Heegner Points and periods over $\mathcal{H}_p \times \mathcal{H}_q$

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Offen im Denken

We define the following subset of $M_2(\mathbb{Z})$

$$M_0(N) := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z}) \text{ s.th. } \begin{pmatrix} a & b \\ c & d \end{pmatrix} \equiv \begin{pmatrix} * & * \\ 0 & * \end{pmatrix} \pmod N \right\}$$

and the congruent subgroup

$$\Gamma_0(N) := M_0(N) \cap \mathrm{SL}_2(\mathbb{Z}).$$

The modular curve of degree N will be denoted as

$$X_0(N)$$

and the Shimura curve associated to the decomposition $N = N^+N^-$ as

$$X_{N^+,N^-}$$

CM points

Let $E/\mathbb{Q}$ be an elliptic curve of conductor N. We consider an order $\mathcal{O}$ of a quadratic imaginary field $K/\mathbb{Q}$.
We consider the complex parametrization given by the modularity theorem

$$\Phi : X_0(N) \to E.$$

Moduli spaces: We use the fact that the modular curve is a compactification of a moduli space $Y_0(N)$

$$X_0(N)(\mathbb{C}) \supseteq Y_0(N)(\mathbb{C}) := \{(E, E') \text{ ell. curves} \mid E \to E' \text{ N-isogeny}\}/\cong.$$

CM points are the points in $X_0(N)$ whose coordinates have the same complex multiplication

$$\mathrm{CM}(\mathcal{O}) = \{(E, E') \in X_0(N) \mod \mathrm{End}(E) \cong \mathrm{End}(E') \cong \mathcal{O}\}.$$

CM points

Riemann surface: We consider the following isomorphism of curves

$$X_0(N)(\mathbb{C}) \cong \mathcal{H}^*/\Gamma_0(N) := (\mathcal{H} \cup \mathbb{P}^1(\mathbb{Q}))/\Gamma_0(N)$$

We define the associated order to an element $\tau \in \mathcal{H}/\Gamma_0(N)$

$$\mathcal{O}_\tau := \{\gamma \in M_0(N) \text{ such that } \gamma\tau = \tau\} \cup \{0_2\}.$$

With this last concept, the definition of CM points given before is equivalent to

$$\mathrm{CM}(\mathcal{O}) = \{\tau \in \mathcal{H}/\Gamma_0(N) \mid \mathcal{O}_\tau = \mathcal{O}\}.$$

Proposition (Heegner hypothesis).

The collection of CM points attached to an order $\mathcal{O}$ is non-empty of and only if the ideal $N\mathcal{O}$ factors as a product $\mathcal{N}\overline{\mathcal{N}}$ of cyclic ideals of norm N.

We consider the following map

$$\begin{aligned} \delta : \mathrm{Pic}(\mathcal{O}) &\to X_0(N)(\mathbb{C}) \\ \mathfrak{a} &\mapsto (\mathbb{C}/\mathfrak{a}, \mathbb{C}/\mathcal{N}^{-1}\mathfrak{a}) \end{aligned}$$

Considering this maps for all the possible orientations of $\mathcal{O}$ characterizes all the CM points of $X_0(N)$ with respect to $\mathcal{O}$.
Let H be the ring class field attached to $\mathcal{O}$. We consider the reciprocity map given by Class Field Theory

$$\mathrm{rec} : \mathrm{Pic}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Gal}(H/K)$$

Theorem. (Shimura reciprocity law)

If we fin an orientation $\mathcal{N}$, and we consider a point $P \in \mathrm{CM}(\mathcal{O})$ such that $P = \delta(\mathfrak{a})$. Then $\Phi(P) \in E(H)$ and, for all $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$, we have

$$\Phi(\delta(\mathfrak{b})P) = \mathrm{rec}(\mathfrak{b})^{-1}\Phi(P).$$

Optimal oriented embeddings

Definition.

An embedding $\Psi : K \to M_2(\mathbb{Q})$ is an optimal oriented embedding if it satisfies the following conditions

(i) (Optimal) The embedding satisfies $\Psi(K) \cap M_0(N) = \Psi(\mathcal{O})$.

(ii) (Oriented) The embedding is oriented with respect to $\mathcal{N}$ if the following diagram commutes

$$\begin{array}{ccc} \mathcal{O} & \xrightarrow{\Psi} & M_0(N) \\ \downarrow & & \downarrow \\ \mathcal{O}/\mathcal{N} & \xrightarrow{\sim} & \mathbb{Z}/N\mathbb{Z} \end{array}$$

(iii) The action induced by $\Psi(K^\times)$ has a unique fixed point $\tau_\Psi \in K$ satisfying

$$\Psi(\lambda)\begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix} = \lambda \begin{pmatrix} \tau_\Psi \\ 1 \end{pmatrix}.$$

We say that the embedding Ψ is oriented at ∞ if $\tau_\Psi \in \mathcal{H}$.

We define the assignment that sends an optimal oriented embedding to the class in $\mathrm{Pic}(\mathcal{O})$ represented by the lattice $\mathbb{Z}\tau_\Psi + \mathbb{Z} \subset \mathbb{C}$ which we will denote as $\mathfrak{a}_\Psi$.

Lemma.

The assignment $\Psi \mapsto [\mathfrak{a}_\Psi]$ induces a bijection

$$\mathrm{Emb}(\mathcal{O}, M_0(N)) \longleftrightarrow \mathrm{Pic}(\mathcal{O}).$$

This helps us induce a left action

$$\mathrm{Pic}(\mathcal{O}) \circlearrowright \mathrm{Emb}(\mathcal{O}, M_0(N)) \text{ we will identified as } \Psi^{\mathfrak{a}}$$

Definition.

A cusp form of weight k for a congruence subgroup $\Gamma \subseteq \mathrm{SL}_2(\mathbb{Z})$ is an holomorphic function $f : \mathcal{H} \to \mathbb{C}$ which is satisfies the modular condition

$$f(\gamma z) = j(\gamma, z)^k f(z),$$

it is holomorphic at infinity and $f(\infty) = 0$. We denote the set of cusp forms of this type as $S_k(\Gamma)$.

We consider the cusp form associated to the elliptic curve E given by the modularity theorem

$$f(\tau) = \sum a_n q^n \text{ with } q = \exp(2\pi i \tau).$$

We define the integral period of f and the fixed point of the optimal oriented embedding

$$J_\Psi := \int_{i\infty}^{\tau_\Psi} 2\pi i f(t) dt.$$

Relation CM points and periods

We fix a Weierstraß uniformization associated to the lattice Λ_E

$$\eta : \mathbb{C}/\Lambda_E \xrightarrow{\sim} E(\mathbb{C}).$$

Theorem

For all optimal oriented embedding Ψ the point $\eta(J_\Psi)$ belongs to $E(H)$, and for all $\mathfrak{b} \in \mathrm{Pic}(\mathcal{O})$, we have

$$\eta(J_{\Psi^{\mathfrak{b}}}) = \mathrm{rec}(\mathfrak{b})^{-1}\eta(J_\Psi).$$

The proof of this theorem is by giving an explicit relation between the periods J_Ψ and the CM points in $X_0(N)$.

Objectives of my thesis

Question 1: Can we find analytic expressions of CM points over p-adics?

$$\{\text{periods in } \mathcal{H}_p\} \longleftrightarrow \{\text{CM point over Shimura curves}\}$$

Question 2: Can we find periods over $\mathcal{H}_p \times \mathcal{H}_q$ for arbitrary primes p and q which are related to CM points?
Question 3: Can we extend this results for finite products $\mathcal{H}_{p_1} \times \cdots \mathcal{H}_{p_n}$?

Heegner points

Let $E/\mathbb{Q}$ an elliptic curve of conductor $N = pN^+N^-$ with all the factors prime pairwise, N^- square-free and p prime. We consider an $\mathbb{Z}[1/p]$-order $\mathcal{O}$ of a quadratic imaginary field $K/\mathbb{Q}$ of conductor prime to N.
Let B be the quaternion algebra over $\mathbb{Q}$ of discriminant N^- and R the Eichler $\mathbb{Z}[1/p]$-order of level N^+.
Moduli spaces: Shimura curves are moduli spaces, therefore, we have the following expression

$$X_{pN^+,N^-}(\mathbb{C}_p) = \left\{ \big((A,\iota),(A',\iota')\big) \mid \exists (A,\iota) \to (A,\iota') \text{ isogeny of deg } N^+ \right\} / \cong,$$

where A, A' are abelian surfaces and ι, ι' are embeddings of the form $\mathcal{O}_B \hookrightarrow \mathrm{End}(A)$.
In this case, the CM points are the ones which have the same ring of endomorphisms

$$\mathrm{CM}(\mathcal{O}) = \left\{ (A, A') \in X_{pN^+,N^-}(\mathbb{C}_p) \mid \mathrm{End}(A) \cong \mathrm{End}(A') \cong \mathcal{O} \right\}.$$

Heegner points

Riemann Surfaces: We consider the embedding

$$\iota : B \to M_2(\mathbb{Q}_p).$$

Let $\Gamma := \iota(R_1^\times)$. The theory of p-adic uniformization shows us that

$$X_{pN^+,N^-}(\mathbb{C}_p) \cong \mathcal{H}_p/\Gamma, \text{ where } \mathcal{H}_p := \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$$

For every $\tau \in \mathcal{H}_p$ we define

$$\mathcal{O}_\tau := \{\gamma \in R \text{ such that } \iota(\gamma)\tau = \tau\} \cup \{0\}.$$

Similarly to the modular curve case, CM points can be defined as

$$\mathrm{CM}(\mathcal{O}) := \{\tau \in \mathcal{H}_p/\Gamma \mid \mathcal{O}_\tau = \mathcal{O}\}.$$

Theorem (Characterization of Heegner points)

Given an order $\mathcal{O}$, the set of Heegner points $\mathrm{CM}(\mathcal{O})$ is non-empty if and only if the following conditions are satisfied

(i) all primes dividing pN^+ are inert in K,

(ii) all primes dividing N^- are split in K.

Using the Modularity theorem and the Jacquet-Langlands correspondence, we get a complex parametrization

$$\Phi : X_{pN^+,N^-} \to E.$$

Let H the ring class field attached to $\mathcal{O}$ and the reciprocity map of class field theory

$$\mathrm{rec}: \mathrm{Pic}(\mathcal{O}) \xrightarrow{\sim} \mathrm{Gal}(H/K).$$

Similarly to the last case, if we fix orientations for $\mathcal{O}$ and R we have the following property.

Theorem

If we fix orientations, $\mathrm{Pic}(\mathcal{O})$ has an induced action in a concrete subset of $\mathrm{CM}(\mathcal{O})$. Given a CM point P on this subset, we have $\Phi(P) \in E(H)$ and, for all $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$

$$\Phi(\mathfrak{a}P) = \mathrm{rec}(\mathfrak{a})^{-1}\Phi(P).$$

We consider the Bruhat-Tits $\mathcal{T}_p$ (edges denoted as $\mathcal{E}(\mathcal{T}_p)$ and vertices $\mathcal{V}(\mathcal{T}_p)$ the vertices) and the p-adic upper half plane $\mathcal{H}_p$. We have a reduction map

$$r : \mathcal{H}_p \to \mathcal{E}(\mathcal{T}_p) \cup \mathcal{V}(\mathcal{T}_p).$$

Definition.

The cusp forms of weight 2 over are functions $f : \mathcal{H}_p \to \mathbb{C}_p$ satisfying the following conditions

- The restrictions to each affinoid subset $A_{[e]}$, for an edge $e \in \mathcal{E}(\mathcal{T}_p)$, are uniform limits of rational functions with poles outside $A_{[e]}$.
- The function f satisfies the modular equation for a given $\gamma \in \Gamma$ i.e.

$$f(\gamma z) = j(\gamma, z)^2 f(z).$$

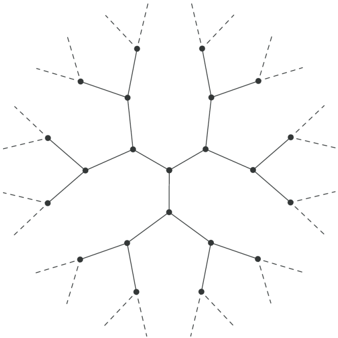
We denote the collection of cusp forms of weight 2 attached to Γ as $S_2(\Gamma)$.

Definition.

A function $\kappa : \mathcal{E}(\mathcal{T}_p)$ is an harmonic cocycle if

- $\sum_{s(e)=v} \kappa(e) = 0$ for all vertex $v \in \mathcal{V}(\mathcal{T}_p)$.
- $\kappa(\overline{e}) = -\kappa(e)$ for all edges $e \in \mathcal{E}(\mathcal{T}_p)$.

Γ induces an action over Harmonic cocycles. We denote the set of Harmoic cocycles invariant by Γ as $\mathrm{Har}(\Gamma)$.



There exists a bijection between Harmonic cocycles and cusp forms of weight 2

$$S_2(\Gamma) \longleftrightarrow \mathrm{Har}(\Gamma).$$

To every Harmonic cocycle we can associate a measure μ_κ (the one associated to f through the bijection is μ_f).

Theorem.

Any form $f \in S_2(\Gamma)$ has the following expression

$$f(z) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{1}{z-t} \right) d\mu_f(t).$$

Line integral

We choose a p-adic logarithm

$$\log_p : \mathbb{C}_p^\times \to \mathbb{C}_p.$$

Definition.

Given a cusp form $f \in S_2(\Gamma)$, de define the line integral between two points $\tau_1, \tau_2 \in \mathcal{H}_p$ as

$$\int_{\tau_1}^{\tau_2} f(z)dz := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_f(t).$$

This definition agrees with the definition we have given before

$$\int_{\tau_1}^{\tau_2} f(z)dz = \int_{\tau_1}^{\tau_1} \int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{1}{z-t}\right) d\mu_f(t) = \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_f(t).$$

Multiplicative integral

From the work of Cerendik-Drienfeld and Jacquet-Langlands, there exists cusp form $f \in S_2(\Gamma)$ which has the same L-function as the elliptic curve E. We determine f, up to a sign, by imposing that κ_f takes $\mathbb{Z}$ values.

Definition.

We define the multiplicative integral of f between the points $\tau_1, \tau_2 \in \mathcal{H}_p$ as

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} f(z)dz := \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_p)} \left(\frac{t-\tau_2}{t-\tau_1} \right) d\mu_f(t),$$

where the multiplicative integral over $\mathbb{P}^1(\mathbb{Q}_p)$ is defined by formally exponientalite the common definition.

We can extend the definition of this integral to give a definition of a semi-definite p-adic multiplicative integral for a $\tau \in \mathcal{H}_p$

$$\times\!\!\!\!\!\!\int^{\tau} f(z)dz.$$

Heegner points and p-adic periods

We assume the Heegner Hypothesis

- all primes dividing pN^+ are inert in K,
- all primes dividing N^- are split in K.

We consider an optimal oriented embedding $\Psi : K \to B$ (with respect to the order $\mathcal{O}$ and orientations of $\mathcal{O}$ and T). Let $\tau_\Psi \in \mathcal{H}_p$ the fixed point associated to the action induced by $\iota\Psi$ i.e. satisfies

$$\iota\Psi(\alpha)\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix} = \alpha\begin{pmatrix}\tau_\Psi \\ 1\end{pmatrix}, \quad \text{for all } \alpha \in (K \otimes \mathbb{Q})^\times.$$

We define the period integral associated to Ψ

$$J_\Psi := \times\!\!\!\!\!\!\int^{\tau_\Psi} f(z)dz.$$

We consider the Tate p-adic uniformization for the elliptic curve E

$$\eta_p : \mathbb{C}_p^\times / q_T^{\mathbb{Z}} \xrightarrow{\sim} E(\mathbb{C}_p).$$

Theorem.

For all optimal oriented embedding Ψ of $\mathcal{O}$ to R, we have that the point $\eta_p(J_\Psi)$ is a global point in $E(H)$ and for all $\mathfrak{a} \in \mathrm{Pic}(\mathcal{O})$, we have

$$\eta_p(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1} \eta_p(J_\Psi).$$

The theorem follows from Drinfeld's moduli interpretation of the p-adic upper half plane. The interpretation implies that the points J_Ψ is a Heegner point in the Shimura curve.

Periods over $\mathcal{H}_p \times \mathcal{H}_q$

Let $E/\mathbb{Q}$ an elliptic curve of conductor $N = pqN^+N^-$, B the quaternion algebra over $\mathbb{Q}$ of discriminant N^- and R the Eichler $\mathbb{Z}[1/pq]$-order of B of level N^+.

We consider the embedding

$$\iota : B \to M_2(\mathbb{Q}_p) \times M_2(\mathbb{Q}_q).$$

We define the group $\Gamma := \iota(R_1^\times)$.

Definition

A cusp form of weight 2 on $(\mathcal{T}_p \times \mathcal{H}_p)/\Gamma$ is a function of the form $f : \mathcal{E}(\mathcal{T}_p) \times \mathcal{H}_q \to \mathbb{C}_q$ satisfying

- $f(\gamma_p e, \gamma_q z) = j(\gamma_q, z)^2 f(e, z)$ for all $\gamma \in \Gamma$,
- For each $z \in \mathcal{H}_p$ the function $f_z(e) := f(e, z)$ is an harmonic cocycle,
- For each edge $e \in \mathcal{E}(\mathcal{T}_p)$ the function $f_e(z) = f(e, z)$ is a q-adic cusp form.

There exists an isomorphism $S_2((\mathcal{T}_p \times \mathcal{H}_q)/\Gamma) \cong S_2^{p-new}(\mathcal{H}_p/\Gamma_p)$, which allow us to extend the definition of f. Given two vertices $v_1, v_2 \in \mathcal{V}(\mathcal{T}_q)$ we define the following harmonic cocycle

$$\kappa_f\{v_1 \to v_2\}(e) = \sum_{\epsilon : v_1 \to v_2} \kappa_{f_e}(\epsilon).$$

This allows us to define the double-integral with respect to the points $\tau_1, \tau_2 \in \mathcal{H}_p$

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_1}^{v_2} \omega_f = \times\!\!\!\!\!\!\int_{\mathbb{P}^1(\mathbb{Q}_{\shortmid})} \left(\frac{t - \tau_2}{t - \tau_1} \right) d\mu_f\{v_1 \to v_2\}(t)$$

If we consider a function $g \in S_2((\mathcal{H}_p \times \mathcal{T}_q)/\Gamma)$, we can define the double integral as

$$\int_{v_1}^{v_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega_g = \prod_{\epsilon : v_1 \to v_2} \times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} g_\epsilon^{\#}(z) dz.$$

Theorem (Fubini Theorem).

There exits an explicit function $f^{\#} \in S_2((\mathcal{H}_p \times \mathcal{T}_q)/\Gamma)$ such that

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \int_{v_1}^{v_2} \omega_f = \int_{v_1}^{v_2} \times\!\!\!\!\!\!\int_{\tau}^{\tau_2} \omega_{f^{\#}}.$$

The definition of the integral on the right can be extended to get a semi-indefinite integral

$$\int_{v_1}^{v_2} \times\!\!\!\!\!\!\int^{\tau_1} \omega_{f^{\#}} := \prod_{\epsilon : v_1 \to v_2} \times\!\!\!\!\!\!\int^{\tau} f_{\epsilon}^{\#}(z)dz \in \mathbb{C}_p^{\times}/q_T^{\mathbb{Z}}.$$

Theorem of Bertolini, Darmon, and Green

We assume the following generalized Heegner Hypothesis for the quadratic imaginary field $K/\mathbb{Q}$

- all primes dividing N^- are inert in K,
- all primes dividing N^+ are split in K,
- p is inert in K,
- q is split in K.

Let $\mathcal{O}$ be a $\mathbb{Z}[1/pq]$-order of K of conductor prime to N.
Given an optimal oriented embedding Ψ (will exist because of these last assumptions) with fixed point τ_Ψ, we define the period

$$J_\Psi := \int_v^{\gamma_\Psi v} \times\!\!\!\!\!\!\int^{\tau_\Psi} \omega_{f^\#} \in \mathbb{C}_p^\times / q_T^{\mathbb{Z}},$$

where γ_Ψ satisfies that $\iota\Psi(\gamma_\Psi)$ is the generator of the group $\mathcal{O}_1^\times$.

We fix a prime $\mathfrak{q}$ over q and we consider the maximal $\mathbb{Z}[1/p]$-order $\mathcal{O}_0$ inside $\mathcal{O}$.
Let H_0 be the ring class field attached to $\mathcal{O}_0$ and $\sigma_{\mathfrak{q}}$ the element in $\mathrm{Gal}(H_0/K)$ representing $\mathfrak{q}$.
We define H as the subfield of H_0 left invariant by $\sigma_{\mathfrak{q}}^2$.

Theorem (Bertolini, Darmon and Green '07)

The point $\eta_p(J_\Psi)$ is a global point in $E(H)$ and, for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$

$$\eta_p(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

Sketch of proof.

We define $\mathcal{O}_0$ as the maximal $\mathbb{Z}[1/p]$-order inside $\mathcal{O}$ and $R_0 \subseteq R$ the maximal Eichler $\mathbb{Z}[1/p]$-order in R.
We consider an optimal oriented embedding Ψ_0 over $\mathcal{O}_0$ such that

$$\Psi_0^i, \quad i = 1, ..., h.$$

gives rise, by extensions of scalars, to the conjugacy class of Ψ.
Then we prove that there exists $a_i \in \{1, -1, 2\}$ such that

$$J_\Psi = \prod_{i=1}^{h} \left(\times\!\!\!\!\!\!\int^{\tau_{\Psi_0^i}} f_0(z)dz \right)^{a_i},$$

where f_0 is the modular form associated to f on the edge e_0.
This immediately proves the theorem using the theorem that we have seen in the Shimura curve case.

Generalizations

Since the primes p and q divide the conductor N, because of the necessary conditions of Heegner points over Shimura curves, we only have two possible behaviors of such primes in our generalized hypothesis

- (BDG case) p inert and q split in K,
- p and q inert in K.

Theorem (C. '25).

If we assume the generalized Heegner Hypothesis together with the assumption that p and q are inert in K, the point $\eta_p(J_\Psi)$ is a global point in $E(H)$. Furthermore, for all $\mathfrak{a} \in \mathrm{Pic}^+(\mathcal{O})$, we have

$$\eta_p(J_{\Psi^{\mathfrak{a}}}) = \mathrm{rec}(\mathfrak{a})^{-1}\eta_p(J_\Psi).$$

Corollary.

Let q be a prime such that $q \neq p \mid N$ and $q^2 \nmid N$. The set of Global points, which are the images of our double period integrals for a given order $\mathcal{O}$, will be nonempty if and only if the decomposition $N = pN^+N^-$ satisfies the Heegner Hypothesis.

This corollary is an immediate consequence of the theorem we have seen before, together with the theorem of BDG and the impossibility of extending our method for other behaviors of p and q.

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