|N^-$). There exists an analogous result that relates these points to a rigid analytic surface defined over p-adics. Such a relation is called p-adic uniformization, and it will be explained in detail when we introduce non-Archimedean CM points.

2.4 p-adic modular forms over Modular curves

The motivation for the definition of p-adic modular forms can be given using the moduli problems we have just stated. In this section, we will present in detail the underlying construction of p-adic modular forms using the Katz approach for Modular curves, following the paper [How22].
We will start by giving a slightly different definition of the moduli problem of modular curves by allowing arbitrary level structures.

Definition 2.17 (Level K elliptic moduli problem)**.** *Given $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ a closed subgroup, we define the moduli problem*

$$\begin{aligned} Y_K : \mathbb{Q}\text{-Alg} &\to \text{Set} \\ R &\mapsto \{(E, \mathcal{K})\}/\sim \end{aligned}$$

where the tuples satisfy

(i) E/R *elliptic curve*

(ii) $\mathcal{K} \subseteq \underline{Isom}((\mathbb{A}_f)^2, V_{\mathbb{A}_f}(E))$ *a K-torsor*

(iii) the relation is defined up to quasi-isogenies of elliptic curves that preserve the associated torsors $\mathcal{K}$.

Note that when we choose $K = \Gamma_0(N)$, the moduli problem we have just defined coincides with the classical moduli problem defined in the previous section. Moreover, when the subgroup is $K = \{e\}$, we will drop K from our notation.
We also have that the topological constant sheaf $\overline{N_{\mathrm{GL}_2(\mathbb{A}_f)}(K)}$ acts on the functor Y_K. Furthermore, given two closed subgroups $K_1 \leq K_2$, we can define the following map

between the associated functors

$$\begin{aligned} Y_{K_1} &\to Y_{K_2}, \\ (E, \mathcal{K}_1) &\mapsto (E, \mathcal{K}_1 \cdot \underline{K_2}). \end{aligned}$$

We are also interested in defining a similar functor where we disregard the information at the p-coordinate.

Definition 2.18 (Integral level K^p elliptic moduli problem)**.** *Given a closed subgroup $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f)$, we define the moduli problem*

$$\begin{aligned} \mathfrak{Y}_{K^p} : \mathbb{Z}_{(p)}\text{-Alg} &\to \text{Set} \\ R &\mapsto \{(E, \mathcal{K}^p\}/\sim \end{aligned}$$

satisifying

(i) E/R elliptic curve

(ii) $\mathcal{K}^p \subset \underline{Isom}((\mathbb{A}_f^{(p)})^2, V_{\mathbb{A}_f^{(p)}}(E))$ a K^p-torsor

(iii) the relation $\sim$ is defined up to prime-to-p quasi-isogenies of elliptic curves which preserve the associated torsors $\mathcal{K}^p$.

Similarly to the previous case, the topological constant sheaf $\underline{N_{\mathrm{GL}_2(\mathbb{A}_f^{(p)})}(K^p)}$ acts on the functor $\mathfrak{Y}_{K^p}$. Furthermore, given two groups $K_1^p \leq K_2^p$, we can define a map of the associated functors

$$\begin{aligned} \mathfrak{Y}_{K_1^p} &\to \mathfrak{Y}_{K_2^p}, \\ (E, \mathcal{K}_1^p) &\mapsto (E, \mathcal{K}_1^p \cdot \underline{K_2^p}). \end{aligned}$$

As in the last case, when the group is $K^p = \{e\}$, we will drop K^p from the notation. The following lemma gives us a relation between these two types of functors.

Lemma 2.19. *Given a closed subgroup $K^p \leq \mathrm{GL}_2(\mathbb{A}_f^{(p)})$, there exists an isomorphism*

$$\mathfrak{Y}_{K^p,\mathbb{Q}} \xrightarrow{\sim} Y_{\mathrm{GL}_2(\mathbb{Z}_p)K^p}$$

induced by the assignment

$$(E, \mathcal{K}^p) \mapsto (E, \underline{Isom}(\mathbb{Z}_p^2, T_p(E)) \times \mathcal{K}^p).$$

From now on, we will assume that the groups K (respectively K^p) are sufficiently small, this is, if they stabilize a $\hat{\mathbb{Z}}$-lattice $\mathcal{L} \subseteq \mathbb{A}_f^2$ (a $\hat{\mathbb{Z}}^{(o)}$-lattice $\mathcal{L} \subseteq (\mathbb{A}_f^{(p)})^2$) and, for some $n \geq 3$ ($\gcd(n,p) = 1$), it lies in the kernel of them map $\mathrm{GL}(\mathcal{L}) \to \mathrm{GL}(\mathcal{L}/n\mathcal{L})$.
One should note that this property is closed under inclusion and conjugation of $\mathrm{GL}_2(\mathbb{A}_f)$ (respectively $\mathrm{GL}_2(\mathbb{A}_f^{(p)})$, which justifies the denomination of such properties.
The following result will assure us of the scheme representation of the two functors we have defined.

Proposition 2.20. *Given a sufficiently small closed subgroup $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ (respectively $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f^{(p)})$), the functor Y_K ($\mathfrak{Y}_{K^p}$) is represented by an affine scheme over* $\operatorname{Spec}\mathbb{Q}$ *(*$\operatorname{Spec}\mathbb{Z}_{(p)}$*), and there exists an $\underline{N_{\mathrm{GL}_2(\mathbb{A}_f)}(K)}$-equivaraint ($\underline{N_{\mathrm{GL}_2(\mathbb{A}_f^{(p)})}(K^p)}$-equivaraint) isomorphism given by the natural map*

$$Y_K \to \lim_{K\subseteq K'\subseteq \mathrm{GL}_2(\mathbb{A}_f) \text{ compact open}} Y_{K'} \quad \left(\mathfrak{Y}_{K^p} \to \lim_{K^p\subseteq K^{p\prime}\subseteq \mathrm{GL}_2(\mathbb{A}_f^{(p)}) \text{ compact open}} \mathfrak{Y}_{K^{p\prime}}\right).$$

From now on, we will refer to the representations of the functors with the same notation as their respective functors. For any $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ (respectively $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f^{(p)})$) we denote as X_K ($\mathfrak{X}_{K^p}$) the smooth compactification of Y_K ($\mathfrak{Y}_{K^p}$). Moreover, the cusps of the modular curves will be the elements on the boundary $X_K \backslash Y_K$ (respectively $\mathfrak{X}_{K^p}\backslash\mathfrak{Y}_{K^p}$).
After introducing the setting we are going to work with in this section, we are in a position to give the algebraic definition of modular forms. For a sufficiently small closed subgroup $K \subseteq \mathrm{GL}(\mathbb{A}_f)$, we denote as E_K/Y_K the universal elliptic curve and ω the line bundle associated to the universal elliptic curve.
As it is mentioned in our reference paper [How22, p. 262], we can extend the definition of ω to the whole X_K by allowing sections with holomorphic q-expansions at each cusp.

Definition 2.21 (Modular Forms)**.** *Given a level $K \subseteq \mathrm{GL}_2(\mathbb{A}_f)$ and a weight k, we define the space of modular forms of weight k and level K as the sections*

$$M_{k,\mathbb{Q}}^K := H^0(X_K, \omega^k).$$

Given a prime p, one might be tempted to define modular forms over $\mathbb{F}_p$ as the sections

$$M_{k,\mathbb{F}_p}^{K^p} := H^0(\mathfrak{X}_{K^p}, \omega^k),$$

however, this definition turns out not to be the desired definition.
In general, the Hasse invariant is defined as a section constructed Zariski locally. Given a pair $(E/R, \alpha)$, where R is an $\mathbb{F}_p$-algebra E/R an elliptic curve and $\alpha \in \omega$ is a non-vanishing invariant differential, we assign an element $\mathrm{Ha}(E/R, \alpha) \in R$ satisfying

$$\mathrm{Ha}(E/R, a\alpha) = a^{-(p-1)}\,\mathrm{Ha}(E/R, \alpha) \text{ for all } a \in A^\times,$$

and which is functorial in based change and isomorphism. For our setting, we consider the invariant differential ∂_α dual to the element α and we define

$$\partial_\alpha^p := \partial_\alpha \circ \cdots \circ \partial_\alpha.$$

One should note that this object is also an invariant differential and, therefore, it is a multiple of ∂_α. We define the value $\mathrm{Ha}(E/R, \alpha)$ as the element that satisfies

$$\partial_\alpha^p = \mathrm{Ha}(E/R, \alpha)\partial_\alpha.$$

One can check that such a definition satisfies the desired properties of the Hase invariant. One can apply this construction to the universal elliptic curve over $\mathfrak{Y}_{\mathbb{F}_p}$. Doing some computations on the Tate curve, one can show that the q-expansions of the elements in the Hasse invariant are equal to 1 in each cusp, and, consequently, we can extend

$$\mathrm{Ha} \in M_{p-1,\mathbb{F}_p}^{\mathrm{GL}_2(\mathbb{A}_f^{(p)})}.$$

In order to define the modular forms over $\mathbb{F}_p$ and $\mathbb{Q}_p$ we need to show that the compactified modular curve $\mathfrak{X}_{K^p}$ decomposes exactly into two loci. We consider the following two matrices

$$b_{ss} = \begin{pmatrix} 0 & p \\ 1 & 0 \end{pmatrix} \text{ and } b_{ord} = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix} \in M_2(\mathbb{Z}_p).$$

We consider the p-divisible groups $\mathbb{X}_{ss}/\mathbb{F}_p$ and $\mathbb{X}_{ord}/\mathbb{F}_p$ associated to the covariant Dieudonné $\mathbb{Z}_p^2$-modules where the Frobenious acts by b_{ss} and b_{ord}, respectively. We know, from the theory of classification of p-divisible groups, that all p-divisible groups of one-dimensional height-two over an algebraically closed field is isomorphic or quasi-isogenous to exactly $\mathbb{X}_{ord} = \mu_p \times \mathbb{Q}_p/\mathbb{Z}_p$ or $\mathbb{X}_{ss}$.
In concrete terms, given a closed algebraic field $k/\mathbb{F}_p$, such a result applies to the groups $E[p^\infty]$ for any elliptic curve E/k. Therefore, we define the ordinary locus of $\mathfrak{Y}_{K^p,\mathbb{F}_p}$ (respectively supersingular locus) to be the ones whose points are ordinary (supersingular) i.e. the elliptic curves representing such points have groups $E[p^\infty]$ are isomorphic or quasi-isogenous to $\mathbb{X}_{ord}$ ($\mathbb{X}_{ss}$). One can show that the supersingular locus is closed i.e. the ordinary locus is open, and that the modular curve $\mathfrak{Y}_{K^p,\mathbb{F}_p}$ can decompose as a strict union of these two loci

$$\mathfrak{Y}_{K^p,\mathbb{F}_p} = \mathfrak{Y}_{K^p,\mathbb{F}_p}^{ss} \sqcup \mathfrak{Y}_{K^p,\mathbb{F}_p}^{ord}.$$

Local computations show us that the supersingular locus $\mathfrak{Y}_{K^p,\mathbb{F}_p}^{ss}$ is in fact the vanishing locus of the Ha. This fact justifies considering cusps as ordinary elements i.e.

$$\mathfrak{X}_{K^p,\mathbb{F}_p}^{ss} = \mathfrak{Y}_{K^p,\mathbb{F}_p}^{ss} = V(\mathrm{Ha}) \text{ and } \mathfrak{X}_{K^p,\mathbb{F}_p}^{ord} = \mathfrak{X}_{K^p,\mathbb{F}_p} \backslash \mathfrak{X}_{K^p,\mathbb{F}_p}^{ss} = \mathfrak{Y}_{K^p,\mathbb{F}_p}^{ord} \sqcup (\mathfrak{X}_{K^p,\mathbb{F}_p} \backslash \mathfrak{Y}_{K^p,\mathbb{F}_p}).$$

After describing these two loci of the compactified modular curve $\mathfrak{X}_{K^p}$, we are in a position to define the moduli problem which will give rise to the modular forms over $\mathbb{F}_p$.

Definition 2.22 (μ_p-Igusa moduli problem)**.** *Given a closed subgroup $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f^{(p)})$, we define the moduli problem*

$$\begin{aligned} Ig_{K^p,\mu_p}^{ord} : \mathbb{F}_p\text{-Alg} &\to \mathrm{Set} \\ R &\mapsto \{(E, \varphi_p, \mathcal{K}^p)\}/\sim \end{aligned}$$

where the triplets satisfy

(i) *E/R is an elliptic curve*

(ii) $\mathcal{K}^p \subseteq \underline{Isom}((\mathbb{A}_f^{(p)})^2, T_{\mathbb{A}_f^{(p)}}(E))$ *a K^p-torsor*

(iii) $\varphi_p : \mu_p \xrightarrow{\sim} \hat{E}[p]$ *an isomorphism*

(iv) *The relation $\sim$ is defined by prime-to-p quasi-isogenies which leave invariant the level $\mathcal{K}^p$ and the ismorphism φ_p.*

One should note that the condition of the isomorphism implies that the compactification of the moduli space attached to this problem, which we will denote as $Ig^{ord,c}_{K^p,\mu_p}$, is a finite étale $(\mathbb{Z}/p\mathbb{Z})^\times$ cover of $\mathfrak{X}^{ord}_{K^p}$

$$\mathfrak{X}^{ord}_{K^p} \leftarrow Ig^{ord,c}_{K^p,\mu_p}.$$

The definition of mod p modular forms will follow a similar approach to the ones defined over $\mathbb{Q}$, but over the étale cover $Ig^{ord,c}_{\mu_p}$.

Definition 2.23 (Modular forms over $\mathbb{F}_p$)**.** *The space of modular forms over $\mathbb{F}_p$ is defined as the section of the moduli space $Ig^{ord,c}_{\mu_p}$*

$$\mathcal{M}_{\mathbb{F}_p} := H^0(Ig^{ord,c}_{\mu_p}, \mathcal{O}).$$

One should note that the line bundle ω actually trivializes in the ordinary part of the compactified modular curve, which explains why we do consider the sections with respect to the generic sheaf $\mathcal{O}$.
As we mentioned before, the general sections of the compactified modular curve $M_{k,\mathbb{F}_p}$ are not the mod p modular forms that we desire; however, there exists a relation between these spaces and the space $\mathcal{M}_{\mathbb{F}_p}$ we have just defined.

Lemma 2.24. *Evaluation along the $(\varphi_p^{-1})^*(dt/t)$ induces a $(\mathbb{Z}/p\mathbb{Z})^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant isomorphism of rings*

$$\left(\bigoplus_{k\geq 0} M_{k,\mathbb{F}_p}\right)/(Ha-1) \cong \mathcal{M}_{\mathbb{F}_p}$$

where $(\mathbb{Z}/p\mathbb{Z})^\times$ acts on the space $\mathcal{M}_{\mathbb{F}_p}$ by the character $z \mapsto z^k$.

One should note that this isomorphism is telling us that the modular forms in $\mathcal{M}_{\mathbb{F}_p}$ are the classes of modular forms in $M_{k,\mathbb{F}_p}$ which are equal on the supersingular locus of $\mathfrak{X}$.
In order to define the modular forms over $\mathbb{Q}_p$, we will consider a formal limit of Igusa-moduli problems.

Definition 2.25 (Katz-Igusa mouli problem)**.** *Given a closed subgroup $K^p \subseteq \mathrm{GL}_2(\mathbb{A}_f^{(p)})$, we define the moduli problem*

$$\begin{aligned} Ig^{ord}_{K^p,\hat{\mathbb{G}}_m} : \mathrm{Nil}_{\mathbb{Z}_p} &\to \mathrm{Set} \\ R &\mapsto \{(E, \varphi_p, \mathcal{K}^p\}/\sim \end{aligned}$$

where the triplets satisfy

(i) *E/R is an elliptic curve*

(ii) *$\mathcal{K}^p \subseteq \underline{Isom}((\mathbb{A}_f^{(p)})^2, T_{\mathbb{A}_f^{(p)}}(E))$ a K^p-torsor*

(iii) *$\varphi_p : \hat{\mathbb{G}}_m \xrightarrow{\sim} \hat{E}$ an isomorphism*

(iv) *The relation $\sim$ is given by prime-to-p quasi-isogenies that preserve the level $\mathcal{K}^p$ and isomorphisms φ_p.*

As we mentioned before, this should be regarded as a limit of moduli problems in the following section, where arrows are given by forgetful functors

$$\mathfrak{X}_{K^p}^{ord} \leftarrow Ig_{K^p,\mu_p}^{ord,c} \leftarrow Ig_{K^p,\mu_{p^2}}^{ord,c} \leftarrow \cdots \leftarrow \lim_n Ig_{K^p,\mu_{p^n}}^{ord,c} =: Ig_{K^p,\hat{\mathbb{G}}_m}^{ord,c}.$$

As before, we will drop the notation when K^p is trivial. The space of p-adic modular forms is also defined as sections of the space $Ig_{\hat{\mathbb{G}}_m}^{ord,c}$.

Definition 2.26 (p-adic modular forms)**.** *The space of p-adic modular forms is defined as unitary $\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$ representation on the $\mathbb{Q}_p$-Banach space*

$$\mathbb{V}_{\mathbb{Q}_p} := \mathbb{V}_{\mathbb{Z}_p}[1/p] \text{ where } \mathbb{V}_{\mathbb{Z}_p} := H^0(Ig_{\hat{\mathbb{G}}_m}^{ord,c}, \mathcal{O}).$$

One can check that if we consider a general sufficiently small closed subgroup K^p, the space of sections associated to such a group is equal to the invariant space of $\mathbb{V}_{\mathbb{Z}_p}$

$$\mathbb{V}_{\mathbb{Z}_p}^{K^p} = H^0(Ig_{K^p,\hat{\mathbb{G}}_m}^{ord,c}, \mathcal{O}).$$

Using the trivialization of ω in $Ig_{K^p,\hat{\mathbb{G}}_m}^{ord}$ by $(\varphi_p^{-1})^*(dt/t)$, allows us to evaluate modular forms to elements in $\mathbb{V}_{\mathbb{Q}_p}$ and get the following result.

Lemma 2.27. *The evaluation on $(\varphi_p^{-1})^*(dt/t)$ induces a $\mathbb{Z}_p^\times \times \mathrm{GL}_2(\mathbb{A}_f^{(p)})$-equivariant injection*

$$\bigoplus_{k\geq 0} M_{k,\mathbb{Q}_p}^{\mathrm{GL}_2(\mathbb{Z}_p)} \hookrightarrow \mathbb{V}_{\mathbb{Q}_p}.$$

If we rewrite $\mathbb{V}_{\mathbb{F}_p} = \mathbb{V}_{\mathbb{Z}_p}/(p)$, by comparison of moduli problems, we can show that the invariants $\mathbb{V}_{\mathbb{F}_p}^{1+p\mathbb{Z}_p}$ represents the sections of the space $Ig_{\mu_p}^{ord,c}$. Using all the previous lemmas, we get the following diagram that summarizes all the maps we have mentioned

$$\begin{array}{ccccccc}
\bigoplus_{k\geq 0} M_{k,\mathbb{Q}_p}^{\mathrm{GL}_2(\mathbb{Z}_p)} & \longleftarrow & \bigoplus_{k\geq 0} H^0(\mathfrak{X}_{\mathbb{Z}_p}, \omega^k) & & \longrightarrow & & \bigoplus_{k\geq 0} M_{k,\mathbb{F}_p} \\
\downarrow & & \downarrow & & & & \downarrow \\
\mathbb{V}_{\mathbb{Q}_p} & \longleftarrow & \mathbb{V}_{\mathbb{Z}_p} & \longrightarrow & \mathbb{V}_{\mathbb{F}_p} & \longleftarrow & \mathbb{V}_{\mathbb{F}_p}^{1+p\mathbb{Z}_p}
\end{array}$$

Given a sufficiently small compact open group K^p and a collection of cusps $c_1, ..., c_m$ with one cusp in each connected component of $X_{\mathrm{GL}_2(\mathbb{Z}_p)K^p,\mathbb{Q}_p}$, we can consider the following map defined by the q-expansion map

$$M_{k,\mathbb{Q}_p}^{\mathrm{GL}_2(\mathbb{Z}_p)K^p} \to \prod_{i=0}^{m} \breve{\mathbb{Z}}_p[[q]][1/p].$$

One should point out that this approach to the definition of modular forms agrees with a generalization of the definition of Serre, which defines the space of p-adic modular forms as the completion of the span of the images for all k.
The construction we have exposed in this subchapter takes the modular curve as its underlying moduli problem. The p-adic modular forms we are going to use in this document will be defined by taking a Shimura curve as the associated moduli problem. This construction, together with the theory of p-adic uniformization exposed in [BC91], will justify the analytic definition of modular forms that we will give in chapter 4.

2.5 Modularity Theorem

In this subsection, we aim to study the correspondence between elliptic curves over $\mathbb{Q}$ of conductor N and newforms of weight 2 associated to $\Gamma_0(N)$. The following result proves that for every newform, we can associate an elliptic curve with the same L-function.

Theorem 2.28 (Eichler-Shimura reciprocity). *Given a normlaized eigenform $f \in S_2(\Gamma_0(N))$ such that $a_n(f)$ are integers, then there exists an elliptic curve $E_f/\mathbb{Q}$ such that*

$$L(f, s) = L(E_f, s).$$

Proof. We will follow the explicit Eichler-Shimura construction given in [DDT07]. We consider the correspondent subjective algebra homomorphism $\lambda_f : \mathbb{T}_{\mathbb{Q}} \to K_f$ associated to f, where K_f is the field generated by the coefficients of the Fourier expansion of f. We define the ideal $I_f := \ker(\lambda_f) \cap \mathbb{T}_{\mathbb{Z}}$ and we consider the image $I_f J_\Gamma$ that this ideal will produce in J_Γ through the induced action. Using this ideal, we can define the following abelian variety over $\mathbb{Q}$

$$A_f := J_\Gamma / I_f J_\Gamma.$$

Observe that A_f only depends on f and the endomorphism from $\mathbb{T}_{\mathbb{Z}}/I_f$, which is isomorphic to an order in K_f. We define V_f as the subspace of $V = S_2(\Gamma)^\vee$ where the λ_f acts through the map λ_f. Moreover, de denote π_f as the orthogonal projection of V to V_f with respect to the Petersson scalar product.
We consider the set $[f]$ of eigenforms whose Fourier coefficients are Galois conjugates to the coefficients of f. We have that the number of cusps forms in $[f]$ is equal to the

degree $[K_f : \mathbb{Q}]$. We define

$$V_{[f]} = \bigoplus_{g \in [f]} V_g \text{ and } \pi_{[f]} = \sum \pi_g.$$

Note that the map $\pi_{[f]}$ is the orthogonal projection of V to $V_{[f]}$. From Lemma 1.46. in [DDT07, p. 45], we have that the abelian variety A_f is isomorphic over $\mathbb{C}$ to the complex torus $V_{[f]}/\pi_{[f]}(\Lambda)$ where the map

$$\widetilde{\pi_{[f]}} : V/\Lambda \to V_{[f]}/\pi_{[f]}(\Lambda)$$

is the natural projection form J_Γ to A_f. In concrete, we have that A_f is an abelian variety of dimension $d = [K_f : \mathbb{Q}]$, however, since we are assuming that the coefficients of f are integers, we have $d = q$ i.e. A_f is an elliptic curve.
We have left to prove that this elliptic curve and the modular form have the same associated L-function. We shall point out that the conductor of the abelian variety A_f is N, for more details, please read [DDT07, p. 47].
Therefore, we consider the setting of primes that have good reduction on A_f. One should observe that since J_Γ has good reduction at primes not dividing N, we have that A_f also has good reduction at such primes. Given a prime of good reduction p, we define $N_{f,p}$ as the number of points on A_f over the finite field $\mathbb{F}_p$. We define the Tate module $\mathcal{T}_\ell(A_f)$ of the abelian variety A_f

$$\mathcal{T}_\ell(A_f) := \lim_{\leftarrow}(A_f)[\ell^n],$$

where the inverse limit is taken over the multiplication of ℓ maps. From Weil's theory [Wei48], we have that the element $N_{f,p}$ is given by the equation

$$N_{f,p} = \det(1 - F),$$

where F is the Frobenius endomorphism associated to the ℓ-adic Tate module $\mathcal{T}_\ell(A_f)$. From Theorem 1.41. [DDT07, p. 42], we have

$$\det(1 - F) = \mathrm{Norm}_{K_f/\mathbb{Q}}(\lambda_f(1 - a_p(f) + \langle p \rangle\, p)).$$

If we use both of the last equations, we get the following expression for $N_{f,p}$

$$N_{f,p} = \mathrm{Norm}_{K_f/\mathbb{Q}}(\lambda_f(1 - a_p(f) + \langle p \rangle\, p)).$$

In concrete, if we define the local Hasse-Weil L-function of A_f over $\mathbb{F}_p$ as

$$L(A_f/\mathbb{F}_p, s) = \det(1 - Fp^{-s})^{-1}.$$

Using this definition and the results relation the numbers $N_{f,p}$, we have that the L-function of $A_f/\mathbb{F}_p$ satisfies

$$L(A_f/\mathbb{F}_p, s) = \prod_{\sigma} L_p(f^{\sigma}, s),$$

where the product is taken over all the embeddings $\sigma : K_f \hookrightarrow \mathbb{C}$. Since f has integer Fourier coefficients and is a modular form over $\Gamma_0(N)$, we have the following relation

$$\#E(\mathbb{F}_p) = p + 1 - a_p(f).$$

This last equation immediately implies that the following L-functions associated to the elliptic curve A_f and f will be equal

$$L(A_f, s) = L(f, s).$$

□

We shall point out that the elliptic curve A_f defined in the proof is normally called a strong modular elliptic curve associated to the cusp form f and, from now onwards, it will be denoted as E_f.
The immediate question that comes to mind after seeing this last result is if there exists a correspondence in the other direction i.e. if given an elliptic curve $E/\mathbb{Q}$, there exists a newform such that their L-functions are the same. The modularity theorem essentially gives this result.

Theorem 2.29 (Modularity Theorem). *Given an elliptic curve $E/\mathbb{Q}$ of conductor N, there exist $f \in S_2(\Gamma_0(N))$ such that*

$$L(E, s) = L(f, s),$$

and the elliptic curve E is isomorphic to the strong modular elliptic curve E_f.

As a historical remark, this result was first proven by Wiles for semistable elliptic curves in [Wil95] and [TW95], giving de facto a proof of Fermat's last theorem. This result was generalized for elliptic curves that are semistable under reduction by 3 and 5 by Conrad, Diamond, and Taylor, in [Dia96] and [CDT99]. Finally, the general result was given by Breuil, Conrad, and Diamond in [Bre+01].
A useful result of the modularity theorem proves that the elliptic curves of conductor N can be parametrized over the complex numbers $\mathbb{C}$ by the modular curve $X_0(N)$.

Corollary 2.30. *Given an elliptic curve $E/\mathbb{Q}$ of conductor N, there exists a complex parametrization of curves*

$$\Phi : X_0(N) \to E.$$

Proof. We define as $f \in S_2(\Gamma_0(N))$ the newform associated to E, given by the modularity theorem. The modular curve $X_0(N)$ can be embedded in its Jacobian $J_0(N)$ by sending each point to $(P) - (i\infty)$. If we compose this map with the natural projection $J_0(N) \to E_f$ given by the Eichler-Shimura construction, we get the following modular parametrization

$$\Phi_f : X_0(N) \to E_f.$$

Joining this map with an isogeny $E_f \to E$, we get the desire complex parametrization

$$\Phi : X_0(N)(\mathbb{C}) \xrightarrow{\Phi_f} E_f(\mathbb{C}) \to E(\mathbb{C}).$$

□

This correspondence can be generalized to Shimura curves. Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = N^+N^-$ with $(N^+, N^-) = 1$ and N^- a square-free integer with an odd number of prime divisors. Given an eigenform $f \in S_2(\Gamma_{N^+,N^-})$ having integer $a_n(f)$, we can associate an elliptic curve, similarly to the modular curve case

Theorem 2.31 (Eichler-Shimura construction)**.** *Given a new form $f \in S_2(\Gamma_{N^+,N^-})$, there exists an elliptic E_f over $\mathbb{Q}$ such that*

$$L(f, s) = L(E_f, s).$$

Proof. Let $\mathbb{T}$ be the algebra generated by the Hecke operators T_n with $(n, N) = 1$. These operators can be seen as algebraic correspondences of the Shimura curve X_{N^+,N^-} and, therefore, give us the endomorphisms of the Jacobian J_{N^+,N^-} of the Shimura curve. The eigenform f defines the following homomorphism

$$\psi_f : \mathbb{T} \to \mathbb{Z}$$

that sends T_n to $a_n(f)$. We denote as I_f the kernel of the function ψ_f. The multiplicity result (expand) allows us to define the following elliptic curve over $\mathbb{Q}$

$$E_f := J_{N^+,N^-}/I_f.$$

Proving an analogous result of the Eichler-Shimura congruence for the correspondence T_p on $X^2_{N^+,N^-}$ gives us the desired equation of L-functions (expand)

$$L(E_f, s) = L(E, s).$$

□

We aim to give a similar result to the one proportionate by the Modularity theorem, in order to do so, we will announce the following correspondence between newforms in $\Gamma_0(N)$ and newforms in Γ_{N^+,N^-} given by the Jacquet-Langlands correspondence.

Theorem 2.32 (Jacquet-Langlands Correspondance)**.** *Given a newform $f \in S_2(\Gamma_0(N))$, there exists a newform $g \in S_2(\Gamma_{N^+,N^-})$ such that*

$$L(f,s) = L(g,s).$$

Observe that this result, together with the modularity theorem, gives us the inverse correspondence between modular elliptic curves and newforms in $S_2(\Gamma_{N^+,N^-})$, generalizing this correspondence for Shimura varieties. Similarly to the modular case, a complex parametrization can be proven for this setting

Corollary 2.33. *Given an elliptic curve $E/\mathbb{Q}$ of conductor $N = N^+N^-$ (satisfying the hypothesis described before), there exists a complex parametrization of curves*

$$\Phi : \mathrm{Div}^0(X_{N^+,N^-}) \to E.$$

Proof. This corollary is proven following the same method as the one of Corollary 2.30. □

3 Archimedean construction of Heegner points

After studying the modularity theorem for both modular curves and Shimura curves, we are in a position to define CM points for these two settings and their fundamental properties.

3.1 CM points

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. The modularity theorem assures us of the existence of a complex parametrization

$$\Phi : X_0(N) \to E.$$

Let $K/\mathbb{Q}$ be a quadratic imaginary field and $\mathcal{O}$ an order. We use the moduli interpretation of the modular curve to define the CM points associated to the order $\mathcal{O}$

$$\mathrm{CM}(\mathcal{O}) = \{(E,E') \in X_0(N)(\mathbb{C}) \mid \mathrm{End}(E) \cong \mathrm{End}(E') \cong \mathcal{O}\}.$$

One should observe that the points that we have just describe are pairs of elliptic curves which have Complex Multiplication with respect to the same quadratic imaginary order, hence, the name given for these points. A priori, it seems to be difficult to find points on the modular curve such that their image through Φ has algebraic coordinates. However, the following theorem gives a nice result regarding the image of CM points.

Theorem 3.1. *Let $\mathcal{O}$ be a quadratic imaginary order of K and H/K the ring class field associated to $\mathcal{O}$. For all $P \in \mathrm{CM}(\mathcal{O})$, we have $\Phi(P) \in E(H)$.*

Proof. This statement is essentially proven by showing that elliptic curves defined over $\mathbb{C}$ with complex multiplication with respect to $\mathcal{O}$ are extensions of elliptic curves defined over H. Then one uses the fact that the map Φ is an algebraic map of curves defined over $\mathbb{Q}$ to conclude the proof. $\square$

The group $\mathrm{Pic}(\mathcal{O})$ induces an action in $\mathrm{CM}(\mathcal{O})$ by multiplying the associated lattices of the two elliptic curves of a given point by a representative of the fractional ideal class chosen. Let H be the ring class field associated to $\mathcal{O}$, we denote as

$$\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \to \mathrm{Gal}(H/K)$$

the Artin reciprocity law (it is an isomorphism). This induced action gives us the following property of CM points.

Theorem 3.2 (Shimura reciprocity law). *Let $P \in \mathrm{CM}(\mathcal{O})$ and $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we have the following equation*

$$\Phi(\mathfrak{a}P) = \mathrm{rec}(\mathfrak{a})^{-1}\Phi(P).$$

Proof. The proof follows from the constructions of canonical models given by Shimura in [Shi63]. $\square$

The joint results of the last two theorems are commonly referred to as the theorem of complex multiplication. One can also explicitly a characterization of CM points with respect to an order $\mathcal{O}$ by imposing some conditions on the behavior of the primes that divide N with respect to the extension K.

Theorem 3.3 (Heegner Hypothesis). *Let $\mathcal{O}$ be an order of a quadratic imaginary field K of conductor prime to N. The set of CM points associated to $\mathcal{O}$ if and only if the ideal $N\mathcal{O}$ can decompose as a product $\mathcal{N}\overline{\mathcal{N}}$ of cyclic ideals of norm N.*

Proof. If $\mathrm{CM}(\mathcal{O}) \neq \varnothing$, then the order can be realized as a subring of $M_0(N)$. Then we can define a ring homomorphism $\mathcal{O} \to \mathbb{Z}/N\mathbb{Z}$. We define the ideal

$$\mathcal{N} := \ker(\mathcal{O} \to \mathbb{Z}/N\mathbb{Z}).$$

Since the discriminant is coprime to the conductor, we have that $N\mathcal{O} = \mathcal{N}\overline{\mathcal{N}}$ and the ideal $\mathcal{N}$ has norm N. This proves the first implication.
If the decomposition hypothesis of $N\mathcal{O}$ is satisfied, we can define the following map using the moduli interpretation of $Y_0(N) \subset X_0(N)$

$$\begin{aligned} \delta : \mathrm{Pic}(\mathcal{O}) &\to X_0(N)(\mathbb{C}) \\ \mathfrak{a} &\mapsto (\mathbb{C}/\mathfrak{a}, \mathbb{C}/\mathcal{N}^{-1}\mathfrak{a}) \end{aligned}$$

It is easy to check that such map is well defined and that the points in $\mathrm{Im}(\delta)$ are CM points. This immediately shows that $\mathrm{CM}(\mathcal{O}) \neq \varnothing$ because $\mathrm{Pic}(\mathcal{O}) \neq \varnothing$, which proves the other implication. $\square$

The reader might have found such a hypothesis in the literature described as the imposition that all the primes dividing N are split in K. It is simple to prove that this hypothesis is equivalent to the one we described here.
One should also point out that the different δ functions for all possible orientations of the order $\mathcal{O}$ will give us all the possible CM points associated to this order i.e.

$$\mathrm{CM}(\mathcal{O}) = \bigcup_{\mathcal{N} \text{ orientation of } \mathcal{O}} \delta_{\mathcal{N}}(\mathrm{Pic}(\mathcal{O})).$$

As we have seen before $X_0(N)(\mathbb{C})$ is normally identified with the Riemann surface $\mathcal{H}^*/\Gamma_0(N)$. This identification allows us to give another equivalent definition of CM points. For any given point $\tau \in \mathcal{H}/\Gamma_0(N)$ (similarly to the moduli case, we do not consider the compactification part), we define the order associated to τ as

$$\mathcal{O}_\tau := \{\gamma \in M_0(N) \mid \gamma\tau = \tau\} \cup \left\{ \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \right\}.$$

With this last concept, the definition of CM points is reduced to the following set

$$\mathrm{CM}(\mathcal{O}) \cong \{\tau \in \mathcal{H}/\Gamma_0(N) \mid \mathcal{O}_\tau \cong \mathcal{O}\}.$$

The equivalence is fairly easy to prove using the isomorphism $X_0(N)(\mathbb{C}) \cong \mathcal{H}^*/\Gamma_0(N)$ specified before; the details will be left to the reader.

3.2 Heegner Systems

We are also interested in considering Heegner points as a system attached to an elliptic curve E over $\mathbb{Q}$ and a quadratic imaginary field K. We consider an elliptic curve $E/\mathbb{Q}$ with conductor N and we assume it satisfies the Heegner Hypothesis for a given quadratic imaginary field K. We will follow the third chapter of [Dar04].
Given an integer n prime to N, we denote as $\mathcal{O}_n$ the order of K of conductor n. Furthermore, we consider the associated ring class field H_n associated to this order.

Definition 3.4 (Heegner point of conductor n)**.** *A point $P \in E(H_n)$ is a Heegner point of conductor n if there exists a CM point $\widetilde{P} \in \mathrm{CM}(\mathcal{O}_n)$ such that*

$$\Phi(\widetilde{P}) = P.$$

We will denote the collection of Heegner points of conductor n as $HP(n) \subseteq E(H_n)$.

Given an integer n, the Heegner points associated to this conductor satisfy the following norm-compatibilities.

Proposition 3.5. *Given a prime ℓ prime to N, for all the points $P_{n\ell} \in HP(n\ell)$ then there exists a points $P_n \in HP(n)$ and, when ℓ divides n, the point $P_{n/\ell} \in HP(N/\ell)$ such that*

$$\mathrm{Tr}_{H_{n\ell}/H_n}(P_{n\ell}) = \begin{cases} a_\ell P_n & \text{if } \ell \nmid n \text{ is inert in } K, \\ (a_\ell - \sigma_\lambda - \sigma_\lambda^{-1})P_n & \text{if } \ell = \lambda\overline{\lambda} \nmid n \text{ is split in } K, \\ (a_\ell - \sigma_\lambda)P_n & \text{if } \ell = \lambda^2 \text{ is ramified in } K, \\ a_\ell P_n - P_{n/\ell} & \text{if } \ell \mid n. \end{cases}$$

Proof. The proof can be found in [Dar04, p. 35]. □

We will say that a map $\tau \in \mathrm{Gal}(H/\mathbb{Q})$ is a reflection with respect to K if its restriction to the field K is not the identity map. One should observe that all the reflections are of order 2 and that two reflection will always be related by multiplication of an element in $\mathrm{Gal}(H/K)$. The following proposition will describe the behavior of the points in $HP(n)$ under the action of reflections.

Proposition 3.6. *Given a reflection $\tau \in \mathrm{Gal}(H/\mathbb{Q})$ there exists a $\sigma \in \mathrm{Gal}(H/K)$ such that*

$$\tau P_n = -\operatorname{sign}(E, \mathbb{Q})\sigma P_n \pmod{E(H)_{\mathrm{tors}}}.$$

Proof. The proof can be found in [Gro84]. □

After describing the behaviour of Heegner points of conductor n with respect to the trace and reflections, we are on a position to give the definition of Heegner Systems.

Definition 3.7 (Heegner System)**.** *Given an elliptic curve E and a quadratic imaginary field $K/\mathbb{Q}$, we define Heegner systems as a collection*

$$(P_n)_n \in \prod_{(n,N)=1} HP(n),$$

which will satisfy the norm compatibilities and the reflection behaviour described before.

Given a Heegner System, we will say that it is non-trivial if one of the associated points P_n is non-torsion. The following theorem assures us of the existence of one non-trivial Heegner system for the setting we are interested in, within the scope of this document.

Theorem 3.8. *If the field K satisfies the Heegner Hypothesis with respect to the elliptic curve E, then there exists a non-trivial Heegner System attached to E and K.*

Before proving this theorem, we shall state a proof a lemma about the torsion group of E over the ring class field H_∞.

Lemma 3.9. *The torsion subgroup $E(H_\infty)$ is finite.*

Proof. All the primes inert at K are either completely split or ramified in all the ring class fields. This implies that given a prime p, the residue field of H_∞ at such a prime is the field $\mathbb{F}_{p^2}$. We have that the prime-to-p torsion in $E(H_\infty)$ inject to $E(\mathbb{F}_{p^2})$, therefore the full p torsion points inject to the group $E(\mathbb{F}_{p_1^2}) \oplus E(\mathbb{F}_{p_2^2})$, where p_1 and p_2 are primes inert in K. □

After proving this lemma, we are ready to give the proof of the theorem, following the proof given by Darmon.

Proof of Theorem 3.13. Let CM(n) be the collection of CM points associated to the order of conductor n. The Heegner condition assure us that the image through the modular parametrization of such a set is dense in $\mathcal{H}$. This fact implies that the image in $E(\mathbb{C})$ is infinite. The statement of the lemma rules out the possibility that all the points in this image are torsion. □

This last theorem will give us a proof for a concrete case of the following conjecture, which expects to have non-trivial Heegner systems associated to elliptic curves with associated sign -1.

Conjecture 3.10. *Let E be an elliptic curve E defined over a number field F and let K be a quadratic extension of F. We define the sign attached to the elliptic curve E over K as the sign of the functional equation of the completed L-function associated to E. If the sign is -1, then there is a non-trivial Heegner system attached to (E, K).*

This conjecture is still remains unproven; however, as we have mentioned before, if we assume that K is a quadratic imaginary field satisfying the Heegner hypothesis, one can construct a non-trivial Heegner system using the modularity Theorem, given, de facto proof for these cases.

3.3 Periods and Heegner points

After defining CM points in this purely algebraic way, it is interesting to prove the relation between this classical definition and some period integrals. To fulfill this approach, we shall start defining optimal oriented embeddings

Definition 3.11 (Optimal oriented embeddings). *Given an order $\mathcal{O} \subseteq K$, we say that an embedding $\Psi : K \to \mathcal{M}_2(\mathbb{Q})$ is optimal oriented if it satisfies the following conditions*

(i) (Optimal) The embedding is called optimal if it satisfies $\Psi(K) \cap M_0(N) = \Psi(\mathcal{O})$.

(ii) (Oriented) We consider an orientation of the Eichler order i.e. a subjective morphism $M_0(N) \to \mathbb{Z}/N\mathbb{Z}$. The embedding is said to be oriented (relative to an orientation $\mathcal{N}$ of $M_0(N)$) if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & M_0(N) \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathcal{N} & \xrightarrow{\sim} & \mathbb{Z}/N\mathbb{Z} \end{array}$$

(iii) *If we consider the action generated by the embedding at K, there exists a unique fixed point $\tau_\Psi \in K$ such that*

$$\Psi(\lambda)\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix} = \lambda\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix}.$$

We say that the embedding is oriented at infinity if τ_Φ belongs to $\mathcal{H}$ under a fixed embedding $K \hookrightarrow \mathbb{C}$

Using this last property, we can define an invertible fractional ideal associated to an optimal oriented embedding Ψ as $\mathfrak{a}_\Psi := \mathbb{Z} + \tau_\Psi\mathbb{Z}$, where τ_Ψ is the fixed point of Ψ. From now onwards, we will fix an order $\mathcal{O} \subseteq K$ satisfying the Heegner Hypothesis i.e. there exists a principal ideal $\mathcal{N}$ such that $N\mathcal{O} = \mathcal{N}\overline{\mathcal{N}}$. This assumption allows us to prove the following bijection.

Lemma 3.12. *There exists a bijection between* $\mathrm{Pic}(\mathcal{O})$ *and* $\mathrm{End}(\mathcal{O}, M_0(N))$ *induced by the assignment* $\Psi \mapsto \mathfrak{a}_\Psi$.

Proof. Given $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, the class $\mathfrak{a}$ has a representative of the form $\langle \tau_\mathfrak{a}, 1\rangle$ with $\tau_\mathfrak{a} \in \mathcal{H}$ such that

$$\mathcal{N}^{-1}\langle \tau_\mathfrak{a}, 1\rangle = \langle \tau_\mathfrak{a}, 1/N\rangle. \tag{3.3.1}$$

The element $\tau_\mathfrak{a}$ is determined by the class $\mathfrak{a}$ up to the action of $\Gamma_0(N)$. We define the optimal embedding $\Psi_\mathfrak{a} : K \hookrightarrow M_2(\mathbb{Q})$ associated to $\mathfrak{a}$ by imposing

$$\Psi_\mathfrak{a}(\tau_\mathfrak{a}) = \begin{pmatrix}\mathrm{Tr}(\tau_\mathfrak{a}) & -\mathrm{Nm}(\tau_\mathfrak{a}) \\ 1 & 0\end{pmatrix}.$$

The class of the embedding $\Psi_\mathfrak{a}$ in $\mathrm{Emb}(\mathcal{O}, M_0(N))$ depends only on the choice of the class $\mathfrak{a}$ and not on the representative of such class. If we fix the orientation map

$$\begin{aligned} M_0(N) &\to \mathbb{Z}/N\mathbb{Z} \\ \begin{pmatrix}a & b \\ c & d\end{pmatrix} &\mapsto \bar{a} \end{aligned}$$

where $\bar{\cdot}$ denotes the natural projection map $\mathbb{Z} \to \mathbb{Z}/N\mathbb{Z}$. The condition 3.3.1 implies that the diagram

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi_\mathfrak{a}} & M_0(N) \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathcal{N} & \xrightarrow{\sim} & \mathbb{Z}/N\mathbb{Z} \end{array}$$

is commutative and, consequently, the embedding $\Psi_\mathfrak{a}$ is oriented. Furthermore, the fixed point of the embedding $\Psi_\mathfrak{a}$ is $\tau_\mathfrak{a}$ which, as we imposed before, is in $\mathcal{H}$. This shows that $\Psi_\mathfrak{a}$ is an optimal oriented embedding.
We shall check that the assignment $\mathfrak{a} \mapsto \Psi_\mathfrak{a}$ induces a map which is the inverse of the map $\Psi \mapsto \mathfrak{a}_\Psi$. The following composition

$$(\Psi \mapsto \mathfrak{a}_\Psi) \circ (\mathfrak{a} \mapsto \Psi_\mathfrak{a}) = id_{\mathrm{End}(\mathcal{O}, M_0(N))}$$

is clearly the identity map. On the other hand, since, given $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, the point $\tau_\mathfrak{a}$ is the fixed point of $\Psi_\mathfrak{a}$, we have

$$(\mathfrak{a} \mapsto \Psi_\mathfrak{a}) \circ (\Psi \mapsto \mathfrak{a}_\Psi) = id_{\mathrm{Pic}(\mathcal{O})}.$$

□

The bijection given by the lemma enables us with an action induced by left multiplication in $\mathrm{Pic}(\mathcal{O})$ in $\mathrm{End}(\mathcal{O}, M_0(N))$ which will be denoted as $\Psi^\mathfrak{a}$ for Ψ an optimal oriented embedding and $\mathfrak{a}$ a class of $\mathrm{Pic}(\mathcal{O})$.
We fix a Néron differential ω_E on E, and we denote Λ the lattice generated by the periods of such differential. We denote as

$$\eta : \mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C})$$

the Weierstraß uniformization associated to the lattice Λ. Let f be the normalized weight 2 cusp form associated to E given by the Modularity Theorem. From Chapter VI.5 of [Sil09], we have the following result.

Proposition 3.13. *Up to a non-zero rational constant, defined as the Mannin constant, which is expected to be* ± 1*, we have that* $(\Phi \circ j)^*\omega_E$ *is equal to* $2\pi i f(t)dt$*.*

Furthermore, in that same paper, the following result is obtained

$$\Phi(\delta(\langle \tau, 1 \rangle)) = \eta\left(\int (\Phi \circ j)^*\omega_E\right).$$

Given an optimal oriented embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, M_0(N))$ we define the following period using the function f

$$J_\Psi := \int_{i\infty}^{\tau_\Psi} 2\pi i f(t)dt.$$

The following theorem relates these periods to the CM points associated to the different elements of $\mathrm{Pic}(\mathcal{O})$ represented by a concrete orientation.

Theorem 3.14. *For all optimal oriented embedding* $\Psi \in \mathrm{Emb}(\mathcal{O}, M_0(N))$ *and* $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$*, the following equation will be satisfied*

$$\eta(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta(J_\Psi).$$

Proof. Fixing the orientation $\mathcal{N}$, we can also fix the map $\delta : \mathrm{Pic}(\mathcal{O}) \to X_0(N)$ described before, which will induces the equation

$$\Phi(\delta(\langle\tau, 1\rangle)) = \eta\left(\int (\Phi \circ j)^* \omega_E\right).$$

proven before. This equation shows that the points $\eta(J_\Psi)$ and $\eta(J_{\Psi^\mathfrak{a}})$ are images of CM points in $X_0(N)$ identified by the map δ. Therefore, the theorem is an immediate result of the Shimura reciprocity law (Theorem 3.2). □

3.4 Heegner points over Shimura curves

Until this moment, we have assumed that the primes dividing the conductor of the elliptic curve behaved in the most convenient manner; however, this is rarely the case. As we have done before, we consider an elliptic curve with conductor N and a quadratic imaginary field $K/\mathbb{Q}$ where all the primes dividing N are not ramified.
We define the pair (N^+, N^-) as the decomposition $N = N^+N^-$ where all the primes dividing N^+ are split in K (i.e. all the primes dividing N^- are split in K. We consider the Shimura curve X_{N^+,N^-} associated to this pair, and we define CM points attached to an order $\mathcal{O} \subseteq K$ over this curve using the moduli interpretation, as we did in the modular curve case

$$\mathrm{CM}(\mathcal{O}) = \{(A, A') \in X_{N^+,N^-}(\mathbb{C}) \ : \ \mathrm{End}(A) \cong \mathrm{End}(A') \cong \mathcal{O}\}.$$

Where the endomorphisms are taken on the sense of the category defined at the construction of Shimura curves given in the section of preliminary notions. Similarly to the case first presented in this subsection, we do have a characterization of the CM over Shimura curves

Proposition 3.15. *The collection of CM points in X_{N^+,N^-} with respect an order $\mathcal{O} \subseteq K$ is non-empty if and only if all the primes dividing N^+ are split in K and all the primes dividing N^- are split in K.*

Proof. The proof is essentially a generalization of the one given in the modular curve one, using the lattice form of abelian surfaces over $\mathbb{C}$. □

Observe that this Characterization assure us that if we fix a quadratic imaginary field, as we have done at the beginning of this section, there is a unique possible datum we can choose in order to have a Shimura curve that has Heegner points.
Following the construction of the Shimura curve X_{N^+,N^-} we consider a quaternion algebra B over $\mathbb{Q}$ of discriminant N^- and an Eichler order R of level N^+ in B. Since the quaternion algebra is split at infinity, we can define an embedding $\iota : B \to M_2(\mathbb{Q})$ and the congruence subgroup

$$\Gamma := \iota(R_1^\times),$$

where $R_1^\times$ are the units of norm 1 in R. As we have shown at the preliminary notions, we can identify the complex points of the Shimura curve with a Riemann surface

$$X_{N^+,N^-}(\mathbb{C}) \cong \mathcal{H}/\Gamma.$$

Observe that this surface is already compact and, therefore, we lose the notion of cusps i.e. the cup forms we will consider will not have a Fourier expression in general. Similarly to the modular curve case, we can give an equivalent definition using this isomorphism by defining the order associated to an element $\tau \in \mathcal{H}/\Gamma$ as

$$\mathcal{O}_\tau := \{\gamma \in R \ : \ \text{norm}(\gamma) = 0 \text{ and } \iota(\gamma)\tau = \tau\} \cup \{0\}.$$

With these orders, one can show that CM points can be identified in the Riemann surface as the collection

$$\text{CM}(\mathcal{O}) = \{\tau \in \mathcal{H}/\Gamma \ : \ \mathcal{O}_\tau \cong \mathcal{O}\}.$$

We use the modularity theorem to identify a cusp form $f_E \in S_2(\Gamma_0(N))$ related to the elliptic curve we have fixed at the beginning of this section. Furthermore, we use the Jacquet-Langlands correspondence to fix a cusp form $f \in S_2(\Gamma)$ which satisfies the following equation of L-functions

$$L(f, s) = L(f_E, s) = L(E, s).$$

The essential difficulty we will face when we try to reproduce the period approach for this setting is that we do not have cusps in $X_\Gamma := \mathcal{H}/\Gamma$ i.e. we do not have the notion of infinity. In consequence, we shall find an alternative definition of semi-defined integral for this setting. One should note that the integral we have defined before of the function f yields a rigid analytic uniformization of the elliptic curve by the Riemann surface X_Γ. If we extend by linearity, we get a map $\text{Div}^0(X_\Gamma) \to \mathbb{C}$, which descends to a map

$$\begin{aligned} &\text{Pic}^0(X_\Gamma) \to \mathbb{C}, \\ &(\tau_2) - (\tau_1) \mapsto \int_{(\tau_2)-(\tau_1)} 2\pi i f(t) dt, \end{aligned}$$

where $\text{Pic}^0(X_\Gamma)$ is the jacobian associated to X_Γ. We chose a correspondence $\theta \in \mathbb{T}$ which induces a map $\text{Div}(X_\Gamma) \to \text{Div}^0(X_\Gamma)$. Using this map, we redefine the integral we are considering

$$\int_{\tau_1}^{\tau_2} 2\pi i f(t) dt := \int_{\theta((\tau_2)-(\tau_1))} 2\pi i f(t) dt.$$

We can extend the definition of this integral defined over $\text{Div}^0(X_\Gamma)$ to the whole variety $\text{Pic}(X_\Gamma)$ as

$$(\tau) \mapsto \int^{\tau} 2\pi i f(t) dt := \int_{\theta(\tau)} 2\pi i f(t) dt.$$

One should see this integral as the extension of the semi-indefinite integral over the Riemann surface X_Γ. In this sense, the integral satisfies the common semi-indefinite relation of integrals

$$\int^{\tau_2} 2\pi i f(t)dt = \left(\int_{\tau_1}^{\tau_2} 2\pi i f(t)dt\right)\left(\int^{\tau_1} 2\pi i f(t)dt\right).$$

We will face a similar problem for the non-Archimedean CM points, which we will solve with a similar approach. Given a quadratic imaginary field $K/\mathbb{Q}$ and an order $\mathcal{O} \subseteq K$, we introduce the notion of optimal oriented embeddings for this setting.

Definition 3.16 (Optimal oriented embedding)**.** *An embedding $\Psi : K \to B$ is considered optimal oriented if it satisfies the following properties*

(i) (Optimal) The embedding is optimal with respect to R and $\mathcal{O}$ if $\Psi(O) = \Psi(K) \cap R$,

(ii) (Oriented) Fixing orientation maps $\mathcal{O}, R \to (\mathbb{Z}/N^+\mathbb{Z}) \times \prod_{l|N^-} \mathbb{F}_{l^2}$, we say that the embedding is oriented if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & R \\ & \searrow & \downarrow \\ & & \left(\mathbb{Z}/N^+\mathbb{Z}\right) \times \prod_{l|N^-} \mathbb{F}_{l^2} \end{array}$$

(iii) The induced action of K though the embedding $\iota\Psi$ has a unique fixed point $\tau_\Psi \in K$. We say that the embedding is oriented at infinity if this point belongs to the upper half-plane under a fixed embedding $K \to \mathbb{C}$.

One can prove that such optimal oriented embeddings exist under the assumption of the Heegner Hypothesis for the Shimura curve X_{N^+,N^-}.

Proposition 3.17. *There exists an optimal oriented embedding $\Psi : K \to B$ with respect to $\mathcal{O}$ and R if the Heegner Hypothesis is satisfied by K.*

Proof. The proof follows from the theory shown in [Voi21]. □

Similarly to the last case, given an optimal oriented embedding Ψ we can define a period using the function f as

$$J_\Psi := \int^{\tau_\Psi} 2\pi i f(t)dt$$

The reader should recall that in order to relate these periods with Heegner points we shall fix a Néron differential (with its associated lattice Λ) and consider the associated Weierstraß uniformization

$$\eta : \mathbb{C}/\Lambda \xrightarrow{\sim} E(\mathbb{C}).$$

Let H be the ring class field associated to the order $\mathcal{O}$. The following theorem gives us the analogous result to the one we proved for modular curves

Theorem 3.18. *Given an optimal oriented embedding Ψ, we have that $\eta(J_\Psi) \in E(H)$. Furthermore, given any element $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$ the following equation will be satisfied*

$$\eta(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta(J_\Psi).$$

Proof. The proof follows from the main theorem of complex multiplication and realizing that the values $\eta(J_\Psi)$ are the images of the CM points associated to each ψ through the map given by the modularity theorem. □

3.5 Famous solutions given by Heegner points

To underpin the importance of Heegner's points in modern Algebraic Number Theory, and our motivation to study them, we will describe three famous problems where Heegner points have provided a partial solution.

Hilbert's 12 problem

The Kronecker-Weber theorem gives us an explicit characterization of the maximal abelian extension of $\mathbb{Q}$

$$\mathbb{Q}^{ab} = \bigcup_{n\in\mathbb{N}} \mathbb{Q}(\zeta_n).$$

One might ask a similar question for a different based field $K/\mathbb{Q}$. The characterization in this case is given by narrow class fields

$$K^{ab} = \left(\bigcup_{n\geq 1} K_n\right) \cup \left(\bigcup_{n\geq 1} K(\zeta_n)\right)$$

where K_n are the narrow class fields of conductor n. However, finding explicit generators of the narrow class fields is a difficult problem, which is normally referred to as explicit class field theory.

The construction of algebraic generators has only worked out for quadratic imaginary fields K using the theory of complex multiplication and Heegner points. One can identify a concrete Heegner point by a quadratic order $\mathcal{O}$ inside K and a class $[\mathfrak{a}] \in \mathrm{Pic}(\mathcal{O})$. Using this notation, we define the following

$$j(\mathfrak{a}) \in \mathbb{Q}^{ab},$$

using the common map $j : X_0(1) \to \mathbb{P}^1(\mathbb{C})$. We shall start proving that these integers are indeed algebraic integers.

Proposition 3.19. *Given a class $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, one has that $j(\mathfrak{a})$ is an algebraic number.*

Proof. Given a $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we fix $j = j(\mathfrak{a})$ and E an elliptic curve representing the isomorphism class associated to $\mathfrak{a}$ (through the natural map $\mathrm{Pic}(\mathfrak{O}) \to X_0(1)$). This elliptic curve is isomorphic over $\mathbb{C}$ to a curve with the following equation

$$y^2 + xy = x^3 + \frac{36}{j-1728}x - \frac{1}{j-1728}.$$

This last elliptic curve is defined over $\mathbb{Q}(j)$. If j is transcendental, the field $\mathbb{Q}(j)$ has a subfield which admits infinitely many different homomorphism to $\mathbb{C}$. This, when considering the different images of $\mathrm{Pic}(\mathcal{O})$ implies that there are infinitely many non-isomorphic elliptic curves with complex multiplication $\mathcal{O}$. However, this contradicts the finitness of $\mathrm{Pic}(\mathcal{O})$. □

The characterization of the field generated by the numbers associated to a given order $\mathcal{O}$, gives us a solution for Hilbert's 12th problem for quadratic imaginary extension.
One has that the action induced by $\mathrm{Pic}(\mathcal{O})$ on the modular curve $X_0(1)$ commutes with the action of $G_K = \mathrm{Gal}(\overline{K}/K)$. If we fix an elliptic curve E of Complex Multiplication $\mathcal{O}$, one has the existence of the following homomorphism

$$\eta : G_K \to \mathrm{Pic}(\mathcal{O}) \text{ such that } E^\sigma = \eta(\sigma) * E \text{ for all } \sigma \in G_K.$$

Observe that the definition of this map does not depend on the base elliptic curve because of the commutativity of the $\mathrm{Pic}(\mathcal{O})$. Using this map, we have that the different $j(\mathfrak{a})$ for $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$ will generate the field $H := \overline{K}^{\ker(\eta)}$. The following theorem gives a partial solution to Hilbert's 12th problem.

Theorem 3.20 (Hilbert 12 problem for quadratic imaginary extensions)**.** *The integers $j(\mathfrak{a})$ associated to CM points generate the ring class fields of the same conductor.*

Proof. In order to prove this theorem, we will prove that $H = H_{\mathcal{O}}$, where $H_{\mathcal{O}}$ is the ring class field associated to $\mathcal{O}$. We fix an elliptic curve E associated to an element $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$ and we define the set of primes Σ of $\mathcal{O}_K$ satisfying the following conditions

(i) $\mathfrak{p}$ is unramified in H/K,

(ii) The curve E has good reduction at all the primes above $\mathfrak{p}$,

(iii) The norm of the prime $\mathfrak{p}$ does not divide the element $j(\mathfrak{a}) - j(\mathfrak{b})$ for all $\mathfrak{a}, \mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$ such that $\mathfrak{a} \neq \mathfrak{b}$,

(iv) The norm of $\mathfrak{p}$ is a rational prime.

The set of primes Σ has Dirichlet density one. Using the Cebotarev density theorem, we have that the corresponding Frobenius elements will generate the group $\mathrm{Gal}(H/K)$. Given a prime $\mathfrak{p} \in \Sigma$, we denote as $\sigma_{\mathfrak{p}}$ the Frobenius element associated to such prime. We fix a prime $\tilde{\mathfrak{p}}$ over $\mathfrak{p}$ and we denote as $\overline{E}$ the reduction of the elliptic curve E at $\tilde{\mathfrak{p}}$. We point out the next two following observations, which are proven in detail in Exercise 7 of [Dar04, p. 42].

(i) There is a unique inseparable isogeny of degree p (up to automorphism) from $\overline{E}$ given by the Fromenious morphism

$$Frob : \overline{E} \to \overline{E}^p$$

where $\overline{E}^p$ is the elliptic curve obtained from applying the common Frobenius map to the coefficients of $\overline{E}$ and the map $Frob$ sends the points (x, y) to (x^p, y^p).

(ii) The elliptic curve $E/E[\mathfrak{p}]$ is defined over H and the natural projection

$$E \to E/E[\mathfrak{p}]$$

is a purely iseparable morphism modulo $\tilde{\mathfrak{p}}$.

One deduces from this two facts that $E/E[\mathfrak{p}]$ is congruent to $\sigma_{\mathfrak{p}}(E) \equiv \overline{E}^p$ module $\tilde{\mathfrak{p}}$. Since all the elliptic curve representing the different element of $\text{Pic}(\mathcal{O})$ are different modulo $\tilde{\mathfrak{p}}$, we have that $\eta(\sigma_{\mathfrak{p}}) = [\mathfrak{p}]$. This property will be satisfied by a set of primes of K of density one; consequently, this will be satisfied for all primes which are unramified in H. This immediately proves that $H = H_{\mathcal{O}}$. □

Observe that this result gives us a solution to Hilbert's 12th problem for quadratic imaginary fields. This is the only partial solution known for the time being, however, the Stark conjecture [Sta80] and the Charollois-Darmon conjecture [CD08] are predicted to give solutions for other extensions.

Birch and Swinnerton-Dyer Conjecture

The most famous problem, where advances have been made using Heegner points, is the Birch and Swinnerton-Dyer (BSD) Conjecture. Given an elliptic curve $E/\mathbb{C}$, from the Mordell-Weil Theorem, we have that there exists $r \in \mathbb{N}$ and a finite group G such that

$$E(\mathbb{Q}) \cong G \times \mathbb{Z}^r.$$

We define the rank of the elliptic curve E as the exponent r of this last equation. Furthermore, we can define the analytic rank of the elliptic curve considering the associated L-function attached to E

$$\text{rank}_a(E) = \text{ord}_{s=1} L(E, s).$$

The BSD conjecture proposes a relation between those two definitions of ranks attached to elliptic curves and, in its more strong statement, proposes an equation for the first coefficient of the Taylor expansion of the L-function attached to E. We will give the statement presented by Darmon in [Dar04].

Conjecture 3.21 (Birch and Swinnerton-Dyer). *Given an elliptic curve $E/\mathbb{Q}$, the following statements conjecture a relation between the analytic and algebraic ranks of E in order of increasing strength*

BSD1 $L(E,s) \neq 0$ *if and only if* $\#E(\mathbb{Q}) < \infty$.

BSD2 *The analytic and algebraic ranks associated to* E *are equal*

$$\mathrm{rank}(E) = \mathrm{rank}_a(E).$$

BSD3 *Let* R_E *the regulator of the elliptic curve,* c_p *the local terms defined in the elliptic curves section and* $Ш(E/\mathbb{Q})$ *the Shafarevich-Tate group. We have the following expression for the first non-zero coefficient of the Taylor expansion of* $L(E,s)$ *at* $s=1$

$$\lim_{s\to 1} \frac{L(E,s)}{(s-r)^{\mathrm{rank}(E)}} = Ш(E/\mathbb{Q}) R_E \left(\prod_{p|N} c_p \right) c_\infty.$$

The conjecture has not been generally proven but using Heegner points, the statement can be proven for analytic ranks 0 and 1. Now we aim to give a sketch of such a proof in these two cases.
We consider an elliptic curve $E/\mathbb{Q}$ of conductor N and a quadratic imaginary field K satisfying the Heegner Hypothesis. We define as $\{P_n\}_n$ the Heegner system in E arising from the points in $HP(n)$. We define the point

$$P_K = \mathrm{Tr}_{H/K}(P_1) \in E(K)$$

as the trace of a Heegner point P_1 of conductor one defined over the Hilbert class field H of K. Furthermore, if we consider the ring class field H_n of conductor n and $\chi : \mathrm{Gal}(H_n/K) \to \mathbb{C}^\times$ a primitive character of H_n, we define the general point

$$P_{H_n}^{\chi} = \sum_{\sigma \in \mathrm{Gal}(H_n/K)} \overline{\chi}(\sigma) P_n^{\sigma} \in E(H_n) \otimes \mathbb{C}.$$

We shall give the statement of the most famous result relating the Heegner systems and special values of L-functions associated to elliptic curves over the field K and its twists by characters of ring class fields of conductor n.

Theorem 3.22 (Gross-Zagier formula)**.** *Given a character* $\chi : \mathrm{Gal}(H_n/K) \to \mathbb{C}^\times$, *there exists a relation between the values of the L-function of* E *and its first derivative with inner products of the points described before*

$$\langle P_K, P_K \rangle = L'(E,1),$$
$$\left\langle P_{H_n}^{\chi}, P_{H_n}^{\chi} \right\rangle = L(E,\chi,1).$$

We should point out that these equations are up to a non-zero fudge factor that can be done explicitly; however, the explicit construction of such a factor plays no role in the partial solution of the BSD conjecture that we aim to prove.

One should remark that this theorem immediately proves that $P^{\chi}_{H_n}$ is non-zero if and only if the function $L'(E, \chi, s)$ does not vanish at $s = 1$.
It is relatively easy to find lower bound on the rank of the Mordell-Weil group of an elliptic curve E over a quadratic imaginary field K if we assume that there exists a Heegner system attached to E, however, the following theorem also gives us an upper bound on the size of the Mordell-Weil group $E(K)$ and the Shafarevich-Tate group of E over the field K.

Theorem 3.23 (Kolyvagin)**.** *Given an elliptic curve E over a number field $K/\mathbb{Q}$ and a Heegner system $\{P_n\}_n$, if the point P_K is non-torsion, then the following two statements are true*

(i) The rank of the elliptic curve E is one and P_K generates a finite-index subgroup of $E(K)$,

(ii) The Shafarevich-Tate group associated to the elliptic curve E is finite.

Proof. The proof of this theorem is built with the theory found in the tenth chapter of [Dar04]. □

Using these two last theorems, we can give a proof of the BSD conjecture (for the three statements given before) in the case where the analytic ranks are zero or one.

Theorem 3.24 (Gross-Zagier, Kolyvagin)**.** *If the elliptic curve E has analytic ranks zero or one, then the Birch and Swinnerton-Dyer conjecture is satisfied.*

Proof. We want to start proving the existence of a quadratic Dirichlet character χ satisfying the following properties

(i) $\chi(\ell) = 1$ for all $\ell | N$,

(ii) $\chi(-1) = -1$,

(iii) $L(E, \chi, 1) \neq 0$.

We will prove this case in the two possible signs of the functional equation of the L function $L(E, s)$. If the sign is -1, the work of Waldspurger in [Wal85] assures us the existence of such a character in this case.
On the other hand, if the sign attached to the elliptic curve E is 1, then using the results from [BFH90] and [MMM91], we can guarantee the existence of a character satisfying the properties and that

$$L'(E, \chi, 1) \neq 0.$$

After proving the existence of the Dirichlet character χ satisfying the properties enumerated before, we consider the quadratic imaginary field K associated to the character χ. This field, by definition, will satisfy the following two properties

(i) The field K satisfies the Heegner Hypothsis with respect to the elliptic curve E,

(ii) The order of $L(E/K,s)$ at $s=1$ is one and, consequently, $L'(E/K,1)\neq 0$.

We consider the Heegner System $\{P_n\}_n$ arising from the CM points on $X_0(N)$ attached to the field K. The Gross-Zagier formula and the second property of K imply that this Heegner System is non-trivial and P_K is of non-torsion. Using the Kolyvagin Theorem, we have the following two properties

(i) The Modell-Weil group $E(K)$ has rank one and the quotient $E(K)/\langle P_K\rangle$ is finite,

(ii) The group $\text{Ш}(E/K)$ is finite.

From Proposition 3.6, we can deduce that $P_K\in E(\mathbb{Q})$ if and only if the sign of the elliptic curve E is -1. From this last property, one can prove the statement of BSD2 i.e. the following equation is satisfied in our setting

$$\operatorname{ord}_{s=1} L(E,s)=\operatorname{rank}(E).$$

We can also deduce that $\text{Ш}(E/\mathbb{Q})$ is finite since the map induced by restriction

$$\text{Ш}(E/\mathbb{Q})\to\text{Ш}(E/K)$$

has a finite kernel. The finiteness of $\text{Ш}(E/\mathbb{Q})$ immediately proves BSD3, the strongest statement given as part of the Birch and Swinnerton-Dyer Conjecture. □

As we have mentioned before, this proof is valid for the three statements of the BSD conjecture that we have given; in fact, it proves one of the implications of BSD1. However, it should be pointed out that the other implication of BSD1 remains unproven at the time of writing of this document.
After going through the sketch of this partial proof, a question might have arisen: does there exist a similar relation between the size of the Morder-Weil group and the special values of the L function $L(E,\chi,s)$ for a given character $\chi:\operatorname{Gal}(K)\to\mathbb{C}^\times$? The twisted Birch and Swinnerton-Dyer conjecture tries to give a solution to this question.

Conjecture 3.25 (Twisted Birch and Swinnerton-Dyer)**.** *Let $K/\mathbb{Q}$ be a finite Galois extension and χ a character. Then there exists the following relation*

$$\operatorname{ord}_{s=1} L(E,\chi,s)=\langle\chi,E(K)\otimes\mathbb{C}\rangle.$$

One can easily generalize the partial solution of the classical BSD conjecture for when the L function $L(E,\chi,s)$ has order one at $s=1$. On the other hand, the method can not be generalized for the case $L(E,\chi,1)\neq 0$. To prove the conjecture in this last setting, one has to use the methods developed by Bertolini and Darmon in the paper [BD07].

The partial solution to the BSD conjecture also immediately gives a partial solution to another, less-known conjecture: The Watkins conjecture. Let $E/\mathbb{Q}$ be a modular elliptic curve. From the modularity theorem, there exists a minimal complex parametrization

$$\Phi : X_0(N) \to E,$$

where the minimal notion is defined by taking the parametrization of a minimal degree. This minimal degree is commonly defined as the modular degree, and we will denote it as $\deg(\Phi)$.
In 2002, Marc Watkins computed the modular degree of some elliptic curves and conjectured the following relation between the modular degree of those curves and their algebraic rank [Wat02].

Conjecture 3.26 (Watkins'02)**.** *Let $E/\mathbb{Q}$ be an elliptic curve, $r = \mathrm{rank}(E)$ its algebraic rank, and $\Phi : X_0(N) \to E$ the minimal complex parametrization given by the modularity theorem. There exists the following relation between the analytic rank and the modular degree of the elliptic curve*

$$2^r \,|\, \deg(\Phi).$$

In general, this conjecture has not been proven; however, it is relatively easy to give a partial solution for analytics ranks 0 and 1.

Theorem 3.27. *The Watkins conjecture is satisfied for $r = 0, 1$.*

Proof. We use the partial solution of the Birch and Swinnerton-Dyer Conjecture in the two cases that we consider, we have $r = \mathrm{rank}_a(E)$. For the case when $r = 0$, the statement becomes trivial thanks to this last relation.
For $r = 1$, since $\mathrm{rank}_a(E) = 1$, we can deduce that the sign associated to the L-function of E will be -1. We consider the following commutative diagram

$$\begin{array}{ccc} X_1(N) & \longrightarrow & E \\ {\scriptstyle \omega_N}\uparrow & \nearrow_{\Phi} & \\ X_0(N) & & \end{array}$$

Since the sign of the L-function is -1, we have that the $\deg(\omega_N) = 2$, which from the commutative diagram, proves that $2|\deg(\Psi)$. This result proves the conjecture for analytic rank 1. □

Furthermore, an easy computation shows us that all the Elliptic Curves of analytic rank bigger than 1 indexed in the LMFBD satisfy this conjecture.
The author will like to thank Professor Jan Vonk for proposing a small one-week project to learn about this conjecture in his first year at Universiteit Leiden (Nederland).

4 Non-Archimedean construction of Heegner points

In this section, we will generalize the period construction of Heegner points described over the Modular curve to the setting of Shimura curves. Before moving to the concrete generalization of the periods, we shall introduce the general structure we will use.
Let $E/\mathbb{Q}$ be an elliptic curve with conductor $N = pN^+N^-$, with all the factors prime pairwise, p prime and N^- is squarefree and has an odd number of prime divisors. From the work of Cerednik-Drinfeld and Jacquet-Langlands, there exists a complex parametrization of curves

$$\Phi : \mathrm{Div}^0(X_{pN^+,N^-}) \to E$$

where X_{pN^+,N^-} is the Shimura curve. Let B be the definite quaternion algebra over $\mathbb{Q}$ of discriminant N^- and R the Eichler $\mathbb{Z}[1/p]$-order of B of level N^+. We fix an embedding

$$\iota : B \to M_2(\mathbb{Q}_p).$$

We denote as $R_1^\times$ the collection of units that have norm 1 and we define the following subgroup

$$\Gamma := \iota(R_1^\times) \subseteq \mathrm{SL}_2(\mathbb{Q}_p),$$

which will be essential for the rest of the section.

4.1 Algebraic definition of CM points

Given an elliptic curve $E/\mathbb{Q}$ of conductor N, as we have stated in the preliminary notions there exists an analog of the modular theorem for Shimura curves i.e. there exists a minimal modular parametrization of curves

$$\Phi : X_{N^+,N^-} \longrightarrow E(\mathbb{C}),$$

where N^+N^- is a decomposition of N with $(N^+, N^-) = 1$ and N^- square-free with an odd number of divisors. Let $\mathcal{O}$ be an order in a quadratic imaginary field $K/\mathbb{Q}$. Similarly to the case over the modular curve, we can define CM points using the moduli expression over the field $\mathbb{C}$

$$\mathrm{CM}(\mathcal{O}) = \{(A, A') \in X_{N^+,N^-}(\mathbb{C}) \mid \mathrm{End}(A) \cong \mathrm{End}(A') \cong \mathcal{O}\},$$

we shall point out that the endomorphisms we use in this definition are the ones of the category of abelian surfaces together with embeddings $\mathcal{O}_B \to \mathrm{End}_{\mathbb{C}}(A)$, defined in the preliminary notions.
If we restrict to orders with conductor coprime to N, we can give sufficient conditions for the existance of heegner points associated to this order.

Proposition 4.1 (Heegner Hypothesis). *Given an order $\mathcal{O} \subseteq K$ of conductor prime to N. The collection of CM points is non-empty if and only if the following conditions are satisfied* *(i) All the primes dividing N^- are inert in K,*

(ii) All primes dividing N^+ are split in K.

Proof. If we assume that the set of CM points is non-empty for a given order $\mathcal{O}$, we can view K as a subfield of B, this immediately proves that all the primes dividing N^- are inert in K. Furthermore, we can view $\mathcal{O}$ as a subring of R, which gives us a chain of homomorphisms

$$\mathcal{O} \to \left(\mathbb{Z}/N^+\mathbb{Z}\right) \times \prod_{\ell | N^-} \mathbb{F}_\ell^2 \to \mathbb{Z}/N^+\mathbb{Z}$$

where the second map is the natural projection. Since we have also assumed that the conductor of $\mathcal{O}$ is prime to N, this last map shows us that all the primes dividing N^+ are split in K.

On the other hand, if we assume the Hypothesis given before, we can define an analogous map to the one defined in the Heegner Hypothesis over the Modular curve

$$\delta : \mathrm{Pic}(\mathcal{O}) \to X_{N^+,N^-}(\mathbb{C}).$$

This immediately proves that the set of CM points in this case is non-empty because $\mathrm{Pic}(\mathcal{O}) \neq \varnothing$. □

As it was shown in the preliminary notions, there exists an isomorphism of Riemann surfaces

$$\eta : \mathcal{H} \big/ \Gamma_{N^+,N^-} \xrightarrow{\sim} X_{N^+,N^-}(\mathbb{C}).$$

This helps us redefine CM points using modules. Given an element $\tau \in \mathcal{H} \big/ \Gamma_{N^+,N^-}$, we define the associated order to τ as

$$\mathcal{O}_\tau := \{\alpha \in R \text{ such that } \mathrm{norm}(\gamma) = 1 \text{ and } \iota(\gamma)(\tau) = \tau\} \cup \{0\}.$$

We can redefine the CM points associated to an order $\mathcal{O} \subseteq K$ as

$$\mathrm{CM}(\mathcal{O}) := \left\{\tau \in \mathcal{H} \big/ \Gamma_{N^+,N^-} \text{ such that } \mathcal{O}_\tau = \mathcal{O}\right\}.$$

The theory of complex multiplication over Modular curves can be generalized to the setting of Shimura curves. We can see this idea underpinned in the following theorem, which is proven in a similar way to the case of Modular curves.

Theorem 4.2 (Complex Multiplication for Shimura curves)**.** *Given an order $\mathcal{O} \subseteq K$ of conductor prime to N, we denote as H the ring class field associated to $\mathcal{O}$. We have the following inclusion*

$$\Phi(\mathrm{Div}^0(\mathrm{CM}(\mathcal{O}))) \subset E(H).$$

4.2 Construction of p-adic upper half-plane

In this section, we are going to construct Drinfeld's p-adic upper half-plane using the Bruhat-Tits tree of $PGL_2(\mathbb{Q}_p)$. The upper p-adic upper half plane is a rigid analytic variety over the field $\mathbb{Q}_p$ (can be generalized to any p-adic field K but this process goes out of the scope of this document). We will follow the construction given in [DT07]. Before explicating this construction, we shall introduce the Bruhat-Tits tree of $\mathrm{PGL}_2(\mathbb{Q}_p)$.

Definition 4.3 (Bruhat-Tits tree)**.** *Given a prime p, we define the equivalence relation between two lattices L and L' of rank 2 over $\mathbb{Q}_p^2$ by multiplication of elements of $\mathbb{Q}_p$ i.e. they are related if there exists an $\alpha \in \mathbb{Q}_p$ such that $L = \alpha L'$. The Bruhat-Tits tree is a graph whose vertices are the equivalence classes of lattices and the oriented edges are defined if the associated lattices of the vertices satisfy that their associated finite quotient L/L' has order p.*

From now onwards, given a prime p, we will denote the Bruhat-Tits tree for the prime p as $\mathcal{T}_p$. The following result gives us a clear picture of the structure of these trees.

Proposition 4.4. *Given a prime p, the Bruhat-Tits tree $\mathcal{T}_p$ is an homogeneous tree of degree $p+1$*

Proof. We shall start proving that given a vertex v, with associated lattice L, there are only $p+1$ adjacent vertices. Any other lattice L' satisfying the adjacent condition with respect to L will be in bijection with a subgroup of L/pL. Since there are only $p+1$ subgroups in this quotient, we have proven that v has only $p+1$ adjacent vertices.
We have left to prove that there are no closed loops in the graph $\mathcal{T}_p$. We assume there exists a loop represented by the lattices

$$p^r L \to L_d \to \cdots \to L_1 \to L,$$

where r and d are integers. For every $i \in \{1, ..., d\}$, we have that L_{i-1}/L_{i+1} is not cyclic and consequently, we have $L_{i+1} = pL_{i-1}$. This fact, since there are no loops of 2 vertices, contradicts the minimality of the loop; therefore, $\mathcal{T}_p$ has no loops. □

We will denote the set of vertices of the tree $\mathcal{T}_p$ as $\mathcal{V}(\mathcal{T}_p)$. This set corresponds bijectively to the set of homothety classes of $\mathbb{Z}_p$-latices of rank 2 i $\mathbb{Q}_p^2$. The vertices $v, v' \in \mathcal{V}(\mathcal{T}_p)$ are adjacent if and only if their correspondence lattices L and L', respectively, satisfy that the finite quotient L/L' has order p. We fix our base vertex v° as the one corresponding to the standard lattice $\mathbb{Z}_p^2$. The group $\mathrm{PGL}_2(\mathbb{Q}_p)$ induces a natural left-action on the lattices that represent the elements $\mathcal{T}_p$, which at the same time induces a left-action on the vertices. Given a matrix $g \in \mathrm{PGL}_2(\mathbb{Q}_p)$ the map $g \mapsto gv^\circ$ induces the following isomorphism

$$\mathrm{PGL}_2(\mathbb{Q}_p)/\,\mathrm{PGL}_2(\mathbb{Z}_p) \xrightarrow{\sim} \mathcal{V}(\mathcal{T}_p).$$

Given two adjacent vertices $v, v' \in \mathcal{V}(\mathcal{T}_p)$, we consider the oriented edge $E = (v, v')$. We denote as $\mathcal{E}(\mathcal{T}_p)$ the collection of oriented edges. Moreover, we can define two maps

$$s(e) = v \text{ and } t(e) = v' \text{ for } e = (v, v'),$$

which identify the origin and the target vertices of each edge, respectively. Furthermore, given an edge $e = (v, v')$, we define the opposite edge as $\overline{e} = (v', v)$. We fix the base oriented edge e° as the edge which has $s(e^\circ) = v^\circ$ and its stabilizer in $\mathrm{PGL}_2(\mathbb{Q}_p)$ is the group $\Gamma_0(p\mathbb{Z}_p)$ of upper triangular matrices matrices in $\mathrm{PGL}_2(\mathbb{Q}_p)$ modulo p. Similarly the the vertices, given $g \in \mathrm{PGL}_2(\mathbb{Q}_p)$ the map $g \mapsto ge^\circ$ induces the following isomorphism

$$\mathrm{PGL}_2(\mathbb{Q}_p)/\Gamma_0(p\mathbb{Z}_p) \xrightarrow{\sim} \mathcal{E}(\mathcal{T}_p).$$

We consider the reduction map modulo the maximal ideal of the ring of integers of $\mathbb{C}_p$

$$\mathrm{red} : \mathbb{P}^1(\mathbb{C}_p) \to \mathbb{P}^1(\overline{\mathbb{F}}_p).$$

The aim now is to relate the p-adic upper half-plane, which is a rigid analytic variety over $\mathbb{Q}_p$ where $\mathrm{GL}_2(\mathbb{Q}_p)$ induces an action, with the Bruhat-Tits tree that we have just defined. In concrete, we are interested in the $\mathbb{C}_p$-values of such a variety, which are

$$\mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p) \backslash \mathbb{P}^1(\mathbb{Q}_p).$$

We define the following subset associated to the base vertex v° of $\mathcal{T}_p$

$$A_{v^\circ} := \{z \in \mathbb{P}^1(\mathbb{C}_p) \mid \mathrm{red} \notin \mathbb{P}^1(\mathbb{F}_p)\}.$$

Given any vertex $v \in \mathcal{V}(\mathcal{T}_p)$ there exists $g \in \mathrm{PGL}_2(\mathbb{Q}_p)$ such that $v = gv^\circ$. Using this matrix we extend the definition of this last set to the arbitrary vertex v as $A_v := gA_{v^\circ}$. These sets are called connected affinoid domains in $\mathcal{H}_p$ and are subsets of the projective space $\mathbb{P}^1(\mathbb{C}_p)$ where we excise $p+1$ disjoint open disks. Similarly, we define the following set associated to the base edge e° of $\mathcal{T}_p$

$$W_{]e^\circ[} := \{z \in \mathbb{P}^1(\mathbb{C}_p) \mid 1 < |z|_p < p\} \subset \mathcal{H}_p.$$

Given an edge $e \in \mathcal{E}(\mathcal{T}_p)$, there also exists a matrix $g \in \mathrm{PGL}_2(\mathbb{Q}_p)$ such that $e = ge^\circ$. As we did for vertices, we extend the definition of the last set for any edge e as $W_{]e[} = gW_{]e^\circ[}$. These sets are called oriented wide open annulus attached to the edge e, with orientation corresponding to the disk

$$D_e := \{z \in \mathbb{P}^1(\mathbb{C}_p) \mid |g^{-1}z|_p \geq p\} \subset \mathbb{P}^1(\mathbb{C}_p).$$

Given an oriented edge $e = (v, v') \in \mathcal{E}(\mathcal{T}_p)$, we define the standard affinoid subset attached to the edge e as the set

$$A_{[e]} := A_v \cup W_{]e[} \cup A_{v'}.$$

These affinoids subsets allow us to give a covering of the p-adic upper half plane $\mathcal{H}_p$ where their intersections are reflected in the incidence relations of the tree $\mathcal{T}_p$. Taking this act under consideration, we can define the reduction map

$$r : \mathcal{H}_p \to \mathcal{V}(\mathcal{T}_p) \cup \mathcal{E}(\mathcal{T}_p), \tag{4.2.1}$$

that sends $z \in \mathcal{H}_p$ to v if $z \in A_v$ and to e if $z \in W_{]e[}$. The importance of this map is underpinned by the fact that we can find open compact covers of the p-adic upper half plane by considering the preimage of the edges, which have a fixed origin vertex. We will identify such a process with an exemplification for the case $p = 2$.

4.3 Katz approach to p-adic modular forms

In this subsection, we will justify the definition of p-adic modular forms given two sections forward, more algebraically, using the Katz definition of modular forms following the paper [How22]. In essence, one mimics the process we have established for modular curves, but for Shimura curves. The singularity of this case is that we will not have cusps (at least in the cases we are interested).
We start defining the Shimura curve for a given quaternion algebra $D/\mathbb{Q}$ of discriminant N^-.

Definition 4.5. *Given a $K \subseteq D(\mathbb{A}_f)$, we define the following moduli problem*

$$\begin{aligned} X^K_{N^+,N^-} : \mathbb{Q}\text{-Alg} &\to \text{Set} \\ R &\mapsto \{(A, \mathcal{K})\}/\sim \end{aligned}$$

satisfying

(i) A is an abelian surface over R.

(ii) $\mathcal{K} \subseteq \underline{Isom}((\mathbb{A}_f)^2, T_{\mathbb{A}_f}(A))$ a K-torsor.

(iii) The relation $\sim$ is defined by isogenies that preserve the torsors.

We will denote as $X^K_{N^+,N^-}$ the solution of this moduli problem, and we will drop K when the group is trivial. One should note that the Shimura curve defined before is a special case of this general definition, but it will not be the case $K = \{e\}$. Given a prime $p|N^-$, we define the level p Shimura curve.

Definition 4.6. *Given a $K^p \subseteq D(\mathbb{A}_f^{(p)})$, we define the following moduli problem*

$$\begin{aligned} \mathfrak{X}^{K^p}_{N^+,N^-} : \mathbb{Z}_{(p)}\text{-Alg} &\to \text{Set} \\ R &\mapsto \{(A, \mathcal{K}^p)\}/\sim \end{aligned}$$

satisfying

(i) A is an abelian surface over R.

(ii) $\mathcal{K}^p \subseteq \underline{Isom}((\mathbb{A}_f^{(p)})^2, T_{\mathbb{A}_f^{(p)}}(A))$ a K^p-torsor.

(iii) The relation $\sim$ is defined up to prime-to-p quasi-isogenies which preserve the associated torsor.

As before, we will denote the solution of this moduli problem with the same notation, and we will drop K^p from our notation when the group is trivial.
In the case of abelian surfaces, their p-divisible groups will be classified in more than 2 groups. Nevertheless, we still have a notion of ordinary group and therefore of ordinary locus of our Shimura curve of level p (following a similar construction to the one of modular curves). We denote the ordinary locus of the level p modular curve as $\mathfrak{X}_{N^+,N^-}^{K^p,ord}$ and the associates ordinary p-divisble group as $\mathbb{X}_{ord} = \mu_p^2 \times (\mathbb{Q}_p/\mathbb{Z}_p)^2$. Once we have defined this notion, we are in a position to generalize the Igusa moduli problem to this setting.

Definition 4.7 (Igusa moduli problem)**.** *Given a $K^p \subseteq D(\mathbb{A}_f^{(p)})$, we define the Igusa moduli problem as the functor*

$$\begin{aligned} Ig_{N^+,N^-,\mu_p}^{K^p,ord} : \mathbb{F}_p\text{-Alg} &\to \text{Set} \\ R &\mapsto \{A, \mathcal{K}^p, \varphi_p\}/\sim \end{aligned}$$

satisfying

(i) A is an abelian surface over R

(ii) $\mathcal{K}^p \subseteq \underline{Isom}((\mathbb{A}_f^{(p)})^2, T_{\mathbb{A}_f^{(p)}}(A))$ a K^p-torsor.

(iii) $\varphi_p : \mu_p \times \mu_p \xrightarrow{\sim} \hat{A}[p]$ an isomorphism.

(iv) The relation $\sim$ is defined by prime-to-p quasi-isogenies which preserve both the torsor and the isomorphism.

Similarly to the modular curve case, this moduli problem is an étale $(\mathbb{Z}/p\mathbb{Z})^\times$-cover of $\mathfrak{X}_{N^+,N^-}^{K^p}$. At this point, one can define the $\mathbb{F}_p$ modular forms as sections associated to this moduli problem. The construction is quite similar to the exposition given at the preliminary notions, so we will go directly to the p-adic modular forms, which is the main aim of this chapter.

Definition 4.8 (Katz-Igusa moduli problem)**.** *Given $K^p \subseteq D(\mathbb{A}_f^{(p)}$, we define the Katz-Igusa moduli problem as the functor*

$$\begin{aligned} Ig_{N^+,N^-,\hat{\mathbb{G}}_m}^{K^p,ord} : \text{Nil}_{\mathbb{Z}_p} &\to \text{Set} \\ R &\mapsto \{(A, \mathcal{K}^p, \varphi_p)\}/\sim \end{aligned}$$

satisfying

(i) *A is an abelian surface over R.*

(ii) $\mathcal{K}^p \subseteq \underline{Isom}((\mathbb{A}_f^{(p)})^2, T_{\mathbb{A}_f^{(p)}}(A))$ *a* K^p*-torsor.*

(iii) $\varphi_p : \hat{\mathbb{G}}_m \times \hat{\mathbb{G}}_m \xrightarrow{\sim} \hat{A}$ *an isomorphism.*

(iv) *The relation* $\sim$ *is defined by prime-to-p quasi-isogenies which preserve both the torsor and the isomorphism.*

One should, intuitively, view this last moduli problem as the limit, taken through the powers of p, of the Igusa moduli problem defined previously. Let $\mathcal{O}$ denote the generic sheaf

Definition 4.9 (p-adic modular forms on Shimura Curves)**.** *The ring of p-adic modular forms over Shimura curves is defined as*

$$\mathbb{V}_{\mathbb{Q}_p} := \mathbb{V}_{\mathbb{Z}_p}[1/p] \text{ where } \mathbb{V}_{\mathbb{Z}_p} := H^0(Ig^{K^p,ord}_{N^+,N^-,\hat{\mathbb{G}}_m}, \mathcal{O}).$$

One can check that similar properties to the ones seen at the preliminary notions are satisfied in this setting. However, in order to construct explicit expressions for CM points defined over the $\mathbb{C}_p$ points of a Shimura curve, we will need a more analytic definition of modular forms. The following subchapter will show us that the $\mathbb{C}_p$-points of Shimura curves are related to a rigid analytic space and that our definition of modular forms can be translated to a purely analytic definition in such a space.

4.4 Cerendik-Drinfeld theory of p-adic uniformization

The definition we have given of CM points it is a purely algebraic definition since it identifies points in the moduli interpretation of certain Shimura curves. In order to give explicit expressions, we need to associate a certain rigid analytic space to the $\mathbb{C}_p$ points of our Shimura curve.

Let B be a quaternion algebra over $\mathbb{Q}$ of discriminant N^- split at p (observe we are removing p from the original quaternion algebra) and R be an Eichler $\mathbb{Z}[1/p]$-order of level N^+ in B. Furthermore, we define the congruence subgroup $\iota(R_1^\times)$ where R_1 are the element of R with norm one and ι is the embedding

$$\iota : B \to M_2(\mathbb{Q}_p).$$

The following theorem relates the $\mathbb{C}_p$ points of a certain Shimura curve with the quotient $\mathcal{H}_p/\Gamma$.

Theorem 4.10 (Cerednik, Drienfled)**.** *There exist an isomorphism of curves over* $\mathbb{C}_p$ *of the form*

$$X_{N^+,pN^-}(\mathbb{C}_p) \cong \mathcal{H}_p/\Gamma.$$

The prove of this theorem goes far beyond the scope of this thesis; however we are interested in pointing out a detail of the theory of Cerednik and Drienfeld. They were able to prove that the limit of Igusa towers we have shown before coincides with a certain limit of the p-adic upper half-plane. Therefore, the algebraic definition of p-adic cusp form we have given before has an analytical analogous definition using certain limits on $\mathcal{H}_p$.
More broadly, the theory of p-adic uniformization gives us the ability to give an explicit definition of CM points over $\mathbb{C}_p$ using the isomorphism given before

$$X_{pN^+,N^-}(\mathbb{C}_p) \xrightarrow{\sim} \mathcal{H}_p/\Gamma.$$

Similarly to the CM points over $\mathbb{C}$, we can give an equivalent definition using modules. Given an element $\tau \in \mathcal{H}_p/\Gamma$, we define the module attached to τ as

$$\mathcal{O}_\tau = \{\gamma \in R \text{ such that } \iota(\gamma)\tau = \tau\} \cup \{0\}.$$

After defining this type of module, we can check that the set of CM points is identified under the map η as

$$\mathrm{CM}(\mathcal{O}) = \{\tau \in \mathcal{H}_p/\Gamma \text{ such that } \mathcal{O}_\tau = \mathcal{O}\}.$$

4.5 p-adic cusp forms and p-adic integration

Once we have studied the construction of the p-adic upper half-plane and the Bruhat-Tits tree, we are in a position to define rigid analytic functions, p-adic measures, and line integrals associated to that measure.

Definition 4.11 (Rigid analytic function)**.** *A function $f : \mathcal{H}_p \to \mathbb{C}_p$ is said to be rigid analytic if, for each edge $e \in \mathcal{E}(\mathcal{T}_p)$, the restriction of f to the affinoid $A_{[e]}$ is a uniform limit of rational functions on $\mathbb{P}^1(\mathbb{C}_p)$ having poles outside of $A_{[e]}$.*

Recall the definition of the modular subgroup Γ that we have given at the beginning of this section using the quaternion algebra. We define the group of p-adic modular forms of weight 2 as

$$S_2(\Gamma) := \{f \text{ rigid analytic modular form of weight } 2\}.$$

Given an oriented edge $e \in \mathcal{E}(\mathcal{T}_p)$ and a rigid analytic differential form $f(z)dz$ which an attached sequence of rational functions (f_j) converging uniformly to f on $A_{[e]}$, we can define the p-adic annular residue of f at e as

$$\mathrm{res}_e(f(z)dz) = \lim_{j\to\infty} \sum_{t\in D_e} \mathrm{res}_t(f_j(z)dz).$$

The work of P. Schneider and J. Teitelbaum assures us that this concept is well-defined and that the limit expressed on the right-hand side only depends on the edge e and the function f.

Definition 4.12 (Harmonic cocycle)**.** *A function $\kappa : \mathcal{E}(\mathcal{T}_p) \to \mathbb{C}_p$ is a harmonic cocycle on $\mathcal{T}$ if it satisfies*

(i) $\kappa(e) = -\kappa(\bar{e})$ *for all* $e \in \mathcal{E}(\mathcal{T}_p)$*,*

(ii) $\sum_{s(e)=v} \kappa(e) = 0$ *for all* $v \in \mathcal{V}(\mathcal{T}_p)$*.*

We aim to prove a bijection between the set $S_2(\Gamma)$ and the collection of Harmonic cocycles Γ-invariant, which will be denoted as $\mathrm{Har}(\Gamma)$. The following lemma assures us that we can associate a harmonic cocycle to any rigid analytic modular form of weight 2.

Lemma 4.13. *Let $f : \mathcal{H}_p \to \mathbb{C}_p$ be a rigid analytic modular form of weight 2. The function*

$$\kappa_f : \mathcal{E}(\mathcal{T}_p) \to \mathbb{C}_p, \quad e \mapsto \mathrm{res}_e(f(z)dz),$$

is an harmonic cocycle Γ-invariant i.e. $\kappa_f(\gamma e) = \kappa_f(e)$ for all $\gamma \in \Gamma$.

Proof. The claim that κ_f is a harmonic cocycle is a direct consequence of the residue theorem for rational differentials. The Γ invariance is a direct result of the fact that, for all $\gamma \in \Gamma$,

$$\mathrm{res}_{\gamma e}(f(z)dz) = \mathrm{res}_e(f(\gamma^{-1}y)d(\gamma^{-1}y)) = \mathrm{res}_e(f(z)dz).$$

□

Since the residue map is injective i.e. two functions that coincide in all the residues for all the edges $e \in \mathcal{E}(\mathcal{T}_p)$ they must be the same function. This lemma also proves that the map induced $S_2(\Gamma) \to \mathrm{Har}(\Gamma)$ is injectve.
In order to prove the surjection, we shall introduce the concept of p-adic measure and the integrals that they produce.
Now we aim to associate a p-adic measure to the function f using the harmonic cocycle attached to it. In order to do this, we shall introduce the line integral over $\mathbb{P}^1(\mathbb{Q}_p)$.

Definition 4.14. *A p-adic distribution on $\mathbb{P}^1(\mathbb{Q}_p)$ is a finitely additive function*

$$\mu : \{\text{compact open } U \subseteq \mathbb{P}^1(\mathbb{Q}_p)\} \longrightarrow \mathbb{C}_p.$$

satisfying $\mu(\mathbb{P}^1(\mathbb{Q}_p)) = 0$.

If we consider a p-adic distribution μ, a local function g in $\mathbb{P}^1(\mathbb{Q}_p)$ and an open cover $\{U_1, ..., U_n\} \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ where g takes constant values in each open, we can define the integral of g in $PP^1(\mathbb{Q}_p)$ with respect to the distribution μ following the similar approach as the common Riemann integral

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} g d\mu := \sum_{j=1}^{n} g(t_j)\mu(U_j).$$

In order to extend this definition to a bigger collection of functions, we shall impose stronger conditions on the distribution μ. A p-adic measure is a bounded distribution of $\mathbb{P}^1(\mathbb{Q}_p)$ i.e. a distribution μ such that there exists a real constant $C > 0$ such that

$$|\mu(U)|_p < C \text{ for any compact open } U \subset \mathbb{P}^1(\mathbb{Q}_p).$$

Let μ be a p-adic measure and g a continuous function in $\mathbb{P}^1(\mathbb{Q}_p)$. We define the p-adic line integral as

$$\int_{\mathbb{P}^1(\mathbb{Q}_p)} g(t)d\mu(t) := \lim_{\{U_\alpha\}} \sum_\alpha g(t_\alpha)\mu(U_\alpha).$$

where the limit is taken over increasing fine covers $\{U_\alpha\}$ of $\mathbb{P}^1(\mathbb{Q}_p)$ by disjoint compact open subsets and $t_\alpha \in U_\alpha$ is a sample point.
After recalling the definition of p-adic measures and line integrals, we are in a position to assign a p-adic measure to the function f defined before. For any oriented edge $e \in \mathcal{E}(\mathcal{T}_p)$ we define the compact open $U_e := D_e \cap \mathbb{P}^1(\mathbb{Q}_p)$ of $\mathbb{P}^1(\mathbb{Q}_p)$. This will allow us to associate a measure to each μ_f using the following lemma.

Lemma 4.15. *Given a Harmonic cocycle κ, we can associate a p-adic measure μ to this cocycle.*

Proof. We define the function $\mu : \{\text{compact open } U \subseteq \mathbb{P}^1(\mathbb{Q}_p)\} \to \mathbb{C}_p$ defined as $\mu(U_e) = \kappa(e)$ for all $e \in \mathcal{E}(\mathcal{T}_p)$, using the reduced map 4.2.1 associated to the Bruhat-Tits tree $\mathcal{T}_p$. The additive map follows from the properties of the reduced map we have defined before.
Using the theory developed in the subsection 4.2, we have that we can generate an open compact disjoint cover of $\mathbb{P}^1(\mathbb{Q}_p)$ considering all the open sets U_e for all the edges that have the same initial vertex v. Using this fact, the additive property of the map μ and the second property of the cocycle, we have the following equation

$$\mu(\mathbb{P}^1(\mathbb{Q}_p)) = \mu\left(\bigcup_{s(e)=v} \mu(U_e)\right) = \sum_{s(e)=v} \mu(U_e) = \sum_{s(e)=v} \kappa(e) = 0.$$

With these two last properties, we have proven that μ is a distribution. Since the map κ is Γ-invariant and $\mathcal{T}_p/\Gamma$ is finite, we have that the defined map μ is bounded and, consequently, μ is a measure. □

This result implies that the cocycle κ_f gives rise to a p-adic measure μ_f satisfying

$$\int_{U_e} d\mu_f = \kappa_f(e).$$

Observe that κ_f induces that this measure is Γ-invariant. After fixing the measure attached to the rigid analytic modular form f we want to explicitly develop the p-adic line integral associated to this function.

Proposition 4.16. *The measure attached to the boundary distribution of the function f gives us the following expression*

$$f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{1}{z-t}\right) d\mu_f(t).$$

Proof. Since the residue map is injective, to prove the equation of this exercise, it is enough to check that the function

$$g(z) := \int \left(\frac{1}{y-t}\right) d\mu_f(t)$$

is a weight two modular form in X_Γ that has the same residues as f.
First, one should notice that g is rigid analytic since, from the definition, the line integral is defined as the limit of Riemann sums (which are rational functions) that converge uniformly to g on any affinoid $A_{[e]}$.
Following the computations of Ta, given a $\gamma \in \Gamma$, we have

$$\frac{1}{\gamma z - \gamma x} = \frac{(cz+d)^2}{z-x}.$$

When this result is translated to the definition of g, we get that

$$g(\gamma z) = (cz+d)^2 g(z),$$

which proves that g is a weight 2 modular form with respect to Γ.
Using the definition of the residue defined before, we have that for any $e \in \mathcal{E}(\mathcal{T}_p)$, we have

$$\mathrm{res}_e(g(z)dz) = \int_{U_e} d\mu_f = \kappa_f(e).$$

This proves that g and f have the same residues; consequently, from the injectivity of the map $S_2(\Gamma) \to \mathrm{Har}(\Gamma)$, we have $f = g$ as we wanted. □

Observe that this proposition immediately proves the surjection of the map stated before between weight 2 modular forms and Harmonic Cocycles

$$S_2(\Gamma) \longleftrightarrow \mathrm{Har}(\Gamma).$$

From the Jacquet-Langlands correspondence and p-adic uniformization, we have that the modular form

$$f_E = \sum_{n\geq 1} a_n e^{2\pi i n \tau}$$

attached to E corresponds to a weight 2 rigid analytic modular form on $\mathcal{H}_p$ with respect to the group Γ. From the work of Cerendnik-Drinfeld and Jacquet-Langlands, we have the following result.

Theorem 4.17. *There exists a weight 2 rigid analytic modular form $f : \mathcal{H}_p \to \mathbb{C}_p$ with respect to the group Γ satisfying*

$$T_n f = a_n f \text{ for all } n \geq 1 \text{ with } (n, N) = 1.$$

where T_n is the n-th Hecke correspondence on X_Γ.

We denote as $\mathrm{Har}(\Gamma, R)$ the set of Γ-invariant cocycles with values in a ring $R \subseteq \mathbb{C}_p$. In concrete, we have that $\mathrm{Har}(\Gamma, \mathbb{Q})$ generates a $\mathbb{Q}$-structure in $\mathrm{Har}(\Gamma, \mathbb{C}_p)$ and both spaces are equipped with an induced Hecke algebra $\mathbb{T}_\Gamma$ attached to the curve X_Γ. Moreover, the subspace of $\mathrm{Har}(\Gamma, \mathbb{C}_p)$ where $\mathbb{T}_\Gamma$ acts via the character defined by the form f is generated by an element in $\mathrm{Har}(\Gamma, \mathbb{Q})$.
Since $\mathcal{T}_p/\Gamma$ is a finite graph, the elements in $\mathrm{Har}(\Gamma, \mathbb{Q}_p)$ are determined by its values on a finite set of orbits representatives. This fact allows us to fix an f, up to a sign, by normalizing the function so the set of values representing κ_f are in $\mathbb{Z}$.
We fix a choice of p-adic logarithm

$$\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$$

and we define the Coleman p-adic line integral of the function f.

Definition 4.18 (Coleman p-adic integral)**.** *The Coleman p-adic integral of the rigid analytic function f associated to the boundary measure of this same function is defined, for $\tau_1, \tau_2 \in \mathcal{H}_p$, as*

$$\int_{\tau_1}^{\tau_2} f(z)dz := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_f(t).$$

Observe that this definition is justified by the formal calculation that the equation proven in 4.16 raises

$$\int_{\tau_1}^{\tau_2} f(z)dz = \int_{\tau_1}^{\tau_2} \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{1}{y - t} \right) d\mu_f(t) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_f(t).$$

In order to define the periods that will be associated to the global points of the Shimura curve, we are interested in the values inside the logarithm. Therefore, we are going to define the multiplicative Coleman integral by formally exponentiating the expression in the definition of the integral given before.

Definition 4.19 (Multiplicative Coleman integral)**.** *Given two elements $\tau_1, \tau_2 \in \mathcal{H}_p$, we define the multiplicative Coleman integral of f as*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f(z)dz := \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_f(t)$$

where the integral on the right-hand side of the equation is defined as

$$\times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} g(t)d\mu_f(t) := \lim_{\{U_\alpha\}} \prod_\alpha g(t_\alpha)^{\mu_f(U_\alpha)}$$

the limit is taken over increasingly fine covers $\{U_\alpha\}$ *of* $\mathbb{P}^1(\mathbb{Q}_p)$ *of disjoint open subsets and* $t_\alpha \in U_\alpha$ *is an arbitrary collection of sample points.*

Remark 4.20. *One should observe that this definition would be independent of the choice of p-adic logarithm that we did in the additive integral, since we have forced* κ_f *to take integer values. If we were to use another measure, this definition would depend on the choice of a p-adic exponential.*

The multiplicative Coleman integral induces a rigid analytic uniformization of the elliptic curve E by the curve X_Γ. If we extend linearly the formula on the definition of the multiplicative integral, we will obtain a map from the degree zero divisors of $\mathcal{H}_p$ to $\mathbb{C}_p^\times$. The descend of this map to the Jacobian of X_Γ (denoted as $\mathrm{Pic}^0(X_\Gamma)$) and $\mathbb{C}_p^\times/q_T^{\mathbb{Z}}$ defined as

$$(\tau_2) - (\tau_1) \mapsto \times\!\!\!\!\!\!\int_{(\tau_2)-(\tau_1)} f(z)dz := \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f(z)dz.$$

If the reader compares this process with the one we have used in the Archimedean side, in this setting we are interested in a map which goes to the multiplicative group $\mathbb{C}_p^\times$ rather than the addictive group $\mathbb{C}$ of the Archimedean case. This interest in the multiplicative group $\mathbb{C}_p^\times$ justifies the choice of the Coleman multiplicative integral in front of the Coleman additive integral. We fix a correspondence $\theta \in \mathbb{T}$ that maps $\mathrm{Div}(X_\Gamma)$ to $\mathrm{Div}^0(X_\Gamma)$ for example $T_l - (l+1)$ for a prime $l \nmid N$. We redefine the multiplicative integral as

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f(z)dz := \times\!\!\!\!\!\!\int_{\theta((\tau_2)-(\tau_1))} f(z)dz.$$

This redefine definition can be extended to $\mathrm{Pic}(X_\Gamma)$ as

$$(\tau) \mapsto \times\!\!\!\!\!\!\int^{\tau} f(z)dz := \times\!\!\!\!\!\!\int_{\theta(\tau)} f(z)dz.$$

One can see this definition as an analogous concept to the common semi-indefinite integral, since the integral satisfies the following relation

$$\times\!\!\!\!\!\!\int^{\tau_2} f(z)dz = \left(\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f(z)dz\right)\left(\times\!\!\!\!\!\!\int^{\tau_1} f(z)dz\right).$$

4.6 Heegner points and p-adic periods

Let $K/\mathbb{Q}$ be a quadratic imaginary field contained in $\mathbb{C}_p$ satisfying the Heegner Hypothesis i.e.

(i) the primes dividing N^- are inert in K,

(ii) the primes dividing N^+ are split in K,

(iii) p is inert in K.

Condition (ii) assures us the existence of an embedding from K into the quaternion algebra B. Let $\mathcal{O}$ be any $\mathbb{Z}[1/p]$-order of K of conductor coprime to N. Over this setting an orientation on the rings $T = \mathcal{O}$ or $T = R$ is a surjective homomorphism

$$T \to \left(\mathbb{Z}/N^+\mathbb{Z}\right) \times \prod_{l \mid N^-} \mathbb{F}_{l^2}.$$

The conditions (i) and (ii) imply the existence of an orientation on $\mathcal{O}$. We fix an orientation associated to the order $\mathcal{O}$.
Given an embedding $\Psi : K \to \mathcal{O}$, because of condition (iii), we have that the group $\iota\Psi(K^\times)$ has a fixed vertex $v_\Psi \in \mathcal{T}_p$ through the action induced on the tree. The definition of optimal oriented embedding that we have described over the modular curve case can be extended to this setting by satisfying the following conditions

(i) (Optimal) The embedding Ψ is optimal if it satisfies $\Psi(K) \cap R = \Psi(\mathcal{O})$.

(ii) (Oriented) The embedding Ψ is oriented with respect to the fix orientations in $\mathcal{O}$ and R if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & R \\ & \searrow & \downarrow \\ & & \left(\mathbb{Z}/N^+\mathbb{Z}\right) \times \prod_{l \mid N^-} \mathbb{F}_{l^2} \end{array}$$

(iii) The embedding Ψ is oriented at p (which plays the role of infinity in this setting) if

$$v_\Psi \in \mathrm{SL}_2(\mathbb{Q}_p)v^\circ.$$

The Heegner hypothesis that we have imposed on the field K assures us of the existence of optimal oriented embeddings.
We denote as K_p the field $K \otimes \mathbb{Q}_p$. From the properties of optimal oriented embeddings, we have that $K_p^\times$ acting on $\mathcal{H}_p$ through the embedding $\iota\Psi$ has a unique fixed point τ_Ψ satisfying

$$\iota\Psi(\lambda)\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix} = \lambda\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix}, \text{ for all } \lambda \in K_p^\times.$$

More specifically, this point belongs to $K_p \cap \mathcal{H}_p$. We denote as $\overline{\tau}_p$ the conjugate of τ_p under the action of $\mathrm{Gal}(K_p/\mathbb{Q}_p)$.
Observe that the condition of τ_Ψ, similarly to the case over the modular curve, we have an isomorphism $\mathrm{Pic}(\mathcal{O}) \cong \mathrm{End}(\mathcal{O}, R)$. This endomorphism allows us to define the conjugacy of an embedding Ψ by an element $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, similarly to what we did in that setting.

Definition 4.21. *Given an embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, we define the following periods*

$$I_\Psi := \times\!\!\!\!\!\!\int_{\overline{\tau}_\Psi}^{\tau_\Psi} f(z)dz \in \mathbb{C}_p^\times \text{ and } J_\Psi := \times\!\!\!\!\!\!\int^{\tau_\Psi} f(z)dz \in \mathbb{C}_p^\times \Big/ q_T^{\mathbb{Z}}.$$

The following lemma concretely determines that this period belongs in $\mathbb{C}_p$ and $\mathbb{C}_p \Big/ q_T^{\mathbb{Z}}$, respectively.

Lemma 4.22. *Given $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, the periods I_Ψ and J_Ψ satisfy $I_\Psi \in K_p^\times$ and $J_\Psi \in K_p^\times \Big/ q_T^{\mathbb{Z}}$.*

Proof. This lemma is an immediate consequence of the expression given in the definition of the multiplicative Coleman line integral. □

One should also observe that the property of the semi-indefinite integral gives us the following relation between the periods J_Ψ and I_Ψ

$$J_\Psi / \overline{J_\Psi} = I_\Psi \pmod{q^{\mathbb{Z}}}.$$

Let ω be the sign associated to the elliptic curve $E/\mathbb{Q}_p$ i.e. is 1 if E has split multiplicative reduction in p and -1 otherwise. We denote H as the ring class field of K associated to the fixed order $\mathcal{O}$. We choose an embedding $K_p \hookrightarrow \mathbb{C} - p$, which is equivalent to choosing a prime $\mathfrak{p}$ on H above p. We consider $\sigma_\mathfrak{p}$ the Frobenius element of $\mathfrak{p}$ in $\mathrm{Gal}(H/\mathbb{Q})$. We are in a position to state the following theorem, which states that the image of the periods J_Ψ over η_p are Heegner points in E.

Theorem 4.23. *For any $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, the point $\eta_p(J_\Psi)$ is a Heegner point in $E(H)$. Furthermore $\eta_p(I_\Psi)$ is the Heegner point $\eta_p(J_\Psi) - \omega\sigma_\mathfrak{p}\eta_p(J_\Psi)$ and for all $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we have*

$$\eta_p(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

Proof. The proof follows from Drinfeld's moduli interpretation of the upper half-plane. □

5 Stark-Heegner points

So far, we have defined CM points over Shimura curves (or modular curves) and found an analytical analog to the main theorem of complex multiplication, both in Archimedean and non-Archimedean settings. These theorems mainly follow from the algebraic interpretation of the maps given by the modularity theorem, Jacquet-Langlands correspondence, and p-adic uniformization, and comparing such results with the explicit analytic expressions given by such theorems.
In this chapter, we are going to introduce the notion of real multiplication points. These objects are especially interesting because there is no kind of algebraic intuition in the setting where they are defined.
One should start by noting that there will not be any real multiplication point, over a Riemann surface of the form $\mathcal{H}/\Gamma$. This fact justifies us in using a non-Archimedean setting in order to define real multiplication points. Let $F/\mathbb{Q}$ be a real quadratic field, $pM \in \mathbb{Z}$ an integer such that p is inert in F and all the prime divisors are split in F. We consider the analogous object to $M_0(M)$ with coefficients in $\mathbb{Z}[1/p]$

$$R = M_0(M) \otimes \mathbb{Z}[1/p]$$

and its restriction to $\mathrm{SL}_2(\mathbb{Z}[1/p])$, which will be denoted as Γ. We define real multiplication points for a given $\mathbb{Z}[1/p]$-order $\mathcal{O} \subseteq F$ as

$$\mathrm{RM}(\mathcal{O}) := \{\tau \in \mathcal{H}_p/\Gamma \ : \ \mathcal{O}_\tau = \mathcal{O}\}.$$

One should not that Γ does not act discretely in $\mathcal{H}_p$ and, therefore, in order to get some analytic periods associated to such points one should take a different approach. In this chapter, we will define a map from optimal oriented embeddings, with respect to $\mathcal{O}$ and R, and an elliptic curve $E/\mathbb{Q}$ of conductor pM

$$\mathrm{Emb}(\mathcal{O}, R) \dashrightarrow E(\mathbb{C}_p).$$

More concretely, to each optimal oriented embedding, we will associate the values of a period defined on $\mathcal{H}_p \times \mathcal{H}$ (where the group Γ acts discretely) and we will conjecture the behavior of the images of such embeddings.

5.1 Integration over $\mathcal{H}_p \times \mathcal{H}$

As we have mentioned before, we need to define periods associated to the space $\mathcal{H}_p \times \mathcal{H}$. In order to do so, we need to start defining cusp forms on this space.

Definition 5.1. *A function*

$$f : \mathcal{H} \times \mathcal{E}(\mathcal{T}_q) \to \mathbb{C}$$

is a cusp form of weight 2 on $(\mathcal{H} \times \mathcal{H}_q)/\Gamma$ is it satisfies the following properties.

(i) *For all* $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *the function f satisfies* $f(\gamma z, \gamma e) = (cz+d)^2 f(z,e)$*,*

(ii) *For every edge* $e \in \mathcal{E}(\mathcal{T}_q)$ *the function* $f_e := f(-,e)$ *is a holomorphic function satisfying that* $f_{\bar{e}} = -f_e$ *and for all vertex* $v \in \mathcal{V}(\mathcal{T}_q)$

$$\sum_{s(e)=v} f_e = 0,$$

(iii) *Given an edge* $e \in \mathcal{E}(\mathcal{T}_q)$ *the function* f_e *is a cusp form of weight 2 on* Γ_e*, the stabilizer of e.*

We will denote the collection of cusp forms over $(\mathcal{H} \times \mathcal{T}_q)/\Gamma$ *of weight 2 as* $S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$*.*

We should point out that there exist two degenerative maps between the regular cusp forms associated to the congruence subgroups $\Gamma_0(N)$ and $\Gamma_0(M)$

$$S_2(\mathcal{H}/\Gamma_0(N)) \rightrightarrows S_2(\mathcal{H}/\Gamma_0(M)).$$

As we have pointed out at the preliminary notions, the cusp forms that will be killed by both maps are called p-new cusp forms, and the collection of these forms we will denote as $S_2^{p-new}(\mathcal{H}/\Gamma_0(N))$. The following proposition will give us a correspondence between p-new cusp forms and the cusp forms of $(\mathcal{H} \times \mathcal{H}_q)/\Gamma$.

Proposition 5.2. *There exists an isomorphism*

$$S_2((\mathcal{H} \times \mathcal{T}_q)/\Gamma) \xrightarrow{\sim} S_2^{p-new}(\mathcal{H}/\Gamma_0(N))$$

defined by the correspondence $f \mapsto f_{e^\circ}$*.*

Proof. We should start pointing out that the stabilizer of e° is equal to $\Gamma_0(N)$ and consequently the correspondence gives us, a priori, a map

$$S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma) \to S_2(\mathcal{H}/\Gamma_0(N)).$$

One should note that if $f_{e^\circ} = 0$ then $f_{\bar{e}^\circ}$ is also trivial and, therefore, we have

$$f_e = 0 \text{ for all } e \in \mathcal{E}(\mathcal{T}_q) = \Gamma e^\circ \cup \Gamma_{\bar{e}^\circ}.$$

This last property shows that the assignment is injective; we shall check that all p-new forms will be in the image of the correspondence. Let f_0 be a p-new form. We define

$$f_e(z)dz = f_0(\gamma^{-1}z)d(\gamma^{-1}z) \text{ for } e = \gamma e^\circ \in \Gamma e^\circ.$$

We extend the definition of f to $e \in \Gamma \bar{e}^\circ$ by imposing $f_e = -f_{\bar{e}}$, one should note that this defines a collection $\{f_e\}$ for all $e \in \mathcal{E}(\mathcal{T}_q)$. It is also easy to check that this collection satisfies all the properties of the definition of cusps form over $(\mathcal{H} \times \mathcal{H}_q)/\Gamma$ with the exception of the Harmonic condition. We will prove such a condition. From the definition of p-new forms, we have that

$$\sum_{\gamma \in \Gamma_0(N)/\Gamma_0(M)} f_0(\gamma^{-1}z)d(\gamma^{-1}z) = 0, \tag{5.1.1}$$

$$\sum_{\gamma \in \Gamma_0(N)/\Gamma_0(M)} f_0(\gamma^{-1}\alpha^{-1}z)d(\gamma^{-1}\alpha^{-1}z), \tag{5.1.2}$$

where $\alpha \in \mathrm{GL}_2(\mathbb{Z}[1/q])$ is an element of positive determinant which belongs to the normalizer of $\Gamma_0(N)$ but not to $\Gamma_0(N)$. Observe that from this definition, we have that $e^\circ = \alpha\bar{e}^\circ$ and we denote $v^1 := \alpha v^\circ$ as the target of e°. Using strong approximation, we have that the natural embedding of $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\mathbb{Z}_q)$ identifies the closed space $\Gamma_0(N)/\Gamma_0(M)$ with $\mathrm{SL}_2(\mathbb{Z}_q)/\Gamma_0(q\mathbb{Z}_q)$. This implies that

$$\{\gamma e^\circ \ : \ \gamma \in \Gamma_0(N)/\Gamma_0(M)\}$$

is the list of edges with source v° and

$$\{\alpha\gamma e^\circ \ : \ \gamma \in \Gamma_0(N)/\Gamma_0(M)\}$$

is the list of edges with source v^1. Consequently, the equations 5.1.1 and 5.1.2 imply that the collection $\{f_e\}$ satisfies the harmonic condition for the edges v° and v^1

$$\sum_{\epsilon : s(\epsilon) = v^\circ} f_\epsilon = 0, \quad \sum_{\epsilon : s(\epsilon) = v^1} f_\epsilon = 0. \tag{5.1.3}$$

the Harmonic condition follows from this last property of f and the fact that $\mathcal{V}(\mathcal{T}_q) = \Gamma v^\circ \cup \Gamma v^1$. Consequently, we have proven that f is a cusp form of weight 2 on $(\mathcal{H} \times \mathcal{H}_q)/\Gamma$ such that $f_{e^\circ} = f_0$, consequently, this proves that all p-new forms are in the image of the correspondence. The fact that these are the only forms of the image of the correspondence follows from the equivalences given by the equations 5.1.1, 5.1.2, and 5.1.3. □

This proposition will be fundamental for the development of this section. On one hand, this isomorphism allows us to consider the action of Hecke operators in $S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$ inherited from the action in $S_2^{q-new}(\mathcal{H}/\Gamma_0(N))$. As it is pointed out in [Dar01, p. 18] this action can be explicitly described as

$$(T_\ell f)(e, z)dz = \sum_j f(\gamma_j^{-1}e, \gamma_j^{-1}z)d(\gamma_j^{-1}z)$$

for ℓ a prime prime to N and the collection $\{\gamma_j\}_j$ describe a disjoint union of left cosets such that

$$\Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \bigcup_j \gamma_j \Gamma.$$

On the other hand, this proposition will also allow us to identify a cusp form in $S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$ related to the cusp form in $S_2^{q-new}(\mathcal{H}/\Gamma_0(N))$ related to the elliptic curve E. Before specializing the theory to this concrete cusp form, we need to develop the theory of integration for general cusp forms in $S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$.
For each matrix $\gamma \in \mathrm{PGL}_2(\mathbb{Q}_q)$, we can define the norm $|\gamma|_q := \mathrm{ord}_q(\det(\gamma))$ which is well defined modulo 2. Furthermore, we define ω as the eigenvalue of the Atkin-Lenhner involution W_q action on f_0 i.e. ω is a sign which is equal to 1 if the abelian variety attached to f_0 has split multiplicative reduction over $\mathbb{Q}_q$ and -1 otherwise. The following lemma describes the action of $R_+^\times$, which re the matrices in $R^\times$ with positive determinant, for any cusp form $f \in S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$.

Lemma 5.3. *For all elements $\gamma \in \iota(R_*^\times)$, the function f will satisfy the following expression*

$$f_{\gamma e}(\gamma z) f(\gamma z) = \omega^{|\gamma|_p} f_e(z) dz.$$

Proof. The proof is in Lemma 1.5. of [Dar01]. □

After exploring the definition of Stark-Heegner points, we aim to give an analytic expression of these points using integration over $\mathcal{H}_p \times \mathcal{H}$ for a prime p. Essentially, we are going to generalize the methods of the last section by exchanging the roles of the primes p and ∞.
Fixing the general modular form f of the previous section with $q = p$, we consider the following Harmonic cocycle on the tree $\mathcal{T}_p$, defined for all $e \in \mathcal{E}(\mathcal{T}_p)$ and $\tau_1, \tau_2 \in \mathcal{H}^*$ as

$$\tilde{\kappa}_f\{\tau_3 \to \tau_4\}(e) := \int_{\tau_3}^{\tau_4} f_e(z) dz.$$

The measure $\tilde{\mu}_f\{\tau_3 \to \tau_4\}$ attached to this Harmonic cocycle takes values in $\mathbb{Z}_p$, which does not allow us to follow the natural construction we have used before (see remark 4.20). In order to use this measure in this setting, we need to modify the definition of the measure so that it becomes a p-adic integral, it's bounded, and it takes $\mathbb{Z}$-values. To do so, we have to restricting the points τ_3 and τ_4 to the boundary of $\mathcal{H}^*$ (i.e. in $\mathbb{P}^1(\mathbb{Q})$), after imposing this condition, the measure will be arisen from a cocyle which takes values in the lattice Λ defined in the last section. Using the first property of Lemma 6.2, we can assume that $\tau_3, \tau_4 \in \mathbb{P}^1(\mathbb{Q})$ and, therefore, $\tilde{\mu}_f\{\tau_3 \to \tau_4\}$ can be viewed as a Λ-valued p-adic measure. We consider a surjective group homomorphism

$$\beta : \Lambda \to \mathbb{Z},$$

such that the map is zero on $(\iota\mathbb{R} \cap \Lambda)$ and restricted to $(\Lambda \cap \mathbb{R}^+)$ takes values on the same set. We use this group isomorphism to define the following measure with values in $\mathbb{Z}$

$$\mu_f\{\tau_3 \to \tau_4\} := \beta(\tilde{\mu}_f\{\tau_3 \to \tau_4\}(U)).$$

Since this new measure takes $\mathbb{Z}$-values, we can define the double integral using the multiplicative Coleman integral defined before

$$⨍_{\tau_1}^{\tau_2} \int_x^y \omega_f := ⨍_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_f\{x \to y\}(t),$$

where $\tau_1, \tau_2 \in$ and $x, y \in \mathbb{P}^1(\mathbb{Q})$. Similarly to what we have done before, given $\tau \in \mathcal{H}_p$ and $x \in \mathbb{P}^1(\mathbb{Q})$, we define a 2-cocylce

$$\tilde{d}_{\tau,x}(\gamma_1, \gamma_2) := ⨍_{\tau}^{\gamma_1\tau} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \omega_f.$$

One should note, as we have shown on passed settings, that the class d of this cocycle in $H^2(\Gamma, \mathbb{C}_p^\times)$ will not depend on the choice of τ and x. We want to relate d with an element of the first cohomology of γ with values in a module of M-symbols.

Definition 5.4. *Let A be an abelian group. A function*

$$\begin{aligned} m\{,\} : \mathbb{P}^1(\mathbb{Q}) \times \mathbb{P}^1(\mathbb{Q}) &\to A \\ (x, y) &\mapsto m\{x \to y\} \end{aligned}$$

is an A-valued M-symbol if, given and $x, y, z \in \mathbb{P}^1(\mathbb{Q})$, satisfies the property

$$m\{x \to y\} + m\{y \to z\} = m\{x \to z\}.$$

Let $\mathcal{M}$ be the Γ-module of $\mathbb{C}_p$-valued M-symbols on $\mathbb{P}^1(\mathbb{Q})$ and $\mathcal{F}$ the $\mathbb{C}_p$-valued functions on $\mathbb{P}^1(\mathbb{Q})$. One can consider a general these groups for any abelian group A, and we will denote them as $\mathcal{M}(A)$ and $\mathcal{F}(A)$, respectively. We consider the map $\Lambda : \mathcal{F} \to \mathcal{M}$ defined as

$$(\Lambda f)\{x \to y\} := f(x) - f(y).$$

One should note that this map is surjective and its kernel is the collection of constant functions. From the short exact sequence of $\mathbb{C}_p[\Gamma]$-modules

$$0 \to \mathbb{C}_p \to \mathcal{F} \xrightarrow{\Delta} \mathcal{M} \to 0,$$

we can state the long exact sequence of cohomology groups

$$\cdots \to H^1(\Gamma, \mathcal{F}) \to H^1(\Gamma, \mathcal{M}) \xrightarrow{\delta} H^2(\Gamma, \mathbb{C}_p) \to H^2(\Gamma, \mathcal{F}) \to \cdots.$$

Given an element $\tau \in \mathcal{H}_p$, we define an $\mathcal{M}(\mathbb{C}_p^\times)$-valued one-cocycled as

$$\tilde{c}_\tau(\gamma)\{x \to y\} := \times\!\!\!\!\int_\tau^{\gamma\tau} \int_x^y \omega_f.$$

One can deduce, by directly computing it, that the class c of $\tilde{c}_\tau$ in $H^1(\Gamma, \mathcal{M}(\mathbb{C}_p^\times))$ does not depend on the choice of τ, and it satisfies

$$\delta(c) = d.$$

In section 3.1 of [Dar01] Darmon proves that the cohomology groups $H^1(\Gamma, \mathcal{M})$ and $H^2(\Gamma, \mathbb{C}_p)$ are finite dimensional $\mathbb{C}_p$-vector spaces with an induced action of the Hecke algebra $\mathbb{T}$ and equipped with an involution W_∞ induced by conjugation of

$$\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.$$

We denote as $H^1(\Gamma, \mathcal{M})^{f,+}$ and $H^2(\Gamma, \mathbb{C}_p)^{f,+}$ to the respective f isotypic part of $H^1(\Gamma, \mathcal{M})$ and $H^2(\Gamma, \mathbb{C}_p)$ fixed by W_∞.

Lemma 5.5. *The subgroups $H^1(\Gamma, \mathcal{M})^{f,+}$ and $H^2(\Gamma, \mathbb{C}_p)^{f,+}$ are 1-dimensional $\mathbb{C}_p$-vector spaces.*

Proof. Using the long exact sequence defined before, we get a long exact sequence for their f isotypic part fixed by W_∞ by the following commutative diagram

$$\begin{array}{ccccccc} H^1(\Gamma, \mathcal{F}) & \longrightarrow & H^1(\Gamma, \mathcal{M}) & \xrightarrow{\delta} & H^2(\Gamma, \mathbb{C}_p) & \longrightarrow & H^2(\Gamma, \mathcal{M}) \\ \downarrow & & \downarrow & & \downarrow & & \downarrow \\ H^1(\Gamma, \mathcal{F})^{f,+} & \longrightarrow & H^1(\Gamma, \mathcal{M})^{f,+} & \xrightarrow{\tilde{\delta}} & H^2(\Gamma, \mathbb{C}_p)^{f,+} & \longrightarrow & H^2(\Gamma, \mathcal{M})^{f,+} \end{array}$$

Darmon proved in section 3.1. of [Dar01] that the cohomology groups $H^j(\Gamma, \mathcal{F})^{f,+}$ are trivial for all j. This implies that the map $\tilde{\delta}$ iduces an isomorphism

$$\tilde{\delta} : H^1(\Gamma, \mathcal{M})^{f,+} \xrightarrow{\sim} H^2(\Gamma, \mathbb{C}_p)^{f,+}.$$

The proposition follows from Corollary 3.3 on [Dar01, p. 41]. □

Since the elliptic curve E has multiplicative reduction at p (because p divides the conductor), we can consider the Tate's p-adic period q_T attached to $E/\mathbb{Q}_p$ and the Tate p-adic uniformization

$$\eta_p : \mathbb{C}_p^\times / q_T^{\mathbb{Z}} \to E(\mathbb{C}_p).$$

We defined, as made in previous cases, ω as the sign associated to the p-reduction of the elliptic curve E. The following proposition will help us choose a lattice that satisfies the same properties with respect to the classes c and d as the ones we have stated in previous sections.

Theorem 5.6. *There exists a lattice $\Lambda_p \subseteq \mathbb{C}_p^\times$ commensurable with $q^{\mathbb{Z}}$ such that both images of c and d are trivial in $H^1(\Gamma, \mathcal{M}(\mathbb{C}_p^\times/\Lambda_p))$ and $H^2(\Gamma, \mathbb{C}_p^\times/\Lambda_p)$, respectively.*

Proof. We start choosing the p-adic logarithm function $\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p$ which satisfies $\log_p(p) = 0$. From lemma 3.4. of [Dar01], we have that $\mathrm{ord}_p(c)$ and $\log_p(c)$ both belong to $H^1(\Gamma, \mathcal{M})^{f,+}$ and $\mathrm{ord}_p(c) \neq 0$. From Lemma 5.5, we can deduce that there exists a $\mathcal{L} \in \mathbb{C}_p$ such that

$$\log_p(\tilde{c}_\tau) = \mathcal{L}\,\mathrm{ord}_p(\tilde{c}_\tau) \pmod{B^1(\Gamma, \mathcal{M})}. \tag{5.1.4}$$

Let n be a positive integer prime to N and j the minimum integer such that $p^{2j} \equiv 1 \pmod{n}$. Given an element $\mu \in (\mathbb{Z}/n\mathbb{Z})^\times$, we consider

$$\gamma_\mu := \begin{pmatrix} p^j & \mu\frac{p^j - p^{-j}}{n} \\ 0 & p^{-j} \end{pmatrix},$$

the generator of of the stabilizer in Γ of $(\infty, \mu/c) \in (\mathbb{P}^1(\mathbb{Q}))^2$. We should note that for any 1-coboundary $b \in B^1(\Gamma, \mathcal{M})$, we have

$$b(\gamma_\mu)\{\infty \to \mu/n\} = 0.$$

Consequently, the Equation 5.1.4 implies the relation

$$\log_p(\tilde{c}_\tau)(\gamma_\mu)\{\infty \to \mu/c\} = \mathcal{L}\,\mathrm{ord}_p(\tilde{c}_\tau)(\gamma_\tau)\{\infty \to \mu/c\}. \tag{5.1.5}$$

We consider a Dirichlet character $\chi : (\mathbb{Z}/n\mathbb{Z})^\times \to \mathbb{C}_p^\times$ of conductor n such that $\chi(p) = \omega$ and $\chi(-1) = 1$. In proposition 2.16 of [Dar01], Darmon proves that

$$\sum_{\mu \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(\mu)\,\mathrm{ord}_p(\tilde{c}_\tau)(\gamma_\mu)\{\infty \to \mu/n\} = \frac{2jn}{\tau(\chi)} \frac{L(E/\mathbb{Q}, \chi, 1)}{\Omega^+}, \tag{5.1.6}$$

where τ represents the Gauss sum of Dirichlet characters and Ω^+ is a rational multiple of the real period of the elliptic curve E. Using Proposition 2.18. of [Dar01] and the definition of the Mazur-Swinnerton-Dyer p-adic L-function, denoted as $L_p(E/\mathbb{Q}, \chi, s)$, attached to the elliptic curve $E/\mathbb{Q}$ and the character χ, we have that

$$\sum_{\mu \in (\mathbb{Z}/n\mathbb{Z})^\times} \chi(\mu) \log_p(\tilde{c}_\tau)(\gamma_\mu)\{\infty \to \mu/c\} = L_p'(E/\mathbb{Q}, \chi, 1). \tag{5.1.7}$$

Combining the equations 5.1.5, 5.1.6 and 5.1.7, we have that the difference

$$L(E/\mathbb{Q}, \chi, 1) - L'(E/\mathbb{Q}, \chi, 1),$$

for any Dirichlet character χ such that $\chi(p) = \omega$, is given by a constant $\mathcal{L}$ independent of the choice of χ. This was a conjecture first stated by Mazure, Tate, and Teitelbaum

in [MTT86]. In the same article, the authors conjectured, based on numerical evidence, that
$$\mathcal{L} = \frac{\log_p(q_T)}{\mathrm{ord}_p(q_T)}.$$
It is important to remark that this conjecture was proven by Greenberg and Stevens in [SG93]. From the proof of this conjecture, we get that
$$\log_p(c) = \frac{\log_p(q_T)}{\mathrm{ord}_p(q_T)}\,\mathrm{ord}_p(c) \text{ and } \log_p(d) = \frac{\log_p(q_T)}{\mathrm{ord}_p(q_T)}\,\mathrm{ord}_p(d), \tag{5.1.8}$$
where the second equation is deduced from the first and the fact that $\delta(c) = d$, as shown before. We define $\tilde{e}_{\tau,x}$ a 2-cocylce for $\tau \in \mathcal{H}_p$ and $x \in \mathbb{P}^1(\mathbb{Q})$ as
$$\tilde{e}_{\tau,x} := \frac{\tilde{d}_{\tau,x}^{\mathrm{ord}_p(q_T)}}{q_T^{\mathrm{ord}_p(\tilde{d}_{\tau,x})}} \in Z^2(\Gamma, \mathbb{C}_p^\times),$$
and we denote as e the image of this cocycle in $H^2(\Gamma, \mathbb{C}_p^\times)$. Using the second statement of the equation 5.1.8, we have
$$\mathrm{ord}_p(e) = \log_p(e) = 0$$
i.e. the class e is of torsion. Consequently, there exists an integer $\alpha \in \mathbb{Z}$ such that
$$\tilde{d}_{\tau,x}^{a\,\mathrm{ord}_p(q_T)} = q_T^{a\,\mathrm{ord}_p(\tilde{d}_{\tau,x})} \pmod{B^2(\Gamma, \mathbb{C}_p^\times)}.$$
The statement of the theorem follows from this last equation. □

We consider $\Lambda_p = q_T^{\mathbb{Z}}$ by raising $\tilde{c}$ and $\tilde{d}$ to a common power. Using Theorem 5.6, we can consider an M-symbol $m_\tau \in C^0(\Gamma, \mathcal{M}(\mathbb{C}_p^\times/q_T^{\mathbb{Z}}))$ and a 1-cochain $\xi_{\tau,x} \in C^1(\Gamma, \mathbb{C}_p^\times/q_T^{\mathbb{Z}})$, for $x \in \mathbb{P}^1(\mathbb{Q})$ and $\tau \in \mathcal{H}_p$, such that
$$\tilde{c}_\tau = dm_\tau \text{ and } \tilde{d}_{\tau,x} = d\xi_{\tau,x}.$$
We can use the M-symbol m_τ to give a definition of a semi-definite double integral as
$$⨍^\tau \int_x^y \omega_f := m_\tau\{x \to y\} \in \mathbb{C}_p^\times/q_T^{\mathbb{Z}},$$
for $\tau \in$ and $x, y \in \mathbb{P}^1(\mathbb{Q})$. The semi-indefinite notion is justified by the fact that the integral will satisfy the following properties
$$\left(⨍^\tau \int_x^y \omega_f\right)\left(⨍^\tau \int_x^z \omega_f\right) = ⨍^\tau \int_x^z \omega_f,$$
$$⨍^{\gamma\tau} \int_{\gamma x}^{\gamma y} \omega_f = ⨍^\tau \int_x^y \omega_f,$$
$$⨍^{\tau_2} \int_x^y \omega_f = \left(⨍^{\tau_1} \int_x^y \omega_f\right)\left(⨍_{\tau_1}^{\tau_2} \int_x^y \omega_f\right),$$

for all $\tau, \tau_1, \tau_2 \in \mathcal{H}_p$, $x, y, z \in \mathbb{P}^1(\mathbb{Q})$ and $\gamma \in \Gamma$.

5.2 Periods and Stark-Heegner points

Similarly to the case studied in section 5, we set the constant $M = N/p$. Let $\mathcal{O}$ be a $\mathbb{Z}[1/p]$-order of the real quadratic field F and R the Eichler $\mathbb{Z}[1/p]$-order of $M_2(\mathbb{Q})$. We define the notion of optimal oriented embedding, as we have done previously, for this setting.

Definition 5.7 (Optimal oriented embedding). *An embedding $\Psi : F \to M_2(\mathbb{Q})$ is an optimal oriented embedding if it satisfies the following properties*

(i) The embedding is optimal, with respect to $\mathcal{O}$, if it satisfies $\Psi(\mathcal{O}) = \Psi(F) \cap R$.

(ii) Given two ring homomorphisms $\mathcal{O}, R \to \mathbb{Z}/M\mathbb{Z}$, we say that Ψ is oriented with respect to such maps if the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & R \\ & \searrow & \downarrow \\ & & \mathbb{Z}/M\mathbb{Z} \end{array}$$

(iii) We say that the embedding Ψ is orientated at p if the unique vertex of $\mathcal{T}_p$ fixed by the action of $\Psi(F^\times)$ is $\mathrm{SL}_2(\mathbb{Q}_p)$-equivalent to the vertex v°. This condition is equivalent to requiring that the distance between this vertex and v° is even.

Given an optimal oriented embedding Ψ, we can define the period associated to this embedding as

$$J_\Psi := \xi_{\tau,x}(\gamma_\tau) = \times\!\!\!\!\!\!\int^{\tau} \int_x^{\gamma_\tau x} \omega_f \in \mathbb{C}_p^\times / q_T^{\mathbb{Z}}.$$

The following lemma, which follows from similar computations to the ones we will do in the next chapter, will imply that the period will be well-defined independently of the choices made.

Lemma 5.8. *Given an optimal oriented embedding Ψ, the period J_Ψ only depends on the Γ-conjugancy class of Ψ and not on the choice of x made in the definition.*

Let $\mathcal{O}_0 \subseteq \mathcal{O}$ the maximal $\mathbb{Z}$-order. We define H as the narrow ring class field of F attached to the order $\mathcal{O}$, which is the same as the one attached to the order $\mathcal{O}_0$ since we are supposing that p is split in F. From the theory of class field theory, we can state the reciprocity map

$$\mathrm{rec} : \mathrm{Pic}^+(\mathcal{O}) \to \mathrm{Gal}(H/K),$$

where Pic^+ is the narrow Picard group which is defined by quotiening by the principal ideals which have totally positive norm. The following conjecture, made first by Darmon in [Dar01], would extend the result of 6.9 for totally real fields.

Conjecture 5.9. *For any optimal oriented embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, the point $\eta_q(J_\Psi)$ is in $E(H)$ and, for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$, we have*

$$\eta_q(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_q(J_\Psi).$$

Despite having a similar statement to all the main theorems we have stated in the previous sections, no proof has been found so far for this conjecture. As we have mentioned before, the main obstacle to proving it resided in the fact that we have no algebraic interpretations of the objects defined in this section. In the following chapters, we will introduce settings where algebraicity can be proven, and we will show that similar periods are related to CM points that satisfy the conjecture started by Darmon.

6 Heegner points and integration over $\mathcal{H} \times \mathcal{H}_q$

In this section, we will consider the same cusp form as in the previous section, and we will develop another integration theory exchanging the roles of the primes p and ∞, following the work of [BDG07]. After developing this theory of integration, we will show that the periods associated to certain optimal oriented embeddings are related to the classical CM points over modular curves.
Let $K/\mathbb{Q}$ be a quadratic imaginary field and $pM \in \mathbb{Z}$, with $\gcd(p, M) = 1$ and all its prime divisors split at K. As before, we define the groups

$$R = M_0(M) \otimes \mathbb{Z}[1/p] \text{ and } \Gamma := R \cap \Gamma_0(M).$$

Similarly to the last case, Γ does not act discretely in $\mathcal{H}$, but it does over $\mathcal{H} \times \mathcal{H}_p$. Given an elliptic curve $E/\mathbb{Q}$ of conductor $N := pM$, we will use this fact to construct $\mathbb{C}$-periods over $\mathcal{H} \times \mathcal{H}_p$ in order to construct a map for a given $\mathbb{Z}[1/p]$-order $\mathcal{O} \subseteq K$

$$\mathrm{Emb}(\mathcal{O}, R) \dashrightarrow E(\mathbb{C}).$$

6.1 Cusp forms over $\mathcal{H} \times \mathcal{T}_q$

We use the same definition of cusp forms as the one of the last section, and using the modularity theorem and the isomorphism that identifies $S_2((\mathcal{H} \times \mathcal{H}_p)/\Gamma)$ with $S_2^{p-new}(\Gamma_0(N))$, we consider f associated to the elliptic curve E.
Given a general cusp form $f \in S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$, we will see the integration of this cusp form imagining an integration theory taking values in $\mathbb{C} \otimes \mathbb{C}_q^\times$ and we will view this value through the map $\mathrm{ord}_q : \mathbb{C} \otimes \mathbb{C}_q^\times \to \mathbb{C}$. Consequently, we define the double integral of f for any $\tau_1, \tau_2 \in \mathcal{H}_q^{nr}$ and $x, y \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$ as

$$\int_{\tau_1}^{\tau_2} \int_x^y \omega_f = \sum_{\epsilon : r(\tau_1) \to r(\tau_2)} \int_x^y 2\pi i f_e(z) dz.$$

One should notice that this definition satisfies the common addition properties of integrals and that given any matrix $\gamma \in \Gamma$, we will have the following relation for any $\tau_1, \tau_2 \in \mathcal{H}_q^{nr}$ and $x, y \in \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$

$$\int_{\gamma\tau_1}^{\gamma\tau_2} \int_x^y \omega_f = \int_{\tau_1}^{\tau_2} \int_x^y \omega_f.$$

Following a similar motivation to past chapters, given $x \in \mathcal{H}_q^{nr}$ and $\tau \in \mathcal{H}$, we consider the function $\tilde{d}_{x,\tau} : \Gamma \times \Gamma \to \mathbb{C}$ defined as

$$\tilde{d}_{x,\tau}(\gamma_1, \gamma_2) = \int_{\gamma_1 x}^{\gamma_2\gamma_2 x} \int_\tau^{\gamma_1\tau} \omega_f.$$

One can check, by direct computation, that this function is indeed a 2-cocycle and that the image of $\tilde{d}_{x,\tau}$ in $H^2(\Gamma, \mathbb{C})$ is independent of the choice of the elements x and τ.
As it has been pointed out before, Proposition 5.2 allows us to choose a cusp form $f \in S_2((\mathcal{H} \times \mathcal{H}_q)/\Gamma)$ such that f_{e° is the cusp form associated to the elliptic curve E given by the modularity theorem. We define the group $\lambda \subseteq \mathbb{C}$ as the image of the relative homology $H_1(X_0(N), \text{cusps}; \mathbb{Z})$ under the integration pairing with f_0. The following theorem will enable us with a useful property of Λ, which will be essential for the definition of the periods that we will do at the end of this section.

Theorem 6.1 (Manin-Drinfeld). *The subgroup $\Lambda \subseteq \mathbb{C}$ is a lattice which is commensurable with the Néron lattice Λ_E associated to the elliptic curve E.*

Proof. From [Man72] and [Maz73], we have that the theorem follow from the fact that $\ell + 1 - T_\ell$ maps $H_1(X_0(N), \text{cusps}, \mathbb{Z})$ to $H_1(X_0(N), \mathbb{Z})$ and, consequently, $(\ell + 1 - a_\ell)\Lambda$ is contained in Λ_E. □

The following lemma will give us a useful property of the line integral with respect to the lattice Λ, which will be useful for further results.

Lemma 6.2. *The cusp form f satisfies the following two properties with respect to the lattice Λ*

(i) Given two elements $x, y \in \mathbb{P}^1(\mathbb{Q})$ and $e \in \mathcal{E}(\mathcal{T}_q)$, the complex line integral

$$\int_x^y f_e(z)dz$$

belongs to the lattice Λ.

(ii) Given any $\tau_1, \tau_2 \in \mathcal{H}_q^{nr}$ and any $x, y \in \mathbb{P}^1(\mathbb{Q})$ we have

$$\int_{\tau_1}^{\tau_2} \int_x^y \omega_f \in \Lambda.$$

Proof. Given an edge $e \in \mathcal{E}(\mathcal{T}_q)$, there exists an element $\gamma \in \Gamma$ such that

$$\gamma^{-1}e^\circ = e \text{ or } \gamma^{-1}e^\circ = \bar{e}.$$

Using Lemma 5.3 and the properties of f_0, we have the following equation

$$\int_x^z f_e(z)dz = \pm \int_{\gamma x}^{\gamma y} f_0(z)dz.$$

We clearly have that the second integral of the last expression belongs to λ, which proves the first part of this proposition. The second part of the lemma follows immediately from the first part and the definition of the double integral. □

The lemma we have just proven will be fundamental to proving the following property of the 2-cocycle $\tilde{d}_{x,\tau}$.

Proposition 6.3. *The image of the 2-cocycle $\tilde{d}_{x,\tau}$ through the natural projection in $H^2(\Gamma, \mathbb{C}/\Lambda)$ is trivial.*

Proof. In order to prove this lemma, we will show that the image of $\tilde{d}_{x,\tau}$ will be equal to the class of the coboundary map of the 1-cochain map $\xi : \Gamma \to \mathbb{C}$ defined as

$$\xi_{x,\tau}(\gamma) = \int_x^{\gamma x} \int_\infty^\tau \omega_f.$$

Using the fact that Γ acts trivially on $\mathbb{C}$, we have

$$\begin{aligned}
d\xi_{\tau,x}(\gamma_1, \gamma_2) &= \int_x^{\gamma_1 x} \int_\infty^\tau \omega_f + \int_x^{\gamma_2 x} \int_\infty^\tau \omega_f - \int_x^{\gamma_1\gamma_2 x} \int_\infty^\tau \omega_f \\
&= \int_{\gamma_1\gamma_2 x}^{\gamma_1 x} \int_\infty^\tau \omega_f + \int_x^{\gamma_2 x} \int_\infty^\tau \omega_f \\
&= \int_{\gamma_1\gamma_2 x}^{\gamma_1 x} \int_\infty^\tau \omega_f + \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \int_{\gamma_1\infty}^{\gamma_1\tau} \omega_f \\
&= \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \int_\tau^{\gamma_1 x} \omega_f - \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \int^{\gamma_1\infty} \omega_f \\
&= \tilde{d}_{\tau,x}(\gamma_1, \gamma_2) - \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \int_\infty^{\gamma_1\infty} \omega_f.
\end{aligned}$$

The second part of the Lemma 6.2 implies that the second term of the last expression is in Λ and, consequently, the statement of the proposition is proven. □

The 1-cochain ξ used in the proof of this last proposition depends on a choice of x and τ. In order to make such a choice canonical, we will show that the group of 1-cocycles has finite exponent, so this indetermination can be solved by replacing ξ by an integer multiple of the same cocycle.

Lemma 6.4. *The group of 1-cocycles on Γ with coefficients in the quotient $\mathbb{C}/\Lambda$ has finite exponent.*

Proof. We have that Γ acts trivially on $\mathbb{C}/\Lambda$ and, consequently the group of n-cocycles is

$$\operatorname{Hom}(\Gamma^n, \mathbb{C}/\Lambda) = \operatorname{Hom}((\Gamma^n)^{ab}, \mathbb{C}/\Lambda).$$

The rest follows from the Corollary of Theorem 3 of [Ser70]. □

6.2 Heegner points and complex periods

Let $\mathcal{O} \subseteq K$ a $\mathbb{Z}[1/p]$-order of conductor prime to N and denote $\mathcal{O}_0 := \mathcal{O} \cap \mathcal{O}_K$ as the maximal $\mathbb{Z}$-order in $\mathcal{O}$. We suppose that N satisfies the Heegner hypothesis and, consequently, there exists an orientation $\mathcal{N}$ such that $N\mathcal{O}_0 = \mathcal{N}\overline{\mathcal{N}}$. As we have pointed out before, we consider the standard Eichler order $R = M_0(M) \otimes \mathbb{Z}[1/p]$ and we extend the definition of optimal oriented embeddings for this setting.

Definition 6.5. *Given an embedding $\Psi : K \to B$, we say is optimal oriented with respect to $\mathcal{O}$ and R is it satisfies the following conditions*

(i) The embedding is optimal if it satisfies $\Psi(\mathcal{O}) = R \cap \Psi(K)$,

(ii) Given an Eichler orientation $R \to \mathbb{Z}/N\mathbb{Z}$ and considering the ideal $\mathcal{M} := \mathcal{N}\mathcal{O}$, the embedding is said to be oriented (with respect to $\mathcal{N}$ and the given orientation) if the following diagram is commutative

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & R \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathcal{M} & \xrightarrow{\sim} & \mathbb{Z}/M\mathbb{Z} \end{array}$$

(iii) There exists a fixed point $\tau \in K_q$ associated to the action of $\Psi(K_q^\times)$ such that for any $\lambda \in K_q^\times$, we have

$$\Psi(\lambda)\begin{pmatrix} \tau \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

We say that the embedding Ψ is oriented at q if it belongs to $\mathbb{Z}_q$ under a chosen embedding $K_q \hookrightarrow \mathbb{C}_q$.

As in previous sections, we denote the collection of optimal oriented embeddings in this setting as $\operatorname{Emb}(\mathcal{O}, R)$.

One should observe that $\Psi(K_q^\times)$ leaves invariant the geodesic on the tree $\mathcal{T}_q$ joining the two fixed points of $\Psi(K_q^\times)$ in $\mathbb{P}^1(\mathbb{Q}_q)$. The following lemma will give us a useful result of the stabilizer of τ_Ψ in Γ.

Lemma 6.6. *Given an $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, the stabilizer of τ_Ψ in Γ, which we denote as Γ_τ, has rank one.*

Proof. The generalization of the Dirichlet unit theorem assures us that the group $\mathcal{O}_1^\times$ is of rank one. Since $\Psi(K^\times)$ is the stabilizer of τ in $\mathrm{GL}_2(\mathbb{Q})$ and Ψ is optimal oriented we have that $\Gamma_\tau = \Psi(\mathcal{O}_1^\times)$, which proves the statement. □

Let u be a generator of the group $\mathcal{O}_1^\times$ of units of norm one and consider the matrix $\gamma_\tau = \Psi(u)$. Given $x, y \in \mathcal{H}^{nr}$ and $\tau \in \mathcal{H}$, using the properties of the double integral, we have the equation

$$\int_x^{\gamma_\tau x}\int_\infty^\tau \omega_f - \int_y^{\gamma_\tau y}\int_\infty^\tau \omega_f = \int_{\gamma_\tau y}^{\gamma_\tau x}\int_\infty^\tau \omega_f - \int_x^y\int_\infty^\tau \omega_f = \int_y^x\int_{\gamma_\tau^{-1}}^\tau \omega_f - \int_y^x\int_\infty^\tau \omega_f = 0$$

up to a coefficient in Λ and, for any $\gamma \in \Gamma$ we have

$$\int_x^{\gamma\gamma_\tau\gamma^{-1}x}\int_\infty^{\gamma\tau} \omega_f = \int_{\gamma^{-1}x}^{\gamma_\tau\gamma^{-1}x}\int_{\gamma^{-1}\infty}^{\tau} \omega_f = \int_x^{\gamma_\tau x}\int_\infty^\tau \omega_f \pmod{\Lambda}.$$

which shows that $\xi(\gamma_\tau)$ does not depend on the choice of x and only depends on the Γ-conjugacy class of the embedding Ψ. Given a vertex v, this last fact motivates the definition of the following period associated to the embedding Ψ

$$J_\Psi := \xi(\gamma_\tau) = \sum_{\epsilon: v \to \gamma_\tau v} \int_\infty^\tau 2\pi i f_e(t)dt$$

Note that this definition does not depend on the choice of vertex v. We consider the decomposition $p\mathcal{O}_0 = \mathfrak{p}\overline{\mathfrak{p}}$ and we denote as ϖ the uniformizer of $\mathfrak{p} \otimes \mathbb{Z}_q$. The following lemma will give as a useful property between the optimal oriented embeddings in this setting and the same notion defined on the sense of Section 3.

Lemma 6.7. *For any $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, there exists an optimal oriented embedding (on the sense of Definition) $\Psi_0 \in \mathrm{Emb}(\mathcal{O}_0, M_0(N))$ such that the restriction of Ψ to $\mathcal{O}_0$ is Ψ_0. The choice of Ψ_0 will be characterized by the fact that v° and $\Psi(\varpi)v^\circ$ are adjacent vertices connected by e°.*

Proof. We consider the edge $e \in \mathcal{E}(\mathcal{T}_p)$ such that it lies on the path joining the two fixed points of the action $\Psi(K_\mathfrak{p}^\times)$ in $\mathbb{P}^1(\mathbb{Q}_p)$. There exists an element $\gamma \in \Gamma$ (up to change of orientation if needed) such that $e = \gamma^{-1}e^\circ$. The Embedding satisfying the properties given by the statement is $\Psi_0 = \gamma\Psi\gamma^{-1}$. Furthermore, the lattice $\Psi(\varpi)\mathbb{Z}_p^2$ is homothetic to

$$\begin{pmatrix}\tau_{\Psi^0}\\1\end{pmatrix}\mathbb{Z}_p + \begin{pmatrix}p\\0\end{pmatrix}\mathbb{Z}_p$$

which is in the class of the adjacent vertex of v°. □

In order to have a more clear notation, given an embedding $\Psi_0 \in \mathrm{Emb}(\mathcal{O}_0, M_0(N))$, we will denote as J_{0,Ψ_0} the archimeadian period define in section 3. One should recall that ω is a sign of the abelian variety attached to f_0. We will let t be twice the order of the prime $\mathfrak{p}$ in $\mathrm{Pic}(\mathcal{O}_0)$.

Proposition 6.8. *Given an optimal oriented embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$ and we consider the embedding $\Psi_0 \in \mathrm{Emb}(\mathcal{O}_0, M_0(N))$ given by lemma 6.7, then we have*

$$J_\Psi = \sum_{j=0}^{t-1} \omega^j J_{0,\Psi_0^{\mathfrak{p}^j}} \pmod{\Lambda}.$$

Proof. We consider the consecutive vertices $v_0, v_1, ..., v_t$ of $\mathcal{T}_q$ on the path joining v_0 and $v_t = \gamma_\Psi v_0$ and $e_0, ..., e_{t-1}$ the different edges joining the consecutive vertices. Let R_{v_0,v_t} be the subring of matrices in $M_2(\mathbb{Q}_q)$ which fix both v_0 and v_t. Observe that

$$\Psi(\varpi)(v_j) = v_{j+1} \text{ for } j = 0, ..., t-1.$$

Using strong approximation, we have that there exists an element $\gamma \in R_+^\times$ such that its image in $\mathrm{PGL}_2(\mathbb{Q}_q)/R_{v_0,v_t}^\times$ is $\Psi(\varpi)$. This implies that

$$\gamma(e_j) = e_{j+1} \text{ for } j = 0, ..., t-2.$$

Using the definition of the period attached to Ψ, we have

$$J_\Psi = \sum_{j=0}^{t-1} \int_\infty^{\tau_{\Psi_0}} f_{e_j}(z)dz \pmod{\Lambda}. \tag{6.2.1}$$

Using the property that we have stated before for the edges e and Lemma 6.2, we can state the following equation

$$\int_\infty^{\tau_{\Psi_0}} f_{e_j}(z)dz = \int_\infty^{\tau_{\Psi_0}} f_{\gamma^j e_0}(z)dz = \omega^j \int_\infty^{\gamma^{-j}\tau_{\Psi_0}} f_0(z)dz \pmod{\Lambda}. \tag{6.2.2}$$

Using the conjugation property of optimal oriented embeddings $\mathrm{Emb}(\mathcal{O}_0, M_0(N))$, we have the following equation

$$\int_\infty^{\gamma^{-j}\tau_{\Psi_0}} f_0(z)dz = J_{0,\Psi_0^{\mathfrak{p}^j}}. \tag{6.2.3}$$

The proposition is an immediate consequence of the equations 6.2.1, 6.2.2, and 6.2.3. □

Let H_0 be the ring class field of K attached to the order $\mathcal{O}_0$ and $\sigma_\mathfrak{p}$ the Frobenius element of $\mathrm{Gal}(H_0/K)$ identified with the prime $\mathfrak{p}$. We define the p-narrow class field $H = H_0^{\sigma_\mathfrak{p}^2}$, and we consider the reciprocity given by the theory of class field theory

$$\mathrm{rec} : \mathrm{Pic}^{p+}(\mathcal{O}) \to \mathrm{Gal}(H/K),$$

which $\mathrm{Pic}^{p+}(\mathcal{O})$ is defined by adding the condition that principal ideals must have even valuation at p. If we consider the Weierstrß uniformization η associated to the lattice Λ_E, we can give the main result of this section.

Theorem 6.9. *For all $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$ the point $\eta(J_\Psi)$ is a global point of $E(H)$. Furthermore, for all $\mathfrak{a} \in \mathrm{Pic}^{p+}(\mathcal{O})$, we have*

$$\eta(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta(J_\Psi).$$

Proof. The Proposition 6.8 shows us that the periods J_Ψ are products of periods of optimal oriented embeddings in $\mathrm{Emb}(\mathcal{O}_0, M_0(N))$. Consequently, this Theorem follows from this last result and the main theorem of complex multiplication. $\square$

It is important to remark that this theorem, in concrete terms using the underlying idea of its proof, allows us to find periods over $\mathcal{H} \times \mathcal{H}_q$ for a prime $q|N$, such that $q^2 \nmid N$, as long as N satisfies the Heegner Hypothesis with respect to a given quadratic imaginary field.

7 Heegner points and integration over $\mathcal{H}_p \times \mathcal{H}_q$

After studying the relation between Heegner points and integration periods over $\mathcal{H}_p$, we would like to extend this definition to periods over products of two upper half-planes $\mathcal{H}_p \times \mathcal{H}_q$. Before developing the integration for this setting, we shall introduce exactly the basic objects that we are going to work with.
Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = pqN^+N^-$, with all the factors prime pairwise, p and q prime, and N^- square-free with an odd number of prime divisors.
Let B be the definite quaternion algebra over $\mathbb{Q}$ of discriminant N^- and R the Eichler $\mathbb{Z}[1/pq]$-order of level N^+. We fix the following embedding associated to the algebra B

$$\iota : B \to M_2(\mathbb{Q}_p) \times M_2(\mathbb{Q}_q).$$

Similarly to the last section, we define $R_1^\times$ as the collection of units with norm 1, and we define the following modular subgroup

$$\Gamma := \iota(R_1^\times) \subseteq \mathrm{SL}_2(\mathbb{Q}_p) \times \mathrm{SL}_2(\mathbb{Q}_q).$$

Given an element $\gamma \in \Gamma$, we will represent its coordinates in the matrix product by $\gamma = (\gamma_p, \gamma_q)$.

7.1 Integration of cusp forms on $\mathcal{H}_p \times \mathcal{H}_q$

In order to define the periods, we shall extend the definition of cusp forms to the ones associated to Γ and the definition of the integrals over this product. We can define an action in $\mathcal{H}_p \times \mathcal{H}_q$ by Γ, considering the induced actions in the coordinates i.e. for $\gamma \in \Gamma$ and $(\tau_p, \tau_q) \in \mathcal{H}_p \times \mathcal{H}_q$, we have $\gamma(\tau_p, \tau_q) = (\gamma\tau_p, \gamma\tau_q)$. Furthermore, we have that Γ acts naturally on $\mathcal{E}(\mathcal{T}_p) \times \mathcal{H}_q$.

Definition 7.1 (Cusp form). *A function*

$$f : \mathcal{E}(\mathcal{T}_p) \times \mathcal{H}_q \to \mathbb{C}_q$$

is a cusp form of weight two if it satisfies the following conditions

(i) $f(\gamma_p e, \gamma_q z) = (cz + d)^2 f(e, z)$ *for all* $\gamma \in \Gamma$ *with* $\gamma_q = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

(ii) $f_z(e) := f(e, z)$ *is an harmonic cocycle with values in* $\mathbb{C}_q$ *for all* $z \in \mathcal{H}_q$.

(iii) For all $e \in \mathcal{E}(\mathcal{T}_p)$*, the function* $f_e(z) := f(e, z)$ *is a q-adic rigid analytic modular form of weight two (as defined in section 3) for the stabilizer* Γ_e *of* e *in* Γ.

We will denote as $S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma)$ *the* $\mathbb{C}_q$*-vector space of weight 2 cusp forms over* $(\mathcal{T}_p \times \mathcal{H}_q)/\Gamma$.

We fix an Eichler $\mathbb{Z}[1/q]$-order R_0 of level pN^+. Using this concrete Eichler order, we can consider a base edge e° in $\mathcal{E}(\mathcal{T}_p)$ by imposing the condition that Γ_{e° is equal, via the embedding ι, to the group Γ_q of norm 1 elements in R_0. We denote $S_2^{p-new}(\mathcal{H}_q/\Gamma_q)$ the subspace of p-new cusps forms of weight 2 on Γ_q. The following lemma relates to the two spaces we have just defined.

Lemma 7.2. *There exists an isomorphism from* $S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma)$ *to* $S_2^{p-new}(\mathcal{H}_q/\Gamma_q)$ *which sends* f *to* f_{e°.

Proof. Since the stabilizer of e° in Γ is Γ_q, from the cusp forms definition over $\mathcal{T}_p \times \mathcal{H}_q$ that the following map is well defined

$$\begin{aligned} S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma) &\to S_2(\mathcal{H}_q/\Gamma_q) \\ f &\mapsto f_{e^\circ} \end{aligned}$$

If we assume that $f_{e^\circ} = 0$, then we have that $f_{\bar{e}^0} = 0$, consequently

$$f_e = 0 \text{ for all } e \in \Gamma_{e^\circ} \cup \Gamma_{\bar{e}^0} = \mathcal{E}(\mathcal{T}_p),$$

which proves that the assignment is injective. We will continue proving that the image of this assignment is $S_2^{p-new}(\mathcal{H}_q/\Gamma_q)$. We consider a modular form f^0 in $S_2^{p-new}(\mathcal{H}_q/\Gamma_q)$ and we define

$$f_e(z)dz = f_0(\gamma^{-1}z)d(\gamma^{-1}z) \text{ for } e = \gamma e^\circ.$$

We can extend this definition for edges $e \in \Gamma\bar{e}^0$ by establishing the relation $f_e = -f_{\bar{e}}$. One can easily check that $\{f_e\}$ satisfies all the conditions, with the exception of the

Harmonic condition, of the definitions of cusp forms of weight 2 over $(\mathcal{T}_p \times \mathcal{H}_q)/\Gamma$. We will check that it satisfies the Harmonic condition. Since f_0 is a p-new form, we can establish the following two equations

$$\sum_{\gamma \in \Gamma_0(N)/\Gamma_0(M)} f_0(\gamma^{-1}z)d(\gamma^{-1}z) = 0, \tag{7.1.1}$$

$$\sum_{\gamma \in \Gamma_0(N)/\Gamma_0(M)} f_0(\gamma^{-1}\alpha^{-1}z)d(\gamma^{-1}\alpha^{-1}z) = 0, \tag{7.1.2}$$

where $\alpha \in \mathrm{GL}_2(\mathbb{Z}[1/p])$ such that it has positive determinant, it belongs to the normalizer of $\Gamma_0(N)$ but not to $\Gamma_0(N)$ itself. One should realize that $\alpha e^\circ = \overline{e}^0$, and consequently $v^1 = \alpha v^0$ is the target of e°. Using strong approximation, we have that the natural embedding of $\mathrm{SL}_2(\mathbb{Z})$ into $\mathrm{SL}_2(\mathbb{Z}_p)$ identifies the coset space $\Gamma_0(N)/\Gamma_0(M)$ with $\mathrm{SL}_2(\mathbb{Z}_p)/\Gamma_0(p\mathbb{Z}_p)$, which implies that the set

$$\{\gamma e^\circ \ : \ \gamma \in \Gamma_0(N)/\Gamma_0(M)\}$$

is the list of edges with source v^0 and

$$\{\alpha\gamma e^\circ \ : \ \gamma \in \Gamma(N)/\Gamma_0(M)\}$$

is the list of edges with source v^1. This implies that the equations 7.1.1 and 7.1.2, deduced from the definition of f_0, imply that the collection $\{f_e\}$ is harmonic at v^0 and v^1 i.e. we have the equations

$$\sum_{\epsilon : s(e) = v^0} f_\epsilon = 0 \text{ and } \sum_{\epsilon : s(e) = v^1} f_\epsilon = 0. \tag{7.1.3}$$

Using the fact that $\mathcal{V}(\mathcal{T}_p) = \Gamma v^0 \cup \Gamma v^1$ and the Γ-equivariance property from f, we can establish the Harmonic property.
Consequently, f is a cusp form of weight 2 on $(\mathcal{H}_p \times \mathcal{T}_q)/\Gamma$. This shows us that the assignment $f \mapsto f_{e^\circ}$ has all the new-forms at p on its image. The fact that all the functions in the image will be p new forms comes from the fact that the statement given in equation 7.1.3 is equivalent to the equation 7.1.1 and 7.1.2. □

Using this lemma, we can find a correspondent function f on $(\mathcal{T}_p \times \mathcal{H}_q)/\Gamma$ to the function f defined in section 3. Recall that the function of section 3 is defined using the Jacquet-Langlands and the Modularity Theorem, and we force the associate κ_f to have integer values. Note that this function will be determined up to a sign, and f_{e° will be the normalized cusp form of Section 3.
Following the same notation as the one introduced in the last section, we denote as $\mathcal{H}_q^{nr}$ the unramified q-adic upper half-plane. Given two points $x_1, x_2 \in \mathcal{H}_q^{nr}$ which have

associated vertices v_1 and v_2 respectively, we define the $\mathbb{Z}$-valued harmonic cocycle on $\mathcal{E}(\mathcal{T}_q)$ as

$$\kappa_f\{x_1 \to x_2\}(e) = \sum_{\epsilon: v_1 \to v_2} \kappa_{f_e}(\epsilon),$$

where the sum is defined over the edges that join the vertices v_1 and v_2. Using the lemma 7.2, after fixing two points $x_1, x_2 \in \mathcal{H}_q^{nr}$ we have the following diagram

$$\begin{array}{ccc} S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma) & \longrightarrow & \mathrm{Har}(\Gamma_q) \\ \big\downarrow \sim & & \sim \big\uparrow \\ S_2^{p-new}(\mathcal{H}_q/\Gamma_q) & \longrightarrow & \mathrm{Har}(\Gamma_q) \end{array}$$

where the right side is given by the multiplication of an integer that depends on x_1 and x_2. Therefore, using the correspondence between harmonic cocycles and rigid analytic cusp forms of weight 2, we can see that with this definition, fixed $x_1, x_2 \in \mathcal{H}_q^{nr}$, we are identifying a subgroup of $\mathrm{Har}(\Gamma_q)$. In concrete, since we have fixed f so its associated harmonic cocycles have integer values, we have that the harmonic cocycles $\kappa_f\{x_1 \to x_2\}$ also have integer values. As we proved in Lemma 4.15, Harmonic cocycles stable over Γ_q generate a q-adic measure, which in our case will be denoted as $\mu_f\{x_1 \to x_2\}$, such that

$$\int_{U_e} d\mu_f\{x_1 \to x_2\}(t) = \kappa_f\{x_1 \to x_2\}(e).$$

Following the same methods as the last section, especially motivated by Proposition 4.16, we define the following integral.

Definition 7.3. *Let $\tau_1, \tau_2 \in \mathcal{H}_p$ and $x_1, x_2 \in \mathcal{H}_q^{nr}$. We define the double integral*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{x_1}^{x_2} \omega_f = \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_f\{x_1 \to x_2\}(t) \in \mathbb{C}_p^{\times}.$$

This definition should be viewed as a defined double integral on $\mathcal{H}_p \times \mathcal{H}_q$. Given a function $g \in S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma)$, we define the Γ-ivariant cocycle

$$\lambda_f : \mathcal{E}(\mathcal{T}_p) \times \mathcal{E}(\mathcal{T}_q) \to \mathbb{C},$$

defined by the correspondence $(e, \epsilon) \mapsto \kappa_{f_e}(\epsilon)$. Oberve that from the correspondence between $S_2(\mathcal{H}_q/\Gamma_q)$ and $\mathrm{Har}(\Gamma_q)$, we have that the harmonic cocycle will uniquely determine g. Furthermore, we have that in the case of f, the Harmonic cocycle λ_f will take integer values. This fact is formalized in the following lemma, where we denote as $\mathrm{Har}_2(\Gamma, R)$, the Γ-invariant harmonic cocycles on $\mathcal{T}_p \times \mathcal{T}_q$ with values on a given ring $R \subseteq \mathbb{C}$.

Lemma 7.4. *The map $S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma) \to \mathrm{Har}_2(\Gamma, \mathbb{C}_p)$, defined by sending a cusp form of weight 2 on $(\mathcal{T}_p \times \mathcal{H}_q)/\Gamma$ to a $\mathbb{C}_p$-valued harmonic cocycle through the residue map, is a Hecke-equivariant isomorphism of $\mathbb{C}_p$-vector spaces.*

Proof. The proof follows immediately from the correspondence given by Proposition 7.2 and the correspondence between one-dimensional harmonic cocycles and rigid analytic cusps forms of weight 2. □

Exchanging the roles of p and q, we get the analogous definitions for cusps forms of weight 2 on $(\mathcal{H}_p \times \mathcal{T}_q)/\Gamma$ and the correspondent Γ-invariant harmonic cocycles λ_g for all $g \in S_2((\mathcal{H}_p \times \mathcal{T}_q)/\Gamma)$, defined by the correspondence $(e, \epsilon) \mapsto \kappa_{g_\epsilon}(e)$.

Corollary 7.5. *For all $g \in S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma)$ such that λ_g is $\mathbb{Z}$-valued, there exists a cusp form $gf^{\#}$ in $S_2((\mathcal{H}_p \times \mathcal{T}_q)/\Gamma)$ such that $\lambda_g = \lambda_{g^{\#}}$.*

Proof. The corollary is an immediate consequence of the last lemma, the fact that λ_f is $\mathbb{Z}$-valued and that $(\mathcal{T}_p \times \mathcal{T}_q)/\Gamma$ is a finite graph. □

From now on, we will denote $f^{\#}$ as the cusp form given by this last corollary associated to the cusp form f.
The following definition gives us the analogous definition of the double definite integral, where we switch the order given in the previous definition.

Definition 7.6. *Given $x_1, x_2 \in \mathcal{H}_p^{nr}$ with associated vertices v_1 and v_2 respectively, and $\tau_1, \tau_2 \in \mathcal{H}_p$, we define the double integral*

$$\int_{x_1}^{x_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}} = \prod_{\epsilon: v_1 \to v_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f_\epsilon^{\#}(z) dz \in \mathbb{C}_p^{\times},$$

where the product is taken over the oriented edges that join v_1 and v_2.

The following theorem relates the two definitions of double definite integrals that we have given, where the correspondence given by Corollary 7.5 plays an essential role.

Theorem 7.7 (Fubini Theorem)**.** *We have the following equation of integrals for all $\tau_1, \tau_2 \in \mathcal{H}_p$ and $x_1, x_2 \in \mathcal{H}_q^{un}$*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{x_1}^{x_2} \omega_f = \int_{x_1}^{x_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}}.$$

Proof. Using the definition of the double integral on the right-hand side and Proposition 4.16, we have the following equation

$$\int_{x_1}^{x_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}} = \prod_{\epsilon: v_1 \to v_2} \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \frac{z - \tau_2}{z - \tau_1} d\mu_{f_\epsilon^{\#}}(z).$$

From the definition of the other double integral, to prove the statement, it is enough to show that

$$d\mu_f\{x_1 \to x_2\} = \sum_{\epsilon: v_1 \to v_2} d\mu_{f_\epsilon^\#}.$$

Using the definition of the double harmonic cocycle, this is essentially reduced to proving

$$\kappa_{f_e}(\epsilon) = \kappa_{f_\epsilon^\#}(e),$$

which is given by Corollary 7.5. □

To give an analogous definition to the period J_Ψ defined in the last section, we need to introduce a semi-indefinite double integral. Using the semi-indefinite integral defined in Section 3, we give the following definition.

Definition 7.8 (semi-indefinite double integral)**.** *Given $\tau \in \mathcal{H}_p$ and $x_1, x_2 \in \mathcal{H}_q^{un}$, we define the semi-indefinite double integral, using the semi-indefinite integral defined in section 3, as*

$$\int_{x_1}^{x_2} ⨍^{\tau} \omega_{f^\#} = \prod_{\epsilon: r(x_1) \to r(x_2)} ⨍^{\tau} f_\epsilon^\#(z)dz \in \mathbb{C}_p^\times / q_T^{\mathbb{Z}}.$$

The semi-indefinite double integral attached to the cusp form f satisfies the following property, which will be useful in further proofs.

Lemma 7.9. *Given a $\tau \in \mathcal{H}_p$ and $x_1, x_2 \in \mathcal{H}_q^{un}$, we have the following two properties of the integrals associated to $f^\#$*

(i) For all edges $e \in \mathcal{E}(\mathcal{T}_q)$, if $\tau \in \mathbb{P}^1(\mathbb{Q}_p)$, then

$$⨍^{\tau} f_e^\#(z)dz \in q^{\mathbb{Z}}.$$

(ii) If $\tau \in \mathbb{P}^1(\mathbb{Q}_p)$, then the double integral of $f^\#$ satisfies

$$\int_{x_1}^{x_2} ⨍^{\tau} \omega_{f^\#} \in q^{\mathbb{Z}}.$$

Proof. We start proving the first part of the lemma. Given an arbitrary edge $e \in \mathcal{T}_q$, there exists an element $\gamma \in \Gamma$ such that

$$\gamma^{-1}e^\circ = e \text{ or } \gamma^{-1}e^\circ = \bar{e}.$$

In either of the two possible definitions, we get that

$$⨍^{\tau} f_e^\#(z)dz = \left(⨍^{\gamma\tau} f_0(z)dz\right)^{\omega_q},$$

where ω_q is the sign of the elliptic curve E at the prime q. The multiplicative integral on the right of the last equation is clearly in $q^{\mathbb{Z}}$ and therefore, we have proven the first part of the lemma. The second part follows immediately using the definition of the double integral and the first part that we have just proven. $\square$

Given $\tau \in \mathcal{H}_p$ and $x \in \mathcal{H}_q^{nr}$, we define the two-cocycle $\tilde{d}_{\tau,x} \in Z^2(\Gamma, \mathbb{C}_p^\times)$ as

$$\tilde{d}_{\tau,x}(\gamma_1,\gamma_2) := \times\!\!\!\!\!\!\int_{\tau}^{\gamma_1\tau} \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \omega_f = \int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \times\!\!\!\!\!\!\int_{\tau}^{\gamma_1\tau} \omega_{f\#}.$$

Observe that the cohomology class representing this last function in $H^2(\Gamma, \mathbb{C}_p^\times)$ does not depend on the choice of τ and x. We finish this subsection by proving the following claim of the image of $\tilde{d}_{\tau,x}$ in $H^2(\Gamma, \mathbb{C}_p^\times/q_T^{\mathbb{Z}})$.

Proposition 7.10. *The image of $\tilde{d}_{\tau,x}$ in $H^2(\Gamma, \mathbb{C}_p^\times/q_T^{\mathbb{Z}})$ is trivial.*

Proof. We define the one-cochain $\xi_{\tau,x} : \Gamma \to \mathbb{C}_p^\times/q^{\mathbb{Z}}$ as

$$\xi_{\tau,x}(\gamma) := \int_x^{\gamma x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#}.$$

We will directly compute that $d\xi_{\tau,x}(\gamma_1,\gamma_2)$ is equal to the image of $\tilde{d}_{\tau,x}(\gamma_1,\gamma_2)$ in $H^2(\Gamma, \mathbb{C}_p^\times/q^{\mathbb{Z}})$ modulo $q^{\mathbb{Z}}$. Using the properties of the double integrals, we can find the following equation

$$\begin{aligned}
d\xi_{\tau,x}(\gamma_1,\gamma_2) &= \int_x^{\gamma_1 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} + \int_x^{\gamma_2 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} - \int_x^{\gamma_1\gamma_2 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} \\
&= \int_{\gamma_1\gamma_2 x}^{\gamma_1 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} + \int_x^{\gamma_2 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} \\
&= \int_{\gamma_1\gamma_2 x}^{\gamma_1 x} \times\!\!\!\!\!\!\int^{\tau} \omega_{f\#} + \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \times\!\!\!\!\!\!\int_{\gamma_1\infty}^{\gamma_1\tau} \omega_{f\#} \\
&= \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \times\!\!\!\!\!\!\int_{\tau}^{\gamma_1 x} \omega_{f\#} - \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \times\!\!\!\!\!\!\int^{\gamma_1\infty} \omega_{f\#} \\
&= \tilde{d}_{\tau,x}(\gamma_1,\gamma_2) - \int_{\gamma_1 x}^{\gamma_2\gamma_1 x} \times\!\!\!\!\!\!\int^{\gamma_1\infty} \omega_{f\#}
\end{aligned}$$

Since $\gamma_1\infty \in \mathbb{P}^1(\mathbb{Q}_p)$, from the second part of Lemma 7.9, we have that the second term of the last equation is in $q^{\mathbb{Z}}$ and consequently the proposition is proven. $\square$

7.2 Periods and Heegner points

To underpin the main idea of the methods that we are going to use to construct global points over the elliptic curve E using periods of $\mathcal{H}_p \times \mathcal{H}_q$, we are going to describe this process following the example of Bertolini, Darmon, and Green.

As we have mentioned before, we assume that the conductor of the elliptic curve is of the form $N = pqN^-N^+$ and that it satisfies the following generalized Heegner Hypothesis for a fixed $K/\mathbb{Q}$ quadratic imaginary field

(i) all primes dividing N^- are inert in K,

(ii) all primes dividing N^+ are split in K,

(iii) p is inert in K,

(iv) q is split in K

As we have done at the beginning of this section, we define B as the quaternion algebra over $\mathbb{Q}$ of discriminant N^-, and R an Eichler $\mathbb{Z}[1/pq]$-order of level N^+. We choose a $\mathcal{O}$ a $\mathbb{Z}[1/pq]$-order of K of conductor prime to N. The Heegner Hypothesis imposed above implies that there exists an optimal oriented embedding (with respect to $\mathcal{O}$, R, and the Eichler orientations)

$$\Psi : \mathcal{O} \to B.$$

We define the local algebras $K_p = K \otimes \mathbb{Q}_p$ and $K_q = K \otimes \mathbb{Q}_q$. The group $K_p^\times$ acting on the upper half-plane $\mathcal{H}_p$ via the embedding $\iota\Psi$ has a unique fixed point τ_Ψ normalized so it satisfies

$$\iota\Psi(\alpha)\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix} = \alpha\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix}, \text{ for all } \alpha \in K_p^\times.$$

Moreover, we define $\gamma_\Psi \in \Gamma$ as the image of a generator modulo torsion of the group $\mathcal{O}_1^\times$ (units of norm 1).

We fix an element $x \in \mathcal{H}_q^{nr}$. Using the cocycle $\xi_{\tau,x}$ defined in the last subsection, we define the multiplicative period integral associated to the embedding Ψ as

$$J_\Psi := \xi_{\tau_\Psi,x}(\gamma_\Psi) = \int_x^{\gamma_\Psi x} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f\#} \in \mathbb{C}_p^\times \Big/ q_T^{\mathbb{Z}}.$$

Similarly to the last section, we define the signs ω_p and ω_q of the elliptic curve E with respect to the primes p and q (they are 1 if E has split multiplicative reduction on the respective prime and -1 otherwise). We denote as $\mathcal{O}_0$ as the maximal $\mathbb{Z}[1/p]$-order contained in $\mathcal{O}$, we fic a prime $\mathfrak{q}$ over q and we denote as h the order of the subgroup generated by such a prime in $\mathrm{Pic}(\mathcal{O}_0)$.

Lemma 7.11. *If ω_q is equal to -1 and h is odd, then K_Ψ is equal to ± 1.*

Proof. We take an element $v \in \mathcal{O}_0$ of norm uq^h with u a unit of the ring of units of K. Let δ_Ψ the image of v through $\iota\Psi$. Simple computations give us

$$\int_{\delta_\Psi x}^{\delta_\Psi \gamma_\Psi x} \times\!\!\!\!\!\!\int^{\delta_\Psi \tau_\Psi} \omega_{f\#} = \left(\int_x^{\gamma_\Psi x} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f\#}\right)^{\omega_q^h}$$

and

$$\int_{\delta_\Psi x}^{\delta_\Psi \gamma_\Psi x} \times\!\!\!\!\!\!\int^{\delta_\Psi \tau_\Psi} \omega_{f^\#} = \int_{\delta_\Psi x}^{\gamma_\Psi \delta_\Psi x} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f^\#}.$$

The lemma follows from these two equations and the fact that the definition of J_Ψ does not depend on the choice of x. □

Let H_0 be the ring class field associated to the order $\mathcal{O}_0$ and $\sigma_\mathfrak{q}$ the eleemnt of the Galois group $\mathrm{Gal}(H_0/K)$ corresponding to $\mathfrak{q}$. We define H as the subgroup of H_0 which is fixed by $\sigma_\mathfrak{q}^2$. Recall that in the last section, we defined $\sigma_\mathfrak{p}$ as the Frobenius element of $\mathrm{Gal}(H/\mathbb{Q})$ of a fixed prime $\mathfrak{p}$ over p.

Theorem 7.12. *The point $\eta_p(J_\Psi)$ is a global point in $E(H)$ on which the involution $\sigma_\mathfrak{q}$ acts via ω_q. In concrete, $\eta_p(I_\Psi)$ is $\eta_p(J_\Psi) - \omega_p\sigma_\mathfrak{p}\eta_p(J_\Psi)$. Moreover, for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$ the priods satisfy*

$$\eta_p(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

Proof. Using the definition of the double integral, we have the following expression of the period J_Ψ

$$J_\Psi = \prod_{\epsilon: v \to \gamma_\Psi v} \times\!\!\!\!\!\!\int^{\tau_\Psi} f_\epsilon^\#(z)dz.$$

Let $R_0 \subset R$ be an Eichelr $\mathbb{Z}[1/p]$-order of level N^+q in B. We denote as

$$\Psi_0^i, i \in \{1, ..., h\}$$

a set of representatives for the conjugacy classes of $\mathrm{Emb}(\mathcal{O}_0, R_0)$ which give rise, by extension of scalars, to the conjugation class of Ψ. We denote $\tau_{\Psi_0^i} \in K_p$ the normalized fixed point associated to each embedding. We also define f_0 as the rigid analytic modular form associated to $f^\#$.
We will distinguish two cases depending on the parity of h. If h is even we denote as $v_0, .., v_h$ the consecutive vertices joining v and $v_h = \gamma_\Psi v$ and $\{e_1, ..., e_h\}$ the oriented edges joining this vertices. Using the definition of the double integral, we have

$$J_\Psi = \prod_{j=1}^{h} \times\!\!\!\!\!\!\int^{\tau_{\Psi_0}} f_{e_j}^\#(z)dz.$$

Let ϖ be the uniformizor associated to the prime $\mathfrak{q}$ i.e. $\mathfrak{q} \otimes \mathbb{Z}_q = \varpi(\mathcal{O}_0 \otimes \mathbb{Z}_q)$. Observe that we have the following property

$$\Psi(\varpi)(v_j) = v_{j+1}, j \in \{0, ..., h-1\}.$$

Let R_v be the subring of $M_2(\mathbb{Q}_q)$ of elements that fix v_0 and v_h. By strong approximation, there exists an element $\gamma \in R_v^\times$ whose image in $PGL_2(\mathbb{Q}_p)/R_v^\times$ is $\Psi(\varpi)$ such that

$$\gamma(e_j) = e_{j+1}, j \in \{1, ..., h-1\}.$$

Using the properties of the element γ, we can produce the following computations

$$\oint^{\tau_{\Psi_0}} f^{\#}_{e_j}(z)dz = \oint^{\tau_{\Psi_0}} f^{\#}_{\gamma^j e_0}(z)dz = \left(\oint^{\gamma^{-j}\tau_{\Psi_0}} f^{\#}_{e_o}(z)dz\right)^{\omega_q^j} = \left(\oint^{\tau_{\Psi_0^j}} f^{\#}_{0}(z)dz\right)^{\omega_q^j}.$$

When we insert this last equation into the expression of J_Ψ, we get

$$J_\Psi = \prod_{i=1}^{h}\left(\oint^{\tau_{\Psi_0^i}} f_0(z)dz\right)^{\omega_q^i}.$$

If h is odd the path $v \to \gamma_\Psi v$ will have $2h$ edges, where two edges will be represented by the same embedding. Using a similar method to the one used in the last case proves the following equation

$$J_\Psi = \prod_{i=1}^{h}\left(\oint^{\tau_{\Psi_0^i}} f_0(z)dz\right)^{2}.$$

However, the factors appearing in both expressions are the periods that we defined through integration over the upper half-plane in section 3, therefore, the statement is true because of the theorem 4.23 in that section. □

8 Heegner points and periods on finite products of upper half-planes

After studying the relation between periods over the products of two upper half-planes and Heegner points, we want to generalize this relation to finite products of upper half-planes. We will start studying the Archimedean case, in concrete, we will start studying such a relation for CM points over the modular curve (the simplest case possible).
Let $E/\mathbb{Q}$ be an elliptic curve of conductor $N = p_1 \cdots p_n M$ with all the factors prime pairwise and all the numbers $p_1, ..., p_n$ prime. Let B be an indefinite quaternion algebra split at infinity and at all the primes $p_1, ..., p_n$ i.e. there exists an embedding

$$\iota : B \to M_2(\mathbb{Q}) \times M_2(\mathbb{Q}_{p_1}) \times \cdots \times M_2(\mathbb{Q}_{p_n}).$$

Similarly to the single product case, we fix the standard Eichler order inside B of level M i.e. the order that we will consider in this setting is

$$R = M_0(M) \otimes \mathbb{Z}[1/p_1] \otimes \cdots \otimes \mathbb{Z}[1/p_n].$$

Using the embedding associated to the quaternion algebra, we can define the congruence subgroup associated to the collection of units in R which have norm 1

$$\Gamma := \iota(R_1^\times) = \left\{\gamma \in \mathrm{SL}_2(\mathbb{Z}[(p_1\cdots p_n)^{-1}]) \ : \ \gamma = \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \mod M\right\}.$$

As we have considered in the other cases, this congruent subroup induces an action over $\mathcal{H}$ (on the projection to the group $M_0(M)$) and ever Bruhat-tits tree $\mathcal{T}_{p_j}$ for all the elements $j \in \{1, ..., n\}$.

8.1 Cusp forms over $\mathcal{H} \times \mathcal{H}_{p_1} \times ... \times \mathcal{H}_{p_n}$

Before moving to the explicit definition of the cycles we are going to consider in this section, it is necessary to extend the definition of cusp forms for an arbitrary product of upper half-planes, where the first upper half-plane is Archimedean.

Definition 8.1. *We say that a function*

$$f : \mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_n}) \to \mathbb{C}$$

is a cusp form of weight 2 on $(\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_n}))/\Gamma$ *if it satisfies*

(i) For all matrices $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma$ *the function* f *satisfies*

$$f(\gamma z, \gamma e_1, ..., \gamma e_n) = (cz + d)^2 f(z, e_1, ..., e_n).$$

(ii) Given any collection of edges $(e_1, ..., e_n) \in \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \mathcal{E}(\mathcal{T}_{p_n})$*, the function* $f_{(e_1,...,e_n)} := f(-, e_1, ..., e_n)$ *is a holomorphic function that satisfies for every coordinate* j*, we have that* $f_{(e_1,...,\overline{e_j},...,e_n)} = -f_{(e_1,...,e_j,...,e_n)}$ *and*

$$\sum_{s(e_j)=v} f_{(e_1,...,e_j,...,e_n)} = 0.$$

(iii) Given any collection of edges $(e_1, ..., e_n) \in \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \mathcal{E}(\mathcal{T}_{p_n})$ *the function* $f_{(e_1,...,e_n)}$ *is a cusp form of weight 2 in* $\Gamma_{e_1} \cap \cdots \cap \Gamma_{e_n}$*, where all the* Γ_{e_j} *are the stabilizer of* e_j *in* Γ.

Following the same notation as the last chapters, the collection of cusp forms of weight 2 over $(\mathcal{H} \times \mathcal{H}_{p_1} \times \cdots \times \mathcal{H}_{p_n})/\Gamma$ *will be denoted as* $S_2((\mathcal{H} \times \mathcal{H}_{p_1} \times \cdots \times \mathcal{H}_{p_n})/\Gamma)$.

One should note that for all $n > 1$, the conditions imposed on the cusp forms over $(\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_n}))/\Gamma$, we have that for all $f \in S_2((\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_n}))/\Gamma)$ and $e \in \mathcal{E}(\mathcal{T}_n)$, we have

$$f(-, e) \in S_2((\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_{n-1}}))/\Gamma_e)$$

where Γ_e is the stabilizer of e in Γ. The following proposition will relate the general cusp forms we have just defined with the common cusp forms of weight two over $\mathcal{H}/\Gamma_0(N)$.

Proposition 8.2. *The correspondence* $f \mapsto f_{e_1^\circ,...,e_n^\circ} := f(-, e_1^\circ, ..., e_n^\circ)$ *induces an isomorphism*

$$S_2((\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_{n-1}})/\Gamma) \xrightarrow{\sim} \bigcap_{j=1,...,n} S_2^{p_j-new}(\mathcal{H}/\Gamma_0(N)).$$

Proof. We start introducing a notation for the different $\mathbb{C}$-vector spaces

$$X_j := S_2\left((\mathcal{H} \times \mathcal{E}(\mathcal{T}_{p_1}) \times \cdots \times \mathcal{E}(\mathcal{T}_{p_j})\Big/\bigcap\nolimits_{j<r\leq n} \Gamma_{e_r}\right) \text{ for } 0 \leq j \leq n.$$

The correspondence $\varphi : X_n \to X_0$ defined by fixing $e_1^\circ, ..., e_n^\circ$ can be decomposed as

$$\varphi : X_n \to X_{n-1} \to \cdots \to X_1 \to X_0$$

by fixing the different e_j° in each step. The methods introduced in Proposition 5.2 imply that all these maps are well-defined and injective, and, consequently, φ is injective. On a similar way, given an $f \in S_2(\mathcal{H}/\Gamma_0(N))$ which is new for all the primes $p_1, ..., p_n$ we can construct recursively functions in X_j which are the image of the previous ones under the maps $X_j \to X_{j-1}$, which proves the inclusion

$$\bigcap_{j=1,...,n} S_2^{p_j-new}(\mathcal{H}/\Gamma_0(N)) \subseteq \varphi(X_n).$$

The equation of this last inclusion follows from the relation between the p_j-new properties and the condition of the Harmonic condition in X_j specified in Proposition 5.2. □

The isomorphism given by this last proposition induces an action of the Hecke algebra from the action in all the $S^{p_j-new}(\mathcal{H}/\Gamma_0(N))$. Following the methods described by Darmon in [Dar01, p. 18], one can give an explicit expression for the action of T_ℓ, for a prime ℓ coprime to N, as

$$(T_\ell)(z, e_1, ..., e_n)dz = \sum_j f(\gamma_j^{-1}z, \gamma_j^{-1}e_1, ..., \gamma_j^{-1}e_n)d(\gamma_j^{-1}z),$$

where the collection of matrices $\{\gamma_j\}_j$ gives us a description of a disjoint union of left cosets such that

$$\Gamma \begin{pmatrix} \ell & 0 \\ 0 & 1 \end{pmatrix} \Gamma = \bigcup_j \gamma_j \Gamma.$$

Moreover, the isomorphism allows us to identify a cusp form $f \in S_2((\mathcal{H} \times \mathcal{H}_{p_1} \times \cdots \times \mathcal{H}_{p_n})/\Gamma)$ which is related to the cusp form f_E associated to the elliptic curve E given by the modularity theorem.

In light of the definition we have given over the product of two upper half planes, we can generalize such construction for $\mathcal{H} \times \mathcal{H}_{p_1} \times \cdots \times \mathcal{H}_{p_n}$ for elements $\tau_{j,1}, \tau_{j,2} \in \mathcal{H}_{p_j}^{nr}$ and $x, y \in \mathcal{H}^*$ as

$$\int_{\tau_{n,1}}^{\tau_{n,2}} \overset{n}{\cdots} \int_{\tau_{1,1}}^{\tau_{1,2}} \int_x^y \omega_f := \sum_{\epsilon: r(\tau_{n,1}) \to r(\tau_{n,2})} \int_{\tau_{n-1,1}}^{\tau_{n-1,2}} \overset{n-1}{\cdots} \int_{\tau_{1,1}}^{\tau_{1,2}} \int_x^y \omega_f$$

One should note that this definition is recursive and, in fact, most of the generalizations we are going to specify will consist of reducing the n-integration case to the one we have studied before. It follows from the definition that the n-integrals of the function f will be Γ-invariant i.e. for all $\gamma \in \Gamma$

$$\int_{\gamma\tau_{n,1}}^{\gamma\tau_{n,2}} \overset{n}{\cdots} \int_{\gamma\tau_{1,1}}^{\gamma\tau_{1,2}} \int_{\gamma x}^{\gamma y} \omega_f = \int_{\tau_{n,1}}^{\tau_{n,2}} \overset{n}{\cdots} \int_{\tau_{1,1}}^{\tau_{1,2}} \int_x^y \omega_f$$

If we fix an element $\tau \in \mathcal{H}$ and a vector $\vec{v} \in \mathcal{T}_{q_1} \times \cdots \times \mathcal{T}_{q_n}$, we can define the n-cocycle $\tilde{d}_{\tau,\vec{v}} : \Gamma^{n+1} \to \mathbb{C}$ as

$$\tilde{d}_{\tau,\vec{v}}(\gamma_1, ..., \gamma_n, \gamma_{n+1}) = \int_{\gamma_1\cdots\gamma_n v_n}^{\gamma_1\cdots\gamma_n\gamma_{n+1}v_n} \overset{n}{\cdots} \int_{\gamma_1 v_1}^{\gamma_1\gamma_2 v_1} \int_{\tau}^{\gamma_1\tau} \omega_f.$$

In order to prove that this cocycle is indeed trivial in its associated cohomology group, we shall check the following property of the n integrals we have defined in this section.

Lemma 8.3. *Given $x, y \in \mathbb{P}^1(\mathbb{Q})$ and a collection*

$$\tau_{t,j} \in \mathcal{H}_{q_t}^{nr} \text{ with } t \in \{1, ..., n\} \text{ and } j \in \{1, 2\},$$

we have that the associated integral satisfies

$$\int_{\tau_{n,1}}^{\tau_{n,2}} \overset{n}{\cdots} \int_{\tau_{1,1}}^{\tau_{1,2}} \int_x^y \omega_f \in \Lambda.$$

Proof. The proof of this lemma follows from the definition of the integral and the first part of Lemma 6.2. □

Similarly to the case where $n = 1$, we are able to prove that the n-cocycle is trivial in its cohomology group.

Proposition 8.4. *The class of $\tilde{d}_{\tau,\vec{v}}$ in $H^{n+1}(\Gamma, \mathbb{C}/\Lambda)$ is trivial.*

Proof. Let $\xi : \Gamma^n \to \mathbb{C}$ an n-cocycle defines as

$$\xi(\gamma_1, ..., \gamma_n) = \int_{x_n}^{\gamma_n x_n} \overset{n}{\cdots} \int_{v_1}^{\gamma_1 v_1} \int_{\infty}^{\tau} \omega_f.$$

The proof follows from showing that the coboundary map of ξ is equal to $\tilde{d}_{\tau,\vec{v}}$ modulo Λ, following a similar computation to the one we have done for $n = 1$ and the previous lemma. □

In order to show that the definition of ξ does not depend on the choice of τ and $\vec{v}$, we should show that the group of n-cocycles has finite exponent.

Lemma 8.5. *The group of n-cocycles on Γ has finite exponent.*

Proof. Similarly to the case shown by Bertolini-Darmon-Green, we have that Γ acts trivially on $\mathbb{C}/\Lambda$ and, consequently the group of n-cocycles is

$$\mathrm{Hom}(\Gamma^n, \mathbb{C}/\Lambda) = \mathrm{Hom}((\Gamma^n)^{ab}, \mathbb{C}/\Lambda).$$

The rest follows from the Corollary of Theorem 3 of [Ser70]. □

8.2 Finite dimensional periods and Heegner points

In this subsection, we will construct certain periods on $\mathcal{H} \times \mathcal{H}_{q_1} \times \cdots \times \mathcal{H}_{q_n}$ and prove that they satisfy similar properties to the ones stated at the main theorem of complex multiplication.

Let $\mathcal{O} \subseteq K$ be an $\mathbb{Z}[1/q_1 \cdots q_n]$-order of discriminant prime to N. The definition of optimal oriented embeddings is easily extended to this setting with respect to $\mathcal{O}$ and R. We denote the collection of these embeddings as

$$\mathrm{Emb}(\mathcal{O}, R).$$

Given an optimal oriented embedding Φ, we consider the collection of matrices $\gamma_{\Psi,1}, ..., \gamma_{\Phi,n}$ which are the images of the generators of $\mathcal{O}_1^\times$ through Ψ. Using this notation, we define the period associated to Ψ as

$$J_\Psi := \xi_{\tau_\Psi, \vec{v}}(\gamma_{\Psi,1}, ..., \gamma_{\Psi,n}) = \int_{x_n}^{\gamma_{\Psi,n} x_n} \overset{n}{\cdots} \int_{v_1}^{\gamma_{\Psi,1} v_1} \int_{\infty}^{\tau_\Psi} \omega_f \in \mathbb{C}/\Lambda.$$

One should recall that we have shown that the definition of this period does not depend on the vector of vertices $\vec{v}$ we have chosen. Let $\mathcal{O}_0 \subseteq \mathcal{O}$ the maximal $\mathbb{Z}[1/q_1 \cdots q_{n-1}]$-order. We fix a prime $\mathfrak{q}|q$ and we define h to be the order of $\mathfrak{q}$ in $\mathrm{Pic}(\mathcal{O}_0)$ if it is even and twice the order if it is odd.

Let H_0 be the ring class of $\mathcal{O}_0$ and $H := H_0^{\mathrm{Frob}_\mathfrak{q}^2}$ the $\mathfrak{q}$-narrow class field. The following Theorem generalizes the main theorem of complex multiplication to the periods we have defined.

Theorem 8.6. *All the periods associated to optimal oriented embeddings $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$ satisfy the statement of the main theorem of complex multiplication. In other words $\eta(J_\Psi) \in E(H)$ is a global point and for all $\mathfrak{a} \in \mathrm{Pic}^{q_1,\ldots,q_n+}(\mathcal{O})$,*

$$\eta(J_{\Psi^\mathfrak{a}}) = \mathrm{rec}(\mathfrak{a})^{-1} \eta(J_\Psi).$$

Proof. We will prove the statement by induction over the variable n. The case $n = 1$ is the common theorem of complex multiplication. For a general $n > 0$, let $R_0 \subseteq R$ be an Eichler $\mathbb{Z}[1/q_1, ..., q_{n-1}]$-order of level N^+q_n and $\Psi_0 \in \mathrm{Emb}(\mathcal{O}_0, R_0)$ be the optimal oriented embedding who's conjugacy classes give rise to the conjugacy classes of Ψ. By direct computation, we get that

$$J_\Psi = \sum_{j=0}^{t-1} \omega_{q_n}^j \int_{x_{n-1}}^{\gamma_{\Psi_0^{\mathfrak{q}^j}, n-1}} {}^{x_{n-1}} \overset{n-1}{\cdots} \int_{v_1}^{\gamma_{\Psi_0^{\mathfrak{q}^j}, 1}} {}^{v_1} \int_{\infty}^{\tau_{\Psi_0^{\mathfrak{q}^j}}} \omega_f.$$

Sine we have reduced to expression of the period to a product of periods for the case $n-1$, the statement follows by the induction hypothesis. □

This result is a priori surprising, since it assures us that the periods constructed over $\mathcal{H} \times \mathcal{H}_{q_1} \times \cdots \mathcal{H}_{q_n}$ will always satisfy the properties of the main statement of Complex Multiplication as long as our data (N, K) satisfies the Heegner Hypothesis.
It is easy to show that if such a hypothesis were not to be satisfied, we would not have optimal oriented embeddings in any of the possible rigid analytic spaces and, consequently, we would not be able to construct the periods. As we have pointed out before, the Heegner hypothesis should be seen as a principle that forces us to choose a concrete Shimura curve depending on our data (in this case, the data forces the modular curve). This result has a similar effect: After choosing the based Modular curve for our Heegner points, one has to choose an appropriate rigid analytic space to construct our periods on. One should notice that the rigid analytic space $\mathcal{H} \times \mathcal{H}_{q_1} \times \cdots \mathcal{H}_{q_n}$ is closely related to the ring associated to the order $\mathcal{O}$ we consider, $\mathbb{Z}[1/q_1 \cdots q_n]$, and that the number of primes is equal to the $\mathbb{Z}$-rank of $\mathcal{O}^\times$.
Now we are going to generalize this result to non-Archimedean settings, and we will see that the value $rk_{\mathbb{Z}}(\mathcal{O}^\times)$ will give us useful information about the values of the periods in the cases where the underlying Shimura curve is not the modular curve.

8.3 Cusp forms over $\mathcal{T}_p \times \mathcal{T}_{q_1} \times ... \times \mathcal{H}_{q_n}$

For this setting, we are going to consider a decomposition of the level N of the form

$$N = pq_1 \cdots q_n N^+ N^-,$$

where $p, q_1, ..., q_n$ are prime and all the factors are prime pairwise. We also consider a quaternion algebra $B/\mathbb{Q}$ of discriminant N^- and split at all primes $p, q_1, ..., q_n$ i.e. there exists an embedding

$$\iota : B \to M_2(\mathbb{Q}_p) \times M_2(\mathbb{Q}_{q_1}) \times \cdots \times M_2(\mathbb{Q}_{q_n}).$$

We also consider $R \subseteq B$ an Eichler $\mathbb{Z}[/pq_1 \cdots q_n]$-order of level N^+ and we define the congruence subgroup $\Gamma := \iota(R_1^\times)$. We follow a similar notation to before, and we will denote the components of the elements of Γ as $(\gamma_p, \gamma_{q_1}, ..., \gamma_{q_n})$. After fixing this notation, we are in a position to define cusp forms over $\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n}$.

Definition 8.7. *We define a cusp form of weight 2 on* $(\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma$ *as a function*

$$f : \mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n} \to \mathbb{C}_{q_n}$$

satisfying

(i) For all $\gamma \in \Gamma$ *such that* $\gamma_{q_n} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, *we have that* $f(\gamma_p e_p, \gamma_{q_1} e_{q_1}, ..., \gamma_{q_n} z) = (cz + d)^2 f(e_p, e_{q_1}, ..., z)$.

(ii) Given $z \in \mathcal{H}_q$ *and* $e_{q_j} \in \mathcal{E}(\mathcal{T}_{q_j})$ *for* $j \in \{1, .., n-1\}$, *we have that the restriction* $f_{(e_{q_1},...,z)} := f(-, e_{q_1}, ..., z)$ *is an Harmonic cocycle.*

(iii) For each $e \in \mathcal{E}(\mathcal{T}_p)$, *the restriction* $f_e := f(e, -)$ *is a cusp form of weight 2 on* $\mathcal{T}_{q_1} \times \cdots \times \mathcal{T}_{q_n} \times \mathcal{H}_q$ *with respect to the stabilizer* Γ_e *of* e *in* Γ.

The collection of cusp forms of weight 2 on $(\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma$ *will be denoted as* $S_2((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma)$.

Recursively, one can find that the space of the cusp forms we have defined is isomorphic to a subgroup of a similar rigid analytic space with $n-1$ primes.

Lemma 8.8. *The assignment that restricts every cusp form to* $e_0 \in \mathcal{E}(\mathcal{T}_{q_1})$ *induces an isomorphism*

$$S_2((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma) \cong S_2^{q_n-new}((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_{n-1}})/\Gamma'),$$

where Γ' *is the Eichler* $\mathbb{Z}[1/pq_1 \cdots q_{n-1}]$*-order of level* $N^+ q_n$.

Proof. The proof follows from similar computations to the ones we have explicitly written before. □

Using this lemma recursively we can consider a cusp form $f \in S_2((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma)$ which is related to the p-adic cusp form we have defined in previous sections associated to the elliptic curve E by the theorey of p-adic uniformization, Jacquet-Langlands and the modularity theorem.

We define the Harmonic cocycle associated to f also recursively. Given $(v_{j,1}, v_{j,2}) \in (\mathcal{V}(\mathcal{T}_{q_j}))^2$ pair of vertices for all $j \in \{1, ..., n\}$ and $e \in \mathcal{E}(\mathcal{T}_p)$, we define

$$\kappa_f\{v_{1,1} \to v_{1,2}\} \cdots \{v_{n,1} \to v_{n,2}\}(e) := \sum_{\epsilon : v_{n,1} \to v_{n,2}} \kappa_{f_e}\{v_{1,1} \to v_{1,2}\} \cdots \{v_{n-1,1} \to v_{n-1,2}\}(\epsilon).$$

We denote the measure of $\mathbb{P}^1(\mathbb{Q}_p)$ associated to the Harmonic cocycle of the type we have just defined as $\mu_f\{v_{1,1} \to v_{1,2}\} \cdots \{v_{n,1} \to v_{n,2}\}$ and we define the following integral.

Definition 8.9. *Given $(v_{j,1}, v_{j,2}) \in (\mathcal{V}(\mathcal{T}_{q_j}))^2$ pair of vertices for all $j \in \{1, ..., n\}$ and $\tau_1, \tau_2 \in \mathcal{H}_p$, we define the $n+1$-integral as*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_{n,1}}^{v_{n,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \omega_f := \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \left(\frac{t-\tau_2}{t-\tau_1} \right) \mu_f\{v_{1,1} \to v_{1,2}\} \cdots \{v_{n,1} \to v_{n,2}\}(t).$$

Given an arbitrary cusp form g, we consider the associated Γ-invariant harmonic cocycle λ_g, defined by the assignment

$$(e_1, ..., e_n, e) \mapsto \kappa_{f_{(e_1,...,e_n)}}(e).$$

The following proposition gives us a similar assignment to the one that Bertolini-Darmon-Green proved for the case $n = 1$.

Proposition 8.10. *There exists a cusp form $f^{\#} \in S_2((\mathcal{H}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{T}_{q_n})/\Gamma)$ such that $\lambda_f = \lambda_{f^{\#}}$.*

Proof. The proof is given by proving an Hecke-equivariant isomorphicm between $_2((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{H}_{q_n})/\Gamma)$ and $S_2((\mathcal{T}_p \times \mathcal{T}_{q_1} \times \cdots \times \mathcal{T}_{q_n})/\Gamma)$ and then use the fact that λ_f takes vlaues in $\mathbb{Z}$. □

The following Theorem relates the integral we have defined with an a new type of $n+1$-integral defined for the cusp form $f^{\#}$.

Theorem 8.11 (Fubini). *We have the following equality*

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_{n,1}}^{v_{n,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \omega_f = \int_{v_{n,1}}^{v_{n,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}}$$

where the right side of the equation is defined as

$$\int_{v_{n,1}}^{v_{n,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}} := \prod_{\epsilon: v_1 \to v_2} \int_{v_{n-1,1}}^{v_{n-1,2}} \overset{n-1}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_{f^{\#}_{\epsilon}}$$

Proof. The proof is given by explicit computations similar to the case $n = 1$. □

The result of this theorem allows us to extend the definition of semidefinite integral (in a recursive way) to the setting we are studying.

Definition 8.12. *Let $\tau \in \mathcal{H}_p$ and $(v_{j,1}, v_{j,2}) \in (\mathcal{V}(\mathcal{T}_{q_j}))^2$ pair of vertices for all $j \in \{1, ..., n\}$. We define the semi-idefinite $n+1$-integral as*

$$\int_{v_{n,1}}^{v_{n,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \times\!\!\!\!\!\!\int^{\tau} \omega_{f^{\#}} := \prod_{\epsilon: v_1 \to v_2} \int_{v_{n-1,1}}^{v_{n-1,2}} \overset{n}{\cdots} \int_{v_{1,1}}^{v_{1,2}} \times\!\!\!\!\!\!\int^{\tau} \omega_{f^{\#}_{\epsilon}},$$

where $f^{\#}_{\epsilon}$ is the restriction of $f^{\#}$ to ϵ in its first coordinate.

Given a $\tau \in \mathcal{H}_p$ and a vector of vectices $\vec{v} \in \mathcal{V}(\mathcal{T}_{q_1}) \times \cdots \times \mathcal{V}(\mathcal{T}_{q_n})$ (we denote its components by the prime subindexes), we define the following $N+1$-cocycle

$$\tilde{d}_{\tau,\vec{v}} := \int_{\gamma_1\cdots\gamma_n v_n}^{\gamma_1\cdots\gamma_n\gamma_{n+1}v_n} \overset{n}{\cdots} \int_{\gamma_1 v_1}^{\gamma_1\gamma_2 v_1} \times\!\!\!\!\!\!\int_{\tau}^{\gamma_1\tau} \omega_{f^\#}.$$

The following proposition will show us that the class of this cocycle in $H^{n+1}(\Gamma, \mathbb{C}_p^\times/q_T^{\mathbb{Z}})$ is trivial, and consequently, we can consider an n-cochain for the definition of our periods in the next subsection.

Proposition 8.13. *The class of $\tilde{d}_{\tau,\vec{v}}$ is trivial in $H^n(\Gamma, \mathbb{C}_p^\times/q_T^{\mathbb{Z}})$.*

Proof. We consider the n-cochain $\xi_{\tau,\vec{v}} : \Gamma^n \to \mathbb{C}_p^\times/q_T^{\mathbb{Z}}$ defined as

$$\xi_{\tau,\vec{v}}(\gamma_1, ..., \gamma_n) := \int_{x_n}^{\gamma_{\Psi,n}x_n} \overset{n}{\cdots} \int_{v_1}^{\gamma_{\Psi,1}v_1} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f^\#}$$

By similar computations to the ones given by Bertolini-Darmon-Green one can prove that $\tilde{d}_{\times,\vec{v}}$ and $\xi_{\tau,\vec{v}}$ are equal modulo $q_T^{\mathbb{Z}}$. □

8.4 Generalization of the Heegner Hypothesis

We will finish this document showing that similar periods to the ones we have defined on the Archimedean construction exist for non-Archimedean settings and proving a generalization of the Heegner hypothesis for the general settings we have studied throughout this thesis.

Let $\mathcal{O} \subseteq K$ be a $\mathbb{Z}[1/pq_1\cdots q_n]$-order of discriminant prime to N. One can easily extend the definition of optimal oriented embeddings with respect to $\mathcal{O}$ and R; we will denote this collection as

$$\mathrm{Emb}(\mathcal{O}, R).$$

Given an optimal oriented embedding Ψ, we consider the collection of matrices $\gamma_{\Psi,1}, ..., \gamma_{\Psi,n}$ which are the images of the different generators of $\mathcal{O}_1^\times$ through the map $\iota\Psi$. Using this, we can define the period associated to an optimal oriented embedding Ψ as

$$J_\Psi = \xi_{\tau_\Psi,\vec{v}}(\gamma_{\Psi,1}, ..., \gamma_{\Psi,n}) = \int_{x_n}^{\gamma_{\Psi,n}x_n} \overset{n}{\cdots} \int_{v_1}^{\gamma_{\Psi,1}v_1} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f^\#} \in \mathbb{C}_p^\times/q_T^{\mathbb{Z}}.$$

As mentioned before, the definition of the period does not depend on the choice of the collection of vertices $\vec{v}$. Before continuing, we will impose the following conditions on the level N

- all primes dividing $q_1\cdots q_n N^+$ are split in K,
- all primes dividing pN^- are inert in K.

Fix a prime $\mathfrak{q}_n | q_n$ in K and $\mathcal{O}_0 \subseteq \mathcal{O}$ the maximal $\mathbb{Z}[1/pq_1 \cdots q_{n-1}]$-order. Consider the sign ω_{q_n} of the elliptic curve E with respect to the prime q_n and let h be the order of the prime in $\mathrm{Pic}(\mathcal{O}_0)$. The following lemma assures us that the periods J_Ψ will be trivial under a concrete assumption of h and ω_{q_n}.

Lemma 8.14. *If h is odd and $\omega_{q_n} = -1$ then $J_\Psi = \pm 1$.*

Proof. Similarly to the two-dimensional case, the proof is given by using the properties of the integral and reaching an equation that only ± 1 can satisfy. □

The following theorem will generalize the theorem of complex multiplication to the periods we have just defined under the conditions we have assumed. Let H_0 be the ring class field of the order $\mathcal{O}_0$ and $\sigma_{\mathfrak{q}_n}$ be the associated element in $\mathrm{Gal}(H_0/K)$ to the prime $\mathfrak{q}_n$. We consider the subfield $H \subseteq H_0$ fixed by the action of the element $\sigma_{\mathfrak{q}_n}^2$.

Theorem 8.15. *Given any $\Phi \in \mathrm{Emb}(\mathcal{O}, R)$, the value $\eta_p(J_\Psi)$ is a global point in $E(H)$ satisfying that for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$,*

$$\eta_p(J_{\Phi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1} \eta_p(J_\Phi).$$

Proof. We will prove the theorem by induction over the set of primes $\{q_1, ..., q_n\}$. The case $n = 0$ is the main theorem of complex multiplication. If we consider an arbitrary $n > 0$ and an optimal oriented embedding $\Phi \in \mathrm{Emb}(\mathcal{O}, R)$, by definition we have that

$$J_\Psi = \prod_{j=1}^{h} \int_{x_n}^{\gamma_n x_n} \overset{n-1}{\cdots} \int_{v_1}^{\gamma_1 v_1} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f_\epsilon^\#}.$$

Consider the Eichler $\mathbb{Z}[1/pq_1 \cdots q_{n-1}]$-order $R_0 \subseteq R$ of level $N^+ q_n$ and a collection

$$\Psi_0^j, \quad j \in \{1, ..., h\}$$

of representatives of the conjugacy classes of $\mathrm{Emb}(\mathcal{O}_0, R_0)$, which gives rise to the conjugacy class of Ψ. Let $\tau_{\Psi_0^j}$ the fixed points associated to the optimal oriented embeddings Ψ_0^j and f_0 the cusp form we obtain by restricting f to $e_0 \in \mathcal{E}(\mathcal{T}_{q_n})$. By explicit computations, one gets that in the case when h is even, the period has the following expression

$$J_\Psi = \prod_{j=0}^{h} \left(\int_{x_{n-1}}^{\gamma_{n-1} x_{n-1}} \overset{n-1}{\cdots} \int_{v_1}^{\gamma_1 v_1} \times\!\!\!\!\!\!\int^{\tau_{\Phi_0^j}} \omega_f \right)^{\omega_{q_n}^j}.$$

On the other hand, if h is odd and $\omega_{q_n} = 1$ (avoiding the trivial case), we have that

$$J_\Psi = \prod_{j=0}^{h} \left(\int_{x_{n-1}}^{\gamma_{n-1} x_{n-1}} \overset{n-1}{\cdots} \int_{v_1}^{\gamma_1 v_1} \times\!\!\!\!\!\!\int^{\tau_{\Phi_0^j}} \omega_f \right)^2.$$

In both cases, we have that the period reduces to a product of periods of the case $n-1$, which by induction satisfies the statement of the theorem. □

This Theorem is a generalization of the main complex multiplication theorem for a very concrete setting. The following theorem explicitly tells us under which conditions we can construct the periods J_Ψ and in which instances we can assure their triviality.

Theorem 8.16 (Generalization of Heegner Hypothesis). *Let $q_1, ...q_n$ be a non-repetitive collection of primes strictly dividing N which are not p and consider the ring $T := \mathbb{Z}[1/pq_1 \cdots q_n]$. Given an T-order $\mathcal{O}$, the values $\eta(J_\Psi)$ associated to the period J_Ψ constructed on the rigid analytic space*

$$\mathcal{H}_p \times \mathcal{H}_{q_1} \times \cdots \times \mathcal{H}_{q_n}$$

as described in this section, for any optimal oriented embedding $\Phi \in \mathrm{Emb}(\mathcal{O}, R)$, it satisfies the main theorem of complex multiplication if and only if K satisfies the Heegner Hypothesis. In other words, the underlying Shimura curve X_{N^+,pN^-} satisfies

(i) all primes dividing N^+ are split in K,

(ii) all primes dividing pN^- are inert in K.

Furthermore, the values $\eta(J_\Psi)$ will be trivial in all the instances were the strict inequality

$$rk_{\mathbb{Z}}(\mathcal{O}^\times) < n$$

is satisfied i.e. at least one of the primes $q_1, ..., q_n$ is inert in K.

Proof. As we have mentioned before we have that the existence of a non-trivial collection of optimal oriented embeddings $\mathrm{Emb}(\mathcal{O}, R)$ happens if and only if the Heegner Hypothesis is satisfied. If one of the primes $q_1, ..., q_n$ is inert in K there is one of the paths of the reduced Bruhat-Tits trees becomes trivial and, consequently, $J_\Psi = 1$ for all $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$, which satisfies the statement trivially. If all the primes are split, then it is the only case where $rk_{\mathbb{Z}}(\mathcal{O}^\times) = n$ and the proof is given by the previous theorem. □

As mentioned before, this result, which can be relatively easily adapted to an Archimedean construction of similar Heegner points over a Shimura curve, should induce us to make two choices: First, one should choose the underlying Shimura curve depending on the data (N, K), and secondly, depending on the $\mathbb{Z}[1/pq_1 \cdots q_n]$-order $\mathcal{O}$, one should consider the periods associated to a rigid analytic space $\mathcal{H}_p \times \mathcal{H}_{q_1} \times \cdots \times \mathcal{H}_{q_n}$ J_Ψ for a given optimal oriented embedding $\Psi \in \mathrm{Emb}(\mathcal{O}, R)$. Furthermore, if one wishes to get interesting periods i.e. aim for periods that might not be trivial, one should restrict to the cases where the rank of the unit group $\mathcal{O}^\times$ is equal to the number of primes that we are considering i.e. by the generalization of the Dirichlet Unit Theorem that all the primes $q_1, ..., q_n$ are split in K.
One should notice that the restrictions to $n = 0$ and $n = 1$ are the common Heegner Hypothesis and the results of Bertolini-Darmon-Green, respectively.

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