A generalization of a Charollois-Darmon conjecture for certain lattice zeta functions associated to ray class field of ATR fields

Universitat Autònoma de Barcelona, Univeristé Laval and McGill University
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Carlos Caralps

Supervised by Professor Hugo Chapdelaine[1], Professor Patrick Allen[2] and Professor Marc Masdeu[3].

Abstract

In the 1970s, Harold Stark proposed a series of conjectures which if true would bring some partial solution to the so-called Hilbert Twelfth problem. In 2008, Pierre Charollois and Henri Darmon (see [CD08]) proposed an interesting refinement of a special case of the rank one Stark conjecture in the setting of ATR fields (Almost Totally Real field). They consider a base field K which is ATR and which is also a relative quadratic extension of a totally real field F of class number one. Their refinement provides a conjectural analytic expression for the complex Stark number itself, and not just its absolute value, in the setting of **F-ring class fields over K**. In this thesis, we provide a generalization of their construction in the setting of **F-ray class fields over K**. We consider a zeta functions of level f over K where f is an element of F and we give a conjectural expression for the complex Stark number associated to this zeta function, and not just its absolute value. Our formula consists in integrating an Eisenstein series of parallel weight 2 over a suitable real analytic chain on the corresponding Hilbert modular variety of F and level f. In the first part of this work, we provide some theoretical background. Secondly, for pedagogical reasons, we then choose to reformulate the version of the Stark conjecture which arises in our setting with the help of an integral formula which computes the logarithm of the absolute value of the Stark number. Finally, at the end of this work, we present our generalization which can be viewed as a natural refinement of the previous integral formula.

Abrégé

Dans les années 70, Harold Stark a proposé une série de conjectures qui, si vraies, apporteraient une résolution partielle au 12e problème de Hilbert. En 2008, Pierre Charollois et Henri Darmon (voir [CD08]) ont proposé un raffinement de la conjecture de Stark en rang un lorsque le corps de base est un corps ATR (Almost Totally Real). Les auteurs considèrent un corps de base ATR K qui est une extension quadratique relative d'un corps F totalement réel ayant un nombre de classe égal à un. Leur raffinement propose de manière conjecturale une expression analytique pour le nombre complexe de Stark lui-même et non pas seulement pour sa valeur absolue dans le contexte des **F-corps d'anneau au-dessus de K**. Dans ce travail nous considérons une généralisation de leur construction dans le contexte des **F-corps de rayon de K**. Nous considérons une fonction zêta de niveau f sur K où f est un élément de F et nous proposons une expression conjecturale pour le nombre complexe associé à cette fonction zêta et non pas seulement pour sa valeur absolue. Notre formule consiste à intégrer une série d'Eisenstein de poids parallèle 2 sur une chaîne réelle analytique convenable d'une certaine variété modulaire de Hilbert associée à F et de niveau f. Dans la première partie de ce travail, nous présentons certaines notions de base qui seront essentielles pour la suite. Deuxièmement, pour des raisons pédagogiques, nous avons choisi de reformuler la version de la conjecture de Stark qui intervient dans notre contexte à l'aide d'une formule intégrale qui calcule la valeur absolue du nombre de Stark. Finalement, comme derrière partie de ce travail nous présentons notre généralisation laquelle peut être vue comme un raffinement de la formule intégrale précédente.

Resum

En els anys 70, Harold Stark va proposar un seguit de conjectures que de ser certes donarien una solució parcial del 12è problema de Hilbert. Al 2008, Peirre Charollois i Henri Darmon (observeu [CD08]) van proposar un refinament interessant de la Conjectura de Stark de rang 1 per a cossos Quasi Totalment Reals. Consideren un cos K Quasi Totalment Real que també sigui una extensió quadràtica de un cos totalment real F amb un nombre de classes trivial. El refinament de Charollois-Darmon proporciona una expressió conjectural per a nombres de Stark Complexes, en comptes de per a el seu valor absolut, de **F-cossos d'anells sobre K**. En aquesta tesis, proposem una generalització de la seva construcció per a **F-cossos de rajos sobre K**. Considerem les funcions zeta de nivell f sobre K on f és un element de F i donem una expressió conjectural dels nombres de Stark complexes associats a aquestes funcions zeta. Les nostres formules es basen en integrar unes series d'Eisenstein de pes 2 sobre un cicle de la corresponent varietat modular de Hilbert de F. En la primera part d'aquest document, proporcionarem el coneixements previs necessaris per entendre els nostres resultats. Segonament, per raons pedagògiques, considerarem una reformulació de la Conjectura de Stark que apareix en la nostre distribució ajudant-nos de una expressió que relaciona la integral de una forma diferenciable amb el valor absolut del nombre de Stark. Finalment, al final d'aquest document, presentem la nostre generalització que pot ser vista com un refinament natural de la formula integral mencionada anteriorment.trivial.

[1] Professeur in the Département de Mathématiques et de Statistique at the Université Laval. Webpage: `https://www.mat.ulaval.ca/hchapdelaine/hugo-chapdelaine`

[2] Professor in the Department of Mathematics and Statistics at the McGill University. Webpage: `https://patrick-allen.github.io`

[3] Professor in the Departament de Matemàtiques at the Universitat Autònoma de Barcelona. Webpage: `https://mat.uab.cat/~masdeu/`

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Intoduction

To introduce the objectives of this document, we shall give a motivational example which underpins the Charollois-Darmon construction paper [CD08] and take this opportunity to introduce some basic concepts. Let K be a number field and let

$$S_\infty = \{v_1, ..., v_t\}$$

be the collection of its archimedean places. We let $n = r_1 + 2r_2$ where r_1 is the number of real embeddings of K and $2r_2$ is the number of complex embeddings of K; so that $t = r_1 + r_2$. For every element $x \in K^\times$ and place $v \in S_\infty$ we define a sign function $s_v : K^\times \to \{\pm 1\}$

$$s_v(x) := \begin{cases} \mathrm{sign}(v(x)) & \text{if } v \text{ is real} \\ 1 & \text{if } v \text{ is complex} \end{cases}$$

Let $\mathcal{O}_K$ be the ring of algebraic integers of K. We say that an element $x \in K^\times$ is *fully positive* if for all $v \in S_\infty$ we have that $s_v(x) = 1$. We shall denote the collection of *fully positive integers* of K and $\mathcal{O}_K$ as K_+ and $\mathcal{O}_{K,+}$, respectively. In particular, we have that

$$x \in \mathcal{O}_{K,+} \iff s_v(x) = 1, \ \ \forall v \in S_\infty$$

Given an $\mathcal{O}_K$-fractional ideal $I \subseteq K$, we shall denote the collection of fully positive units of $\mathcal{O}_K$ congruent to 1 modulo I by $\mathcal{O}^\times_{K,+}(I)$. Given $\mathfrak{a} \subseteq \mathcal{O}_K$ a fractional ideal, we associate to the pair $(\mathfrak{a}, I)$ the partial L-Function of Hurwitz as

$$L(\mathfrak{a}, I, \omega; s) := \frac{1}{(\mathbf{N}\,\mathfrak{a})^s} \sum_{\substack{x \in \left((1 + I\mathfrak{a}^{-1}) \big/ \mathcal{O}^\times_{K,+}(I)\right) \\ x \neq 0}} \frac{\omega(x)}{\left|\mathbf{N}_{K/\mathbb{Q}}(x)\right|^s} \tag{1}$$

where $\omega : K^\times \to \{\pm 1\}$ is a choice of a sign character. We shall point out that we think of I as a level of the zeta function. Furthermore, $\mathfrak{a}^{-1}$ is the inverse ideal of $\mathfrak{a}$ and $s \in \mathbb{C}$ is a complex number such that $\Re(s) > 1$ so that the series converges absolutely.

For the scope of this document, we shall be mainly interested when ω is equal either to $\omega_0 := \mathrm{sign} \circ \mathbf{N}_{K/\mathbb{Q}}$ or $\omega_1 := (\mathrm{sign} \circ \mathbf{N}_{K/\mathbb{Q}})/s_{v_1}$. It can be shown that this L-function admits a meromorphic continuation to the whole complex plane with a possible pole at $s = 1$. For every unit $\epsilon \in \mathcal{O}^\times_K$ it is straightforward to check that

$$L(\epsilon\mathfrak{a}, I, \omega; s) = \omega(\epsilon) \cdot L(\mathfrak{a}, I, \omega; s)$$

In particular, it should be pointed out that if there exists a unit $\epsilon \in \mathcal{O}^\times_K(I)$ such that $\omega(\epsilon) = -1$ then $L(\mathfrak{a}, I, \omega; s) \equiv 0$ (identically equal to zero). Note also that if $t > 2$ and $\omega = \omega_1$, then the L-function in (1) has an order of vanishing at least one at $s = 0$. For any real place $v \in S_\infty$, let $\alpha_v \in \mathcal{O}_K$ be an algebraic integer satisfying the conditions that

$$\alpha_v \equiv 1 \pmod{I}, \ \ s_v(\alpha_v) = -1 \ \text{ and } \ s_{v'}(\alpha_v) = 1 \text{ if } v \neq v'$$

These elements will give rise to complex conjugations in the abelian extension of K prescribed by the ideal I.

In order to give an explicit expression of the ray class group of modulus I, in the narrow sense, we need to introduce a monoid of integral ideals coprime to I

$$\mathcal{I}_{\mathcal{O}_K}(I) := \left\{ \mathfrak{a} \subseteq \mathcal{O}_K \text{ an } \mathcal{O}_K\text{-ideal} \; : \; \mathfrak{a} + I = \mathcal{O}_K \right\}$$

We also need to consider the collection of elements in $\mathcal{O}_K$ fully positive and congruent to 1 mod I

$$P^+_{\mathcal{O}_K,1}(I) := \left\{ \lambda \in \mathcal{O}_K \;\; : \;\; \lambda \equiv 1 \pmod{I} \text{ and } \lambda \gg 0 \right\}$$

The ray class group of K can be expressed as a quotient of the last two sets, understood through the relation of ideals in $\mathcal{I}_{\mathcal{O}_K}$ by multiplication of elements in $P^+_{\mathcal{O}_K,1}(I)$

$$\mathrm{Cl}^+_K(I) \cong \mathcal{I}_{\mathcal{O}_K}(I) \Big/ P^+_{\mathcal{O}_K,1}(I)$$

From class field theory we know that there exists an abelian extension $K(I\infty)$ over K, the ray class field of modulus I in the narrow sense, such that its Galois Group is canonically isomorphic to the class group $\mathrm{Cl}^+_K(I)$. Since I is fixed we let $H := K(I\infty)$ and we consider

$$\mathrm{rec} : \mathrm{Cl}^+_K(I) \to \mathrm{Gal}(H/K)$$

the reciprocity isomorphism given from class fields (it is defined in detail in Section 1.2). We define the complex conjugation associated to an infinite place $v \in S_\infty$ as

$$c_v = \begin{cases} \mathrm{rec}\left(\alpha_v \mathcal{O}_K\right) & \text{if } v \text{ is real} \\ \mathrm{Id} & \text{if } v \text{ is complex} \end{cases}$$

In particular, note that c_v is either trivial or an involution (an element of order two). If $\widetilde{v}$ is a lift of v to H then c_v is an involution precisely when v is real and $\widetilde{v}$ is complex. From now on, we suppose that the place v_1 splits in H (i.e. $c_{v_1} = 1$) and that the remaining places ramify (i.e. c_{v_k} is non-trivial for $2 \leq k \leq t$). Let $\widetilde{v}_1$ be a lift of v_1 to H and let e_H denote the number of roots of unity of H. We are now ready to state the Stark Conjecture which gives a precise relationship between the first derivative of the L-functions at $s = 0$ and the so-called "Stark number".

Conjecture (Stark Conjecture). *Let $\widetilde{v}_1 \in S_\infty$ be a choice of a lift of v_1 to H. Then for any fractional ideal $\mathfrak{a} \subseteq K$ there exists an algebraic number $u_\mathfrak{a} \in H$, called the Stark number, such that*

(i) $L'(\mathfrak{a}, I, \omega_1, 0) = \frac{1}{e_H} \log |\widetilde{v}_1 u_\mathfrak{a}|^2$

(ii) $c_{v_1}(u_\mathfrak{a}) = u_\mathfrak{a}$

(iii) If $t \geq 3$, then $c_{v_2} u_\mathfrak{a} = \cdots = c_{v_t} u_\mathfrak{a} = u_\mathfrak{a}^{-1}$

(iv) For all $[\mathfrak{b}] \in \mathrm{Cl}^+_K(I)$, we have $u_{\mathfrak{a}\mathfrak{b}^{-1}} = \mathrm{rec}(\mathfrak{b})\, u_\mathfrak{a}$

Remark. *In [CD08] the authors assume that the α_v's are units but doing so automatically implies that the potential complex conjugations c_v's are trivial since $\mathrm{rec}(\epsilon_v \mathcal{O}_K) = \mathrm{Id}$ because $\epsilon_v \mathcal{O}_K = \mathcal{O}_K$. Furthermore, if $t > 2$, this assumption when combined with (iii) readily implies that $u_\mathfrak{a} = u_\mathfrak{a}^{-1}$ i.e. $u_\mathfrak{a} \in \{\pm 1\}$.*

The computation of Stark numbers is an interesting and non-trivial problem which allows one to test the conjecture but also to construct explicitly specific abelian extensions over K (relation with the 12th Hilbert Problem). In the case where the place v_1 is real, since $c_{v_1} = \mathrm{Id}$, it follows that $\widetilde{v}_1$ is also real and therefore the quantity $\log |\widetilde{v}_1 u_\mathfrak{a}|$ determines $u_\mathfrak{a}$ up to a sign.
It is possible to rewrite the L-function $L(\mathfrak{a}, I, s)$ in a more uniform way using what we call a lattice zeta function. Given any lattice $\mathfrak{n} \subseteq K$ (not necessarily an $\mathcal{O}_K$-ideal) and a pair of elements $a, b \in K$ we define

$$Z_\mathfrak{n}(a, b, \omega; s) = \mathbf{N}(\mathfrak{n})^s \sum_{\substack{x+a \,\in\, \left(\mathcal{V}^+_{a,b,\mathfrak{n}} \backslash (\mathfrak{n}+a)\right) \\ x+a \neq 0}} \frac{\omega(x+a) \cdot e^{2\pi \mathrm{i} \; \mathbf{Tr}_{K/\mathbb{Q}}(b(x+a))}}{\left|\mathbf{N}_{K/\mathbb{Q}}(x+a)\right|^s} \tag{2}$$

Recall that $\omega : K^\times \to \{\pm 1\}$ is a choice of a sign character, $\Re(s) > 1$ and $\mathcal{V}^+_{a,b,\mathfrak{n}}$ is a certain finite index subgroup of $\mathcal{O}^\times_{K,+}$. It follows from the definition that $L(\mathfrak{a}, I, \omega_1, s) = Z_{I\mathfrak{a}^{-1}}(1, 0, \omega_1; s)$ and one can show that for all $a \in K$

$$\frac{\partial}{\partial s} Z_I(a, 0, \omega_1; s)\Big|_{s=0} = \frac{\pi}{2}\,\mathrm{i}^{r_1}\,|d_K|^{1/2} Z_{I^*}(0, a, \omega_1; 1) \tag{3}$$

where I^* denotes the dual lattice of I with respect to the trace pairing and d_K corresponds to the discriminant of K. In particular, when the identity (3) is combined with the Stark conjecture we obtain that the **absolute value of the Stark number $u_\mathfrak{a}$** can be calculated in terms of the value of a lattice zeta function at $s = 1$ (instead of a first derivative at $s = 0$ of the dual lattice zeta function).

When the place v_1 is complex so that $\widetilde{v}_1$ is automatically complex, the element $\widetilde{v}_i u_\mathfrak{a}$ is a priori a complex number and therefore it is not possible to recover directly this number from its absolute value since the argument of $u_\mathfrak{a}$ is lost. Charollois and Darmon proved a theorem [CD08, p. 671] that relates certain Eisenstein series with the values of $L'(\mathfrak{a}, I, \omega_1, 0)$ when $I = \mathcal{O}_K$ (the trivial level) and when K is a relative quadratic extension of a totally real field F of narrow class number 1. Moreover, in that case, they give a conjectural formula for the complex number $\widetilde{v}_1 u_\mathfrak{a}$ and not just for its absolute value. We should also point out that their construction is more general than what we just described since they can replace the maximal order $\mathcal{O}_K$ of K by non-maximal orders $\mathcal{O} \subseteq \mathcal{O}_K$ relative to F; such orders are of the form $\mathcal{O} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ where $\mathfrak{c} \subseteq \mathcal{O}_K$ is an integral ideal of $\mathcal{O}_F$ (the conductor of $\mathcal{O}$ relative to F). In that setting the Stark number $u_\mathfrak{a}$ is conjectured to lie in a **ring class field of** K associated to that order $\mathcal{O}$.

In this document, we shall present a generalization of the Charollois-Darmon construction for the lattice zeta functions given in (2) by allowing the level I to be non-trivial so that in that case the Stark numbers are conjectured to lie in the **ray class fields of K of modulus I in the narrow sense** (rather than ring class fields as it was the case in the Charollois-Darmon construction). As in the Charollois-Darmon construction, we shall also assume that K/F is a relative quadratic extension of a totally real field F with trivial narrow class group. Moreover, partly because not all the details have been fully worked out and partly to simplify the presentation we shall only formulate a conjecture when the ideal $I \subseteq \mathcal{O}_K$ is of the following specific form: we let $\widetilde{\varpi} \in \mathcal{O}_F$ be a prime element, totally positive, and of absolute degree one which splits in K. This means in particular that $\widetilde{\varpi} = \varpi_1 \cdot \varpi_2$ where ϖ_1 and ϖ_2 are distinct primes in $\mathcal{O}_K$ of absolute degree one. We let $I := \varpi_1\mathcal{O}_K$. In particular, we have that $\mathcal{O}_K/\varpi_1\mathcal{O}_K \simeq \mathcal{O}_F/\widetilde{\varpi}\mathcal{O}_F \simeq \mathbb{F}_p$ where $\mathbf{N}_{F/\mathbb{Q}}(\widetilde{\varpi}) = p$ is a prime element of $\mathbb{Z}$. Our approach will rely heavily on a detailed study made by Hugo Chapdelaine in the preprint [Cha16] on the so-called **lattice Eisenstein series**.

If we let $g := [F : \mathbb{Q}]$ and $\mathfrak{a} \subseteq \mathcal{O}_K$ be a fractional ideal then the Stark conjecture, stated with detail in Section 1.2, predicts

$$\frac{\partial}{\partial s} Z_{\varpi_1\mathfrak{a}^{-1}}(1, 0, \omega_1; s)\Big|_{s=0} = -\frac{1}{e_H}\log|\widetilde{v}_1(u_\mathfrak{a})|^2$$

where $u_\mathfrak{a}$ is an algebraic number in $H = K(I\infty)$. The main objective of this document is to relate the values of lattice zeta functions and their derivatives at $s = 0$ with holomorphic lattice Eisenstein series of parallel weight 2 associated to the number field F, which can be formally defined as

$$E^*_2(b; z) = \text{``}\sum_{\substack{(m,n)\in\left(\mathcal{O}_F\times\mathcal{O}_F\big/\mathcal{V}^+_{0,b,\mathcal{O}_F}\right)\\(m,n)\neq(0,0)}} \frac{e^{2\pi\,\mathrm{i}\,\mathrm{Tr}(\delta b m)}}{\mathbf{N}\left(\delta(mz+n)\right)^2}\text{''} \tag{4}$$

where $b \in \mathcal{O}_F$, $\mathrm{Tr}_{F/\mathbb{Q}}$ corresponds to the trace of F and $\delta \in F$ is a totally positive generator of $\mathcal{O}^*_F$ (the dual lattice of $\mathcal{O}_F$ with respect to the trace pairing). Note that the summation on the right hand side of (4) does not converge absolutely (this is why we have put quotes) but it can be given a meaning using the Fourier series expansion. Let

$$\Gamma := \Gamma^1(\widetilde{\varpi}\mathcal{O}_F) \leq SL_2(\mathcal{O}_F) \text{ and } \mathcal{X} := \Gamma\backslash\mathfrak{h}^g \tag{5}$$

Here $\mathcal{X}$ is a complex analytic space with $p-1$ cusps. It is known that $\mathcal{X}$ can be viewed as the set of $\mathbb{C}$-points of a quasi-projective variety which is called a Hilbert modular variety. Using the holomorphic lattice Eisenstein series of weight 2 we can associate the following closed differential form on $\mathcal{X}$ of degree $g = [F : \mathbb{Q}]$:

$$\Omega_{Eis}(b;z) := \begin{cases} -E_2^*(b;z)\, dz_1 \wedge dz_2 + \frac{R_F^{\widetilde{\varpi}}}{2}\left(\frac{dz_1 \wedge d\bar{z}_1}{y_1^2} - \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}\right) & \text{if } g = 2 \\ \mathrm{i}^{-g}\, E_2^*(b;z)\, dz_1 \wedge \cdots \wedge dz_g & \text{if } g > 2 \end{cases}$$

Here $R_F^{\widetilde{\varpi}}$ is the regulator associated to the group of units $\mathcal{O}_F^\times(\varpi)$. In Sections 2 and 3 we will generalize the analogous theorems from [CD08] for these zeta functions by providing an analytic expression for the Stark number $u_{\mathfrak{a}}$.

1. **K is totally real and $\omega = \omega_0$.** Let us assume that K is a totally real field (i.e. if $\widetilde{v}_1$ is real). In order to simplify the presentation we shall also assume that the ideal $\mathfrak{a}$ is of the form $\mathfrak{a} = \mathcal{O}_F + \tau\mathcal{O}_F$ for some $\tau \in K$. Let $\gamma \in \Gamma^1(\widetilde{\varpi}\mathcal{O}_F)$ be a matrix which stabilizes τ. Since K is totally real it follows that the matrix γ must be totally hyperbolic. It thus follows that for each $1 \leq j \leq g$, the matrix $\gamma^{(j)} \in SL_2(\mathbb{R})$ associated to the j-th embedding of γ has two fixed points in $\mathbb{P}^1(\mathbb{R})$ which we denote by τ_j, τ_j' where $\tau_j < \tau_j'$. Note that the collection $\{\tau_j, \tau_j' : 1 \leq j \leq g\}$ corresponds to the set of all $\mathbb{Q}$-conjugates of $\tau \in K$. In particular, we can associate to τ the following totally geodesic real analytic submanifold of $\mathfrak{h}^g$ of real dimension g:

$$R_\tau = \Upsilon[\tau_1, \tau_1'] \times \cdots \times \Upsilon[\tau_g, \tau_g']$$

We let Δ_τ be the image of of R_τ in $\mathcal{X}$ which is a g-cycle on $\mathcal{X}$ which is topologically homeomorphic to $(S^1)^g$. We shall prove the following Taylor series expansion which relates a lattice zeta functions at $s=0$ to an integral of an Eisenstein series defined over F:

$$Z_{\mathfrak{a}}(a,b,\omega_0;s) = \frac{d_F\sqrt{d_F}}{\mathbf{N}_{K/\mathbb{Q}}(\nu)} \int_{\Delta_\tau} \Omega_{Eis} + O(s) \quad \text{as } s \to 0 \tag{6}$$

2. **K is ATR and $\omega = \omega_1$.** Let us assume now that K is an almost totally real field (i.e. if $\widetilde{v}_1$ is complex). As before let us assume to simplify that the fractional $\mathcal{O}_K$ ideal $\mathfrak{a} \subseteq K$ has the form $\mathfrak{a} = \mathcal{O}_F + \tau\mathcal{O}_F$ for some $\tau \in K$. Let $\gamma \in \Gamma^1(\varpi\mathcal{O}_F)$ be a matrix which stabilizes τ. Since K is ATR it follows that the matrix γ has exactly one elliptic component and $g-1$ hyperbolic ones. Let us assume that $\gamma^{(1)} \in SL_2(\mathbb{R})$ is elliptic with $\tau_1 \in \mathfrak{h}$ as its fixed point. For $j \geq 2$, the matrices $\gamma^{(j)} \in SL_2(\mathbb{R})$ admit two fixed points in $\mathbb{P}^1(\mathbb{R})$ which we denote by τ_j, τ_j' where $\tau_j < \tau_j'$. Note that the collection $\{\tau_j, \tau_j' : 2 \leq j \leq g\} \cup \{\tau_1, \overline{\tau_1}\}$ corresponds to the set of all $\mathbb{Q}$-conjugates of $\tau \in K$. In particular, we can associate to τ the following totally geodesic real analytic submanifold of $\mathfrak{h}^g$ of real dimension $g-1$:

$$R_\tau = \{\tau_1\} \times \Upsilon[\tau_2, \tau_2'] \times \cdots \times \Upsilon[\tau_g, \tau_g']$$

Let Δ_τ be the image of R_τ in $\mathcal{X}$. It is a $(g-1)$-cycle which is topologically homeomorphic to $(S^1)^{g-1}$. It will be proved that Δ_τ, modulo torsion, is a $(g-1)$-boundary so that there exists a g-chain C_τ on $\mathcal{X}$, well-defined up to a g-cycle of $\mathcal{X}$, such that $\partial C_\tau = \Delta_\tau$. Moreover, we shall prove the following Taylor series expansion

$$Z_{\varpi_1\mathfrak{a}^{-1}}(1,0,\omega_1;s) = s\left(\frac{d_F\sqrt{d_F}}{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}\right)\int_{C_\tau} \Omega_{Eis}^+ + O(s^2) \quad \text{as } s \to 0 \tag{7}$$

where Ω^+_{Eis} is the real part of the form Ω_{Eis}.

Since here v_1 is complex (so that $\widetilde{v}_1$ is also complex) we have that $\omega_0 = \omega_1$. We think here of the formula in (7) as being an analogue of (6) where the totally real field K in case 1 is replaced by an ATR field K in case 2.

Remark. *When K is totally real and $\omega = \omega_1$, there is also a formula analogous to* (6) *and* (7) *which involves an Eisenstein series defined over F of parallel weight* 2 *but this time twisted by a non-trivial unitary character of weight* $(1, 0, 0, \dots, 0)$ *of the $\mathbb{C}$-algebra* $(F \otimes_{\mathbb{Q}} \mathbb{C})^{\times}$. *In particular, the resulting Eisenstein series is no longer holomorphic in z. This Eisenstein series needs also to be multiplied by* $\left((\overline{z}_1 - \tau_1)/(z_1 - \tau_1)\right)^{1/2}$ *so that the overall expression has parallel weight* 2. *But we do not pursue this line of investigation in this document.*

As was pointed out earlier, the second theorem only provides an analytic expression for $|u_{\mathfrak{a}}|$ rather than the value of $u_{\mathfrak{a}} \in \mathbb{C}$ itself because the argument of $u_{\mathfrak{a}}$ gets loss when one takes the absolute value. The main contribution of P. Charollois and H. Darmon in [CD08] was to state a conjectural analytic expression for $u_{\mathfrak{a}}$ using an Abel-Jacobi map defined from the classical holomorphic Eisenstein series of parallel weight 2 associated to F. In this work, we generalize this construction. More precisely we show that

$$e_H \cdot \Phi_{\mathrm{Eis}}(\Delta_\tau) = \log(\widetilde{v}_1(u_{\mathfrak{a}})) \pmod{2\pi\,\mathrm{i}\,\mathbb{Z}}$$

If the reader compares the statement of the Stark conjecture given in the paper [CD08] with ours, they will notice that **we do not conjecture** that the Stark numbers $u_{\mathfrak{a}}$ are units in H but rather only algebraic numbers. Our choice of formulation is in accordance with the presentation given in Tate's well-known monograph [Tat84] where S-units appear rather than units. Here S is a finite set of places of H which contains all the finite ramified places in H/K and all the infinite places of H. For the remainder of this introduction, we would like to present a motivating example which illustrates this subtle point.

We let $F = \mathbb{Q}$ and M be an imaginary quadratic extension of discriminant $d_M < 0$. For every positive integer $f \in \mathbb{Z}_{>0}$ there exists a unique order of $\mathcal{O}_M$ of conductor f which corresponds to $\mathcal{O}_f := \mathbb{Z} + f\mathcal{O}_M$. In particular, if $f = 1$ then $\mathcal{O}_1 = \mathcal{O}_M$ is the maximal order of M. If $\mathcal{O} \subseteq M$ is an order it is also convenient to let $c(\mathcal{O})$ be the conductor of $\mathcal{O}$. Given a lattice $\mathcal{L} \subseteq M$ we let $\mathbf{N}\,\mathcal{L} := [\mathcal{O}_M : \mathcal{L}] \in \mathbb{Q}_{>0}$ be the rational index of $\mathcal{L}$ in $\mathcal{O}_M$. If $\mathcal{L}_1, \mathcal{L}_2 \subseteq M$ are two lattices then the product $\mathcal{L}_1\mathcal{L}_2$ is again a lattice inside M. We let $\mathcal{O}_{\mathcal{L}} := \{\lambda \in M : \lambda\mathcal{L} \subseteq \mathcal{L}\}$ be the ring of endomorphisms of $\mathcal{L}$ (it is an order of M which depends only on the homotethy class of $\mathcal{L}$). Furthermore, we let $\mathcal{L}^{-1} := \{\lambda \in M : \lambda\mathcal{L} \subseteq \mathcal{O}_{\mathcal{L}}\}$ be the inverse lattice of $\mathcal{L}$. By definition we have that $\mathcal{L}\mathcal{L}^{-1} \subseteq \mathcal{O}_{\mathcal{L}}$, moreover, since M is quadratic it can be shown that $\mathcal{L}\mathcal{L}^{-1} = \mathcal{O}_{\mathcal{L}}$ (see Theorem 2 on p. 90 of [Lan87]) and therefore $\mathcal{O}_{\mathcal{L}} = \mathcal{O}_{\mathcal{L}^{-1}}$. In other words every lattice $\mathcal{L}$ is invertible with respect to its endomorphism order. To any lattice $\mathcal{L} \subseteq M$ such that $\mathcal{L}^{-1} = \mathbb{Z} + \tau\mathbb{Z}$ with $\tau = x + \mathrm{i}y \in \mathfrak{h}$ we associate the zeta function

$$Z(\mathcal{L}, s) := \frac{1}{e_M\,\mathbf{N}(\mathcal{L})^s} \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{1}{|m\tau + n|^{2s}} \tag{8}$$

where $\Re(s) > 1$. Note that the function $Z(\mathcal{L}, s)$ only depends on the homotethy class of $\mathcal{L}$. Since $4y^2 = \mathbf{N}(\mathcal{L}^{-1})^2 \cdot |d_M| = \frac{1}{\mathbf{N}(\mathcal{L})^2} \cdot |d_M|$ it follows that

$$Z(\mathcal{L}, s) = \frac{1}{e_M} \cdot \left(\frac{2}{\sqrt{|d_M|}}\right)^s \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{y^s}{|m\tau + n|^{2s}} = \frac{1}{w} \cdot \left(\frac{2}{\sqrt{|d_M|}}\right)^s E(\tau, s) \tag{9}$$

where $y = \Im(\tau) \in \mathbb{R}_{>0}$. Given an order $\mathcal{O} \subseteq M$ we let $H_{\mathcal{O}}$ denote the ring class over M associated to $\mathcal{O}$. It can be proved that the conductor of the abelian extension $H_{\mathcal{O}}/M$, namely $\mathfrak{f}(H_{\mathcal{O}}/M)$, is "essentially" equal to $c(\mathcal{O})$ (see exercise 9.20 on p. 196 of [Cox13])

Using the first Kronecker limit formula one finds that for any pair of lattices $\mathcal{L}_1, \mathcal{L}_2 \subseteq M$, the difference of zeta function $Z(\mathcal{L}_1, s) - Z(\mathcal{L}_2, s)$ has no pole at $s = 1$ (which is equivalent, using the functional equation of $E(\tau, s)$ in s, that this difference vanishes at $s = 0$). More precisely one has that

$$Z(\mathcal{L}_1, s) - Z(\mathcal{L}_2, s) = \frac{\pi}{3\sqrt{d_M}} \log\left(\frac{\mathbf{N}(\mathcal{L}_1^{-1})^6 \cdot |\Delta(\mathcal{L}_1^{-1})|}{\mathbf{N}(\mathcal{L}_2^{-1})^6 \cdot |\Delta(\mathcal{L}_2^{-1})|}\right) + O(s-1) \quad \text{as } s \to 1 \tag{10}$$

Here $\Delta(\mathbb{Z}+\tau\mathbb{Z}) = \Delta(\tau)$ corresponds to the modular discriminant function of weight 12 with respect to $SL_2(\mathbb{Z})$. It follows from a theorem of Deuring (see the Corollary on p. 166 of [Lan87]) that if $\mathcal{L}_1$ and $\mathcal{L}_2$ are both $\mathcal{O}$-invertible **for the same order** $\mathcal{O}$ then the positive real number inside the logarithm, namely

$$\frac{\mathbf{N}(\mathcal{L}_1^{-1})^6 \cdot |\Delta(\mathcal{L}_1^{-1})|}{\mathbf{N}(\mathcal{L}_2^{-1})^6 \cdot |\Delta(\mathcal{L}_2^{-1})|} \tag{11}$$

is a unit in $\mathcal{O}_{H_\mathcal{O}}^\times$. This is a priori surprising since the abelian extension $H_\mathcal{O}/M$ is ramified (essentially) at the primes dividing $c(\mathcal{O})$. However, if $\mathcal{O}_{\mathcal{L}_1} \neq \mathcal{O}_{\mathcal{L}_2}$ then the quantity in (11) will not be a unit in general but only an algebraic number. In [CD08, p. 659] the two authors made the conjecture that the Stark number in the setting of ring class fields is expected to be a unit and not just a S-unit which is consistent with our previous observation when M is imaginary quadratic and this seems to us the best reason to trust their conjecture.

Remark. *Our definition of the norm of a lattice $\mathcal{L} \subseteq M$ differs slightly from the definition used in [Lan87] where it is defined instead as $\mathbf{N}\,\mathcal{L} = [\mathcal{O}_\mathcal{L} : \mathcal{L}]$. This is of no consequence for the discussion above since the quantity in* (11) *does not depend on the choice of the definition used as long as $\mathcal{O}_{\mathcal{L}_1} = \mathcal{O}_{\mathcal{L}_2}$.*

Remark. *As a side remark let us point out that the statement made in [CD08, p. 659] to the effect that*

$$\eta(\tau_2)/\eta(\tau_1) \quad \text{or} \quad |\eta(\tau_2)/\eta(\tau_1)|$$

(here $\eta(\tau)^{24} = \Delta(\tau)$) is a unit when $\tau_1, \tau_2 \in M \cap \mathfrak{h}$ is obviously false. Furthermore, note also that it could well be that $|\eta(\tau_2)/\eta(\tau_1)|$ is a unit while $\eta(\tau_2)/\eta(\tau_1)$ fails to be a unit.

Some quite interesting numerical examples for the Stark number were given in [CD08] when K/F is a relative ATR quadratic extension of a real quadratic field F. We would like to emphasize that the numerical examples presented there were for the ring class field $H_\mathcal{O}$ of K associated to the maximal order $\mathcal{O} = \mathcal{O}_F + 1 \cdot \mathcal{O}_K = \mathcal{O}_K$ (i.e. the order of K with trivial conductor). Note that in that case the group of S-units of $H_{\mathcal{O}_K}$ (the Hilbert class field of K in the narrow sense) coincides with the group of units $\mathcal{O}_{H_\mathcal{O}}^\times$ since S contains only the infinite places. It is not completely clear to the present author why the Stark number $u_\mathfrak{a}$ is still expected to be a unit when the order $\mathcal{O} = \mathcal{O}_F + \mathfrak{c}\mathcal{O}_K$ is no longer considered to be maximal i.e. when $\mathfrak{c}$ is an $\mathcal{O}_F$-ideal **distinct from** $\mathcal{O}_F$; even though such a strong statement to the effect that $u_\mathfrak{a}$ would be a unit (and not just an S-unit) would be consistent with our previous observation regarding ring class fields of an imaginary quadratic field M. It would be interesting to provide numerical examples in the situation when $\mathfrak{c} \neq \mathcal{O}_F$ to test their conjecture.
In the setting of ray class fields, if $\zeta(a, f; s)$ denote the partial zeta function over $\mathbb{Q}$ of modulus $f\infty$, it is well known that

$$\zeta'(a; f, 0) = -\frac{1}{2}\log\left|2\sin\left(\frac{\pi a}{f}\right)\right|^2$$

and $2\sin\left(\frac{\pi a}{f}\right) \in \mathbb{Q}(f\infty) = \mathbb{Q}(\zeta_f + \overline{\zeta_f})$ fails to be a unit when f is a power of a prime which justifies the necessity of considering S-units and not just units.

1 Preliminary Notions

Given a number field, which is a finite field extension of $\mathbb{Q}$. It is important to point out that we will normally denote these extensions with a general root of a given irreducible polynomial $p \in \mathbb{Z}[x]$ and denoted them as $\mathbb{Q}[x]/p(x)$ (be aware that this is not the decomposition field of p). Therefore, the extension generated by a root of an irreducible polynomial and a different extension generated by another root of the same polynomial are essentially considered as the same abstract number field. We shall start defining the Archemedian Places of a number field

Definition 1.1 (Places of a Number Field)**.** *Given a Number Field L, its Archimedean places are the equivalent classes of non-trivial Archimedean absolute values. The collection of Archemedian places of L is denoted as*

$$S_\infty^L = \{v_1, ..., v_t\}$$

As André Weil points out in his book [Wei74, p. 44], to each Archemedian place we can assign an embedding using the competition of the given number field through the absolute value chosen. If such completion is included in $\mathbb{C}$ we say the place is complex, otherwise, we say is a real place. The non-trivial fields which only have one complex Hilbert Place are called Almost Totally Real fields, which will be denoted from now on as ATR fields.

It is remarkable that number fields will have the same number of embeddings into $\mathbb{R}$ as real places and will have two times the embeddings into $\mathbb{C}$ as complex places.
If an element of an arbitrary field extension $\lambda \in L/\mathbb{Q}$ has all the images through the real places positive, we say is a fully positive element of L and will be denoted as $\lambda >> 0$. In the concrete case where L does not have any complex places the elements that satisfy this condition are normally said to be totally positive.
For the generalization, we will consider a number field F of degree $g = [F : \mathbb{Q}]$, with all its places real, and a relative quadratic ATR extension K of F. Since all the Places of F will split in K except for the one that will ramify into the complex one, consequently, the field K has $n_1 = 2(g-1)$ real places.

$$\begin{tikzcd}
 & & \mathbb{R}^{n_1} \arrow[d, hook] \\
K \arrow[urr, hook] \arrow[drr, hook] \arrow[dd, dash, "2"] \arrow[dddd, dash, bend right, "2g"'] & & \mathbb{R}^{n_1} \times \mathbb{C} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \\
 & & \mathbb{C} \arrow[u, hook] \\
F \arrow[rr, hook] \arrow[dd, dash, "g"] & & \mathbb{R}^g \cong F \otimes_{\mathbb{Q}} \mathbb{R} \\
 & & \\
\mathbb{Q} & &
\end{tikzcd}$$

Diagram 1

We shall introduce a specific notation to denote the Archimedean places of K (taking under consideration it is an ATR) on the following subsections, we will follow a similar notation as [CD08, p. 656] to facilitate a simultaneous reading of both documents

$$S_\infty^K = \{v_1, v_2, .., v_{2g-1}\}$$

To give a good definition of the zeta functions associated to a ray class field in the next subsection it is necessary to introduce the concept of tensor product which will also be helpful to understand the general picture of the two Number Fields that we are considering.

$$\begin{tikzcd}
V \times W \arrow[rr, "\otimes"] \arrow[drr, "\tau"'] & & V \otimes W \arrow[d, dashed, "\widetilde{\tau}"] \\
 & & Z
\end{tikzcd}$$

Definition 1.2 (Tensor Product of R-Algebras)**.** *The tensor product of two R-algebras V and W, denoted as $V \otimes_R W$, is the unique set that satisfies the decomposition of any bilinear map between τ between $V \times W$ and a R-algebra Z with the R-bilinear map $\otimes(v, w) = v \otimes w$ (i.e. exists a unique bilinear map $\widetilde{\tau}$ satisfying $\tau = \widetilde{\tau} \circ \otimes$).*

This definition is an adapted version of a general statement that uses the concept of universal property in Category theory. If the reader wants to understand better the tensor products the author recommends reading [Wei74], however, the document can be understood without this knowledge because, using the embeddings of K and F, it can be easily shown (it will be left as an advanced exercise to the reader) that

$$\mathbb{R}^{n_1} \times \mathbb{C} \cong K \otimes_{\mathbb{Q}} \mathbb{R} \quad \text{ and } \quad \mathbb{R}^g \cong F \otimes_{\mathbb{Q}} \mathbb{R}$$

Considering these two isomorphisms, we have introduced all the notions that appear in Diagram 1 which gives a visual representation of the general field distribution we will use in this thesis and their respective embeddings. It will also be useful to introduce a notation to denote, given an element $\lambda \in K$, the image of this element through the j embedding as $\lambda^{(j)}$. Introducing an injective map ι from K to the tensor product $K \otimes_{\mathbb{Q}} \mathbb{R}$.

$$\begin{aligned} K &\hookrightarrow \mathbb{R}^{n_1} \times \mathbb{C}^2 \\ \lambda &\mapsto \left(\lambda^{(j)}\right)_{j=1}^{2g} \end{aligned}$$

For the definition of the Zeta function associated to a ray class field, the norm and trace of a general number field L shall be introduced. The norm will be the product of the images through all the embeddings (the reader can check the norm definition as an exercise) and the trace will be defined as the summation of all the images.

$$\begin{aligned} \mathbf{N}_{L/\mathbb{Q}} : K &\to \mathbb{R} \\ \lambda &\mapsto \prod_{j=1}^{2g} \lambda^{(j)} \end{aligned} \qquad\qquad \begin{aligned} \mathbf{Tr}_{L/\mathbb{Q}} : K &\to \mathbb{R} \\ \lambda &\mapsto \sum_{j=1}^{2g} \lambda^{(j)} \end{aligned}$$

For the scope of this document, it is also necessary to introduce the Ring of Integers of a given number field.

Definition 1.3 (Ring of Integers)**.** *Given any number field L, its integers ring is defined as the collection of elements of the field that are zeros of a monic polynomial with integer coefficients. We denote this ring as* $\mathcal{O}_L$.

There are propositions in the literature that give a concrete expression for the rings of integers of relative quadratic extensions but they will not be useful for this document since we focus on the theoretical generalization of the Charollois-Darmon conjecture and they can be easily computed. However, it is interesting to point out that, since $[K : F] = 2$ and F has a trivial class number (defined on the following pages), there will exist $\omega_1, \omega_2 \in K$ such that

$$\mathcal{O}_K = \omega_1\mathcal{O}_F + \omega_2\mathcal{O}_F$$

Since $\mathcal{O}_K$is a free $\mathcal{O}_F$-module and using the fact that $\mathcal{O}_F$ is a Principal Ideal Domain, we have that there exists a number $\tau \in \mathcal{O}_K$ such that

$$\mathcal{O}_K = \mathcal{O}_F + \tau\mathcal{O}_F$$

Despite not giving the general expression of the rings of integers, it is interesting to introduce the expressions for quadratic extensions over $\mathbb{Q}$

Proposition 1.4. *Given a quadratic extension* $R = \mathbb{Q}[x]/(x^2 - \mathcal{D})$*, with* $\mathcal{D}$ *a positive integer free of squares and* $h \in R$ *the general generator of the field, we will have two possible options for the expression of its integer ring.*

(i) If $\mathcal{D} \equiv 2, 3$ *(mod 4), then the integer ring is* $\mathcal{O}_R = \mathbb{Z} + h\mathbb{Z}$.

(ii) If $\mathcal{D} \equiv 1$ *(mod 4), then the integer ring is* $\mathcal{O}_R = \mathbb{Z} + \left(\frac{1+h}{2}\right)\mathbb{Z}$.

We shall also introduce a notation for the collection of all the fully positive elements in the ring of integers of a given number field L.

$$\mathcal{O}_L(\infty) := \left\{\lambda \in \mathcal{O}_K \ : \ \lambda >> 0\right\}$$

Before stating the definition of lattice zeta functions we need to introduce the concept of lattices over K. This concept comes from the lattices in $\mathbb{R}^n$: Given a subgroup $L \subseteq \mathbb{R}^n$ is called a lattice if

$$L \cong \mathbb{Z}^n \quad \text{and} \quad L \text{ is discrete}$$

To generalize the concept of lattice to K, and for the future enunciation of the Stark conjecture, we need to give the definition of the fractional ideal over $\mathcal{O}_K$. These ideals are $\mathcal{O}_K$-modules which satisfy that its multiplication by an element of K is included in the ring of integers.

$$\mathcal{I}(K) := \left\{\varnothing \neq I \ \mathcal{O}_K\text{-module} \ : \ \exists\lambda \in K^\times \text{ s.t. } \lambda I \subseteq \mathcal{O}_K\right\}$$

We can also provide a norm to the collection of fractional ideals in K by considering the absolute value of determinant representative matrix U_I of the map that sends the basis of a concrete fractional ideal $I \subseteq K$ into the basis of $\mathcal{O}_K$

$$\mathbf{N}(I) := |\det(U_I)|$$

We shall point out that, as it is widely known, the norm of a fractional ideal $I \subset K$ can be related to an expression that depends on the element $\alpha \in K$ satisfying $\alpha I \subseteq \mathcal{O}_K$

$$\mathbf{N}(I) = \frac{[\mathcal{O}_K : \alpha I]}{|\mathbf{N}(\alpha)|}$$

After introducing fractional ideals, it is possible to induce a similar object to real latices in an arbitrary number field L using the natural embedding into $K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{2g}$.

Definition 1.5 (Lattice in a number field)**.** *Given a fractional ideal of K, it will be called a lattice if its image through the natural embedding $K \hookrightarrow K \otimes_{\mathbb{Q}} \mathbb{R}$ and the usual isomorphism between $K \otimes_{\mathbb{Q}} \mathbb{R}$ and $\mathbb{R}^{2g}$ is a lattice*

$$K \hookrightarrow K \otimes_{\mathbb{Q}} \mathbb{R} \cong \mathbb{R}^{n_1} \times \mathbb{C} \cong \mathbb{R}^{n_1+2}$$

With the concept of the lattice of a number field defined, it is possible to introduce the dual of a lattice, which will be a fractional ideal (left as an exercise)

$$\mathfrak{n}^* := \left\{ \lambda \in K : \mathbf{Tr}_{K/\mathbb{Q}}(\lambda n) \in \mathbb{Z},\ \forall n \in \mathfrak{n} \right\}$$

The invertible of a lattice can not be defined in general, however, we can give proportionate an invertibility property for any fractional ideal $\mathfrak{n} \subseteq K$

$$\mathfrak{n}^{-1} := \left\{ \lambda \in K \ : \ \lambda \mathfrak{n} \subseteq \mathcal{O}_K \right\} \text{ if } \mathfrak{n} \not\subseteq \mathcal{O}_K \text{ or } \mathfrak{n}^{-1} := \{ \lambda \in K \ : \ \lambda \mathfrak{n} \subseteq \mathfrak{n} \} \text{ if } \mathfrak{n} \subseteq \mathcal{O}_K$$

The reader should observe, as we pointed out in the introduction, that not all fractional ideals satisfy $\mathfrak{n}\mathfrak{n}^{-1} = \mathcal{O}_K$, in fact, if a fractional ideal satisfies this equation we say it is an *invertible ideal* (if we consider the maximal order all ideals are invertible). After specifying the concepts of lattice and fractional ideal of K we are going to give the definition of two interesting values. We start with the discriminant of a number field, which is a value that will be useful in the description of the Fourier series of the lattice Eisenstein series.

Definition 1.6 (Discriminant of a number field)**.** *Let L be a number field and $a_1, ..., a_{2g} \in K$ a $\mathbb{Z}$-basis of $\mathcal{O}_L$, the discriminant of L is defined, denoting it by d_L, as*

$$d_L := \left(\det \begin{pmatrix} a_1^{(1)} & a_2^{(1)} & \cdots & a_{2g}^{(1)} \\ a_1^{(2)} & a_2^{(2)} & \cdots & a_{2g}^{(2)} \\ \vdots & \vdots & \ddots & \vdots \\ a_1^{(2g)} & a_2^{(2g)} & \cdots & a_{2g}^{(2g)} \end{pmatrix} \right)^2$$

In other words, the discriminant of a number field is the square of the area covered by the images of the elements in a given $\mathbb{Z}$-basis of $\mathcal{O}_K$. Similarly to the concept of the ring of integers, we shall give an explicit expression of the discriminant of totally real quadratic extensions which will be useful for computations.

Proposition 1.7. *Given a quadratic extension $R = \mathbb{Q}(x^2 - \mathcal{D})$, with $\mathcal{D}$ a positive integer free of squares, the discriminant will be one of the following two options*

(i) If $\mathcal{D} \equiv 2, 3$ (mod 4), the discriminant of R is $d_R = 4p$.

(ii) If $\mathcal{D} \equiv 1$ (mod 4), the discriminant of R is $d_R = p$.

Proof. This statement is an immediate result of Proposition 1.4 and the definition of the discriminant. □

We shall also introduce the regulator of a number field that will also be useful for the Eisenstein series section.

Definition 1.8 (Regulator of a number field). *Let L be a number field, n_1 the number of real places (we assume they are the first n_1 ones) of L and n_2 the number of complex places. From Dirichlet's unit theorem, we can extract that the rank of $\mathcal{O}_K^\times$ is $n = n_1 + n_2 - 1$, consequently, there exist a collection of generators $\epsilon_1, ..., \epsilon_n \in \mathcal{O}_L^\times$. We define the regulator of the number field L, denoted by R_L, as*

$$R_L := \left| \det \begin{pmatrix} \log|\epsilon_1^{(2)}| & \log|\epsilon_2^{(2)}| & \cdots & \log|\epsilon_n^{(2)}| \\ \log|\epsilon_1^{(3)}| & \log|\epsilon_2^{(3)}| & \cdots & \log|\epsilon_n^{(3)}| \\ \vdots & \vdots & & \vdots \\ \log|\epsilon_1^{(n_1)}| & \log|\epsilon_2^{(n_1)}| & \cdots & \log|\epsilon_n^{(n_1)}| \\ \log|\epsilon_1^{(n_1+1)}|^2 & \log|\epsilon_2^{(n_1+1)}|^2 & \cdots & \log|\epsilon_n^{(n_1+1)}|^2 \\ \vdots & \vdots & & \vdots \\ \log|\epsilon_1^{(n+1)}|^2 & \log|\epsilon_2^{(n+1)}|^2 & \cdots & \log|\epsilon_n^{(n+1)}|^2 \end{pmatrix} \right|$$

1.1 Zeta functions associated to Ray Class Fields

In this section, we will define the lattice zeta functions and, the ones of interest for this thesis, the zeta functions associated to a ray class field over the number field K (defined on the general distribution before).
Given two numbers $a, b \in K$ and a fractional ideal $\mathfrak{n} \subseteq K$, we define the following subset of $\mathcal{O}_K$

$$\mathcal{V}_{a,b,\mathfrak{n}} := \left\{ \varepsilon \in \mathcal{O}_K : (\varepsilon - 1) \in \mathfrak{n},\ (\varepsilon - 1)b \in \mathfrak{n}^*,\ (\varepsilon - 1)ab \in \mathcal{O}_K^* \right\}$$

For the definition of the zeta functions we are only interested on the fully positive units of this last subset, therefore we consider $\mathcal{V}_{a,b,\mathfrak{n}}^+ = \mathcal{V}_{a,b,\mathfrak{n}} \cap \mathcal{O}_K^\times(\infty)$. We shall also consider a sign function: for the rest of the document we consider

$$\omega_K = \text{sign} \circ \mathbf{N}_{K/\mathbb{Q}}$$

In both of the cases we are going to consider, K Totally Real and K ATR, this function works for the expressions that we want to prove, however, the reader should appreciate that the sign functions that we need in order for the zeta function to vanish at $s = 0$ are different but $\omega_0 = \omega_1$ in the ATR case.
After having introduced these two concepts we are in position of giving the definition of lattice zeta functions: Given a lattice $\mathfrak{n} \subseteq K$, two elements $a, b \in K$ and $s \in \mathbb{C}$ such that $\mathfrak{R}(s) > 1$, the lattice zeta function, denoted as $Z_\mathfrak{n}(a, b; s)$, can be defined as the summation

$$Z_\mathfrak{n}(a, b; s) = \mathbf{N}(\mathfrak{n})^s \sum_{\substack{x+a \in \left(\mathcal{V}_{a,b,\mathfrak{n}}^+ \backslash (\mathfrak{n} + a) \right) \\ x+a \neq 0}} \frac{\omega_K(x + a) e^{2\pi i\ \mathbf{Tr}_{K/\mathbb{Q}}(b(x+a))}}{\left| \mathbf{N}_{K/\mathbb{Q}}(x + a) \right|^s}$$

It is remarkable that for some variables $a, b \in K$ the summation of the numerator can be bounded and that the denominator will tend to infinity for $s > 0$ as the norm of the elements in $\mathfrak{n}$ grow. This implies, by Dirichlet theorem, that $Z_\mathfrak{n}(a, b; s)$ is also well defined for $\mathfrak{R}(s) > 0$ which does not happen on the uncompleted Dirichlet L-function case. To prove the meromorphic continuation and, therefore, define the Lattice zeta function over $\mathbb{C}$ with the exception of a possible pole in $s = 1$, we need to introduce a function ϕ

$$\phi(s) = |d_K|^{s/2} \pi^{-(n_1+1)s/2} 2^{1-s} \Gamma(s) \left(\Gamma\left(\frac{s+1}{2} \right) \right)^{n_1}$$

The definition of the Lattice zeta functions comes from the analytic continuation of the product $\varphi\Psi$ which is normally denoted as the Completed zeta function. We will use the following theorem that appears in the paper of Chapdelaine [Cha08, pp. 2, 3] to define these functions (adapted to our case).

Theorem 1.9 (Compleated Lattice zeta functions)**.** *Let*

$$\phi(s) Z_{\mathfrak{n}}(a,b;s)$$

be the completed zeta function of $Z_{\mathfrak{n}}(a,b;s)$. Then $\widehat{Z}_{\mathfrak{n}}(a,b;s)$ admits an analytic continuation to $\mathbb{C}$ and, therefore, has no pole at $s=1,0$. Moreover, $\widehat{Z}_{\mathfrak{n}}(a,b;s)$ satisfies the following functional equation

$$\widehat{Z}_{\mathfrak{n}^*}(-b,a;1-s) = \mathrm{i}^{2(g-1)}\, e^{-2\pi\,\mathrm{i}\,\mathbf{Tr}_{K/\mathbb{Q}}(ab)} \widehat{Z}_{\mathfrak{n}}(a,b;s)$$

The meromorphic continuation of the completed zeta function simultaneously also gives a meromorphic continuation for the lattice zeta function over $\mathbb{C}$ (dividing $\widetilde{Z}_{\mathfrak{n}}(a,b;s)$ by $\phi(s)$). Moreover, the functional equation of $\widetilde{Z}_{\mathfrak{n}}(a,b;s)$ also induces a functional equation for $Z_{\mathfrak{n}}(a,b;s)$

$$Z_{\mathfrak{n}^*}(-b,a;1-s) = \varphi(s)\,\mathrm{i}^{2(g-1)}\, e^{-2\pi\,\mathrm{i}\,\mathbf{Tr}_{K/\mathbb{Q}}(ab)} Z_{\mathfrak{n}}(a,b;s)$$

where the function $\varphi(s)$ is defined as

$$\varphi(s) = |d_K|^{s-1/2}\pi^{-(n_1+1)(s-1/2)}2^{1-2s}\frac{\Gamma(s)}{\Gamma(1-s)}\left(\frac{\Gamma\left(\frac{1+s}{2}\right)}{\Gamma\left(1-\frac{s}{2}\right)}\right)^{n_1}$$

For the main study of this document, the Stark conjecture, it is important to have a good understanding of the first derivative of the functions at $s=0$. For the Dirichlet L-functions, we do not have a general expression of the first derivative, however, the lattice zeta functions are defined over all the complex plane. Consequently, H. Chapdelein worked out an expression of the first derivative in $s=0$ for this type of functions.

$$\frac{\partial}{\partial s} Z_{\mathfrak{n}}(a,b;s)\bigg|_{s=0} = \frac{\pi}{2}\,\mathrm{i}^{2(g-1)}\, e^{2\pi\,\mathrm{i}\,\mathbf{Tr}(ab)}|d_K|^{1/2} Z_{\mathfrak{n}^*}(-b,a;1) \tag{12}$$

It is also interesting to point out that H. Chapdelaine worked out a relationship between the lattice zeta functions and the Hecke L-functions, which are Dirichlet L-functions with a restrictive congruence in their ideals of definition, in [Cha10, p. 812]. The following proposition, extracted from the paper of Chapdelaine, expresses this relation. The proposition uses some concepts we have not introduced (the ideal class group will be introduced in the next chapter) but the reader can appreciate that Hecke L-functions can be expressed as a summation of lattice zeta functions multiplied by some scalars which will imply some relation between the Stark numbers of both functions (will be discussed in the next subsection).

Proposition 1.10 (Relation between the lattice zeta and the Hecke L functions)**.** *Let $\mathcal{O}$ be an arbitrary $\mathbb{Z}$-orde of K and let $\chi : I_{\mathcal{O}}(f)/P_{\mathcal{O},1}(f\infty) \to \mathbb{C}^\times$ be a primitive Hecke character. Then*

$$\sum_{c\in\left(I_{\mathcal{O}}(f)\big/P_{\mathcal{O},1}(f)\right)} \overline{\chi}(\mathfrak{a}_c) Z_{f\mathfrak{a}_c^{-1}}(\chi_\infty;s) = g(\chi,1)L(\chi,s)$$

where $\mathfrak{a}_c \in c$ is an arbitrary chosen integral invertible $\mathcal{O}$-ideal in the class of c.

For the generalisation of the Charollois-Darmon Conjecture that will be presented in this document, we will work with a subcollection of the uncompleted lattice zeta functions. This collection will be given by a lattice that will depend on an element $f \in \mathcal{O}_K\backslash\{0\}$ and $a=0$, which will be called the level of the zeta function,

$$Z^f_{\mathfrak{n}}(b;s) = Z_{f\mathfrak{n}^{-1}}(0,b;s),\ \text{with } \mathfrak{n}\subseteq K \text{ coprime with } f\mathcal{O}_K$$

these functions are called zeta functions associated to a ray class field. From this point forward when I mention the zeta functions without specifying the type, I will be referring to the zeta functions associated to a ray class field of level $f \in \mathcal{O}_K\backslash\{0\}$ and $a=0$. We will finish the subsection by introducing two lemmas that will be useful for the Stark Conjecture.

Lemma 1.11. *If the exponential variable satisfies $b \in \left(\mathfrak{n}^{-1}\right)^*$, then the exponential term will be invariant by any $\lambda \in K$ satisfying $\lambda >> 0$ and $\lambda \equiv 1$ (mod f).*

Proof. Given a $\lambda \in K$ satisfying the properties specified on the statement of the lemma, the basic properties of the exponent function and the trace of K imply the following equation for any $\mu \in \mathfrak{n}^{-1}$

$$e^{2\pi \mathrm{i}\,\mathbf{Tr}(b\lambda\mu)} = e^{2\pi \mathrm{i}\,\mathbf{Tr}(b\mu)} e^{2\pi \mathrm{i}\,\mathbf{Tr}(b(\lambda-1)\mu)}, \quad \forall \mu \in f\mathfrak{n}^{-1}$$

The exponent term will be invariant when the second term of the product is trivial, which is the same as asking the trace of this term to be an integer. Since λ is 1 mod f, we have that $(\lambda - 1)\mu \in \mathfrak{n}^{-1}$ and, therefore, the trace will be an integer if and only if $b \in \left(\mathfrak{n}^{-1}\right)^*$

$$e^{2\pi \mathrm{i}\,\mathbf{Tr}(b(\lambda-1)\mu)} = 1 \iff \mathbf{Tr}(b(\lambda-1)\mu) \in \mathbb{Z} \iff b \in \left(\mathfrak{n}^{-1}\right)^*$$

which directly implies the statement of the proposition. □

After proving this lemma we are in position of giving the conditions that make the zeta functions associated to a ray class field invariant through elements satisfying the same properties as λ, which will give us an interesting property of the Stark Numbers on the following subsection.

Proposition 1.12. *Given a level $f \in \mathcal{O}_K \backslash \{0\}$, a lattice $\mathfrak{n} \subset K$ and assuming that $b \in \left(\mathfrak{n}^{-1}\right)^*$, we have that the zeta function associated to a ray class field of level f will be invariant by multiplication of $\lambda \in K$ satisfying $\lambda >> 0$ and $\lambda \equiv 1$ (mod f)*

Proof. Given any arbitrary $\lambda \in K$ satisfying $\lambda >> 0$ and $\lambda^{-1} \equiv 1$ (*mod* f), the last lemma and the fact that the element is fully positive imply that for $\Re(s) > 1$ the expressions of the zeta functions are the same

$$Z^f_{\lambda\mathfrak{n}}(b; s) = \mathbf{N}\left(\lambda^{-1}\mathfrak{n}^{-1}\right)^s \sum_{\substack{\lambda^{-1}x \in \left(\mathcal{V}^+_{0,b,\mathfrak{n}^{-1}} \backslash (\lambda\mathfrak{n})^{-1}\right) \\ x \neq 0}} \frac{\omega_K(\lambda^{-1}x) e^{2\pi \mathrm{i}\ \mathbf{Tr}_{K/\mathbb{Q}}(\lambda^{-1}bx)}}{\left|\mathbf{N}_{K/\mathbb{Q}}(\lambda^{-1}x)\right|^s} =$$

$$= \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\lambda^{-1})\mathbf{N}\left(\mathfrak{n}^{-1}\right)}{\mathbf{N}_{K/\mathbb{Q}}(\lambda^{-1})}\right)^s \sum_{\substack{\lambda^{-1}x \in \left(\mathcal{V}^+_{0,b,\mathfrak{n}^{-1}} \backslash (\lambda\mathfrak{n})^{-1}\right) \\ x \neq 0}} \frac{\omega_K(\lambda^{-1})\omega_K(x) e^{2\pi \mathrm{i}\ \mathbf{Tr}_{K/\mathbb{Q}}(bx)}}{\left|\mathbf{N}_{K/\mathbb{Q}}(x)\right|^s} = Z^f_{\mathfrak{n}}(b; s)$$

which implies that their meromorphic expressions are also equal and, therefore, we can conclude that they are equivalent by the multiplication of λ. □

1.2 Stark Conjecture

The Stark conjecture predicts general expressions for the first non-zero coefficients of the Taylor expansion of all the L functions at $s = 0$. The rank n Stark conjecture refers to the L functions which vanish in all the derivatives until $n - 1$ and its n derivative is conjectured to be the determinant of a matrix that has as entry logarithms of algebraic numbers $\{u_{1,1}, u_{1,2}, ...\}$ up to a rational coefficient $C \in \mathbb{Q}$. Therefore the Taylor expansion in $s = 0$ is

$$L(\chi, s) = C \left| \det \begin{pmatrix} \log(u_{1,1}) & \log(u_{1,2}) & \cdots & \log(u_{1,n}) \\ \log(u_{2,1}) & \log(u_{2,2}) & \cdots & \log(u_{2,n}) \\ \vdots & \vdots & \ddots & \vdots \\ \log(u_{n,1}) & \log(u_{n,2}) & \cdots & \log(u_{n,n}) \end{pmatrix} \right| s^n + O(s^{n+1})$$

The conjecture does not give an analytic expression for these numbers, which are also not uniquely defined, (C is determined) but gives the concrete fields where they are integers. In our particular case, the lattice zeta functions will vanish for $s = 0$ (citation) and, using the expression we mentioned of the first derivative depending on the

value in $s = 1$ we can also deduce that the first derivative will not vanish, therefore, I will only enunciate the rank one Stark Conjecture in this document.
In order to describe where these algebraic numbers live, it is essential to define the ray class field of a level $f \in \mathcal{O}_K \backslash \{0\}$. For this concept, we need to introduce the following equivalent relation on fractional ideals. Given two fractional ideals $I, J \subseteq K$

$$I \sim_f J \iff \exists \lambda \in K \text{ s.t. } \lambda >> 0,\ \lambda \equiv 1 \ (\mathrm{mod}\ f) \text{ and } I = \lambda J$$

The ray class field, which is simply called narrow class field if $f = 1$, is the quotient of fractional ideals by this relation

$$\mathrm{Cl}_f^+(K) = \mathcal{I}(K) \Big/ {\sim_f}$$

We can also think about the ray class group as a quotient that measures how many essentially different fractional ideals are in K up to multiplication of a full positive element congruent to one modulus f. The elements of this quotient are closely related to the value in $s = 0$ of the associated derivatives as we will see in the following proposition

Proposition 1.13. *The derivatives of all the zeta functions of level $f \in \mathcal{O}_K$ have the same value at $s = 0$ for all the representatives of a given class $[\mathfrak{n}] \in \mathrm{Cl}_f^+(K)$, if $b \in \left(\mathfrak{N}^{-1}\right)^* \supset \mathcal{O}_K$ where $\mathfrak{N}$ is the integer representation of the class.*

Proof. This proof is a direct implication of Proposition 1.12 and the definition of the relation $\sim_f$ on the definition of the Ray Class group. However, if the reader wants to see a more general proof for zeta functions associated to a ray class field the author recommend reading the proof in [Cha10, p. 806]. □

To be consequent with this last result, we will assume from now on that the variable b is in the dual lattice of all the integer representatives of the elements on the ray class field. From Class Field Theory, there exists an abelian extension H/K such that its Galois group is isomorphic to the ray class group

$$\mathrm{Gal}(H/K) \cong \mathrm{Cl}_f^+(K)$$

For the formulation of the Stark Conjecture, we need to relate the ideals of $\mathrm{Cl}_f^+(K)$ with the elements of $\mathrm{Gal}(H/K)$. This relation is described using the Frobenius Symbol which is defined in the following proposition.

Proposition 1.14 (Frobenius Symbol)**.** *Given an unramified prime ideal $\mathfrak{p} \subseteq K$ (i.e. it splits in H) and $\mathcal{P} \subseteq H$ satisfying $\mathfrak{p} = \mathcal{P} \cap \mathcal{O}_K$, there exists a unique element $\sigma \in \mathrm{Gal}(H/K)$ such that for all elements $\alpha \in \mathcal{O}_H$*

$$\sigma(\alpha) \equiv \alpha^{\mathbf{N}(\mathfrak{p})} \ (\mathrm{mod}\ \mathcal{P})$$

Where $\mathbf{N}(\mathfrak{p})$ is the cardinality of the quotient $\mathcal{O}_K/\mathfrak{p}$. The element σ is the Frobenius element of $\mathfrak{p}$ and is denoted as $\left(\frac{H/K}{\mathfrak{p}}\right)$.

Proof. It is not interesting for the scope of this document to specify all the details of this proof. However, in the book of J.S. Milne [Mil08, pp. 117–118], the reader can get a more extensive definition of the Frobenius Symbol, which can be defined explicitly, and the proof of all the statements of this proposition. □

Using the Frobenius Symbol, we can define a function from the ray class group to the Galois Group of the extension H/K.

$$\begin{aligned} \mathrm{rec} : \mathrm{Cl}_f^+(K) &\to \mathrm{Gal}\,(H/K) \\ \mathfrak{p} &\longmapsto \left(\frac{H/K}{\mathfrak{p}}\right) \end{aligned}$$

This function is well defined because given two representatives of a class $[\mathfrak{a}] = [\mathfrak{b}] \in \mathrm{Cl}_f^+(K)$ and the element $\lambda \in K$ that relates both, we have for all $\alpha \in \mathcal{O}_H$

$$\left(\frac{H/K}{\mathfrak{a}}\right)(\alpha) \equiv \alpha^{\mathbf{N}(\mathfrak{a})} \equiv \alpha^{\lambda\mathbf{N}(\mathfrak{b})} \pmod{\lambda\mathfrak{B}} \Longleftrightarrow \left(\frac{H/K}{\mathfrak{a}}\right)(\alpha) \equiv \alpha^{\mathbf{N}(\mathfrak{b})} \equiv \left(\frac{H/K}{\mathfrak{b}}\right)(\alpha) \pmod{\mathfrak{B}}$$

For any real place $v \in S_\infty$ there exists an integer $\alpha_v \in \mathcal{O}_K$ such that $\alpha_v \equiv 1 \pmod f$ and

$$\operatorname{sign}\left(v(\alpha_v)\right) = -1 \text{ and } \operatorname{sign}\left(v'(\alpha_v)\right) \text{ for all } v' \neq v \text{ real}$$

The elements α_v associated to each real place of S_∞ help us inducing a complex conjugation for any place of K. The general expression of each place image is

$$c_v = \begin{cases} \operatorname{rec}\left(\alpha_v \mathcal{O}_K\right) & \text{if } v \text{ is real} \\ \mathrm{Id} & \text{if } v \text{ is complex} \end{cases}$$

Since K is a two-degree extension of F, we have taht any arbitrary fractional ideal $\mathfrak{a} \subseteq K$ is a two degree $\mathcal{O}_F$-module, consequently, there exist $\omega_1, \omega_2 \in K$ such that $\mathfrak{a} = \omega_1\mathcal{O}_F + \omega_2\mathcal{O}_F$. To have a simplified expression of all the elements of the ray class field we shall assume that the narrow class group of F is trivial (i.e. all the fractional ideals of $\mathcal{O}_F$ are principal ideals from a fully positive element of F). Consequently, we can always consider a totally positive element $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$ such that

$$[\mathfrak{a}] = \left[\nu(\mathcal{O}_F + \tau\mathcal{O}_F)\right]$$

After specifying this simplification, we shall consider a prime $\widetilde{\varpi} \in \mathcal{O}_F$ (following the notation of the introduction) that splits in $\mathcal{O}_F$ as $\widetilde{\varpi} = \varpi \cdot \varpi'$ and state the Rank one Stark Conjecture [Sta80, p. 198] for zeta function of level ϖ. We will state a similar conjecture as the one Charollois and Darmon stated [CD08, p. 657] but we add a fifth classical condition that can be seen in the monograph [Tat84].

Conjecture 1.15 (Rank one Stark Conjecture)**.** *Let e_H be the number of unit roots contained in the field H. For all the classes $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$ and $b \in \left(\mathfrak{N}^{-1}\right)^* \supset \mathcal{O}_K$, where $\mathfrak{N}$ is the integer representation of the class; there exists an algebraic number $u_{[\mathfrak{a}]} \in H$ satisfying*

(i) Let $\widetilde{v}_j$ be a choice of a lift of v_j to H and $|\cdot|$ the common $\mathbb{C}$-norm. The zeta function satisfies

$$\frac{\partial}{\partial s} Z_{\mathfrak{a}}^{\varpi}(b;s)\bigg|_{s=0} = -e_H^{-1} \log |\widetilde{v}_j(u_{[\mathfrak{a}]})|^2$$

(ii) For any class $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$, the element $u_{[\mathfrak{a}]}$ satisfies $c_{v_1}(u_{[\mathfrak{a}]}) = u_{[\mathfrak{a}]}$.

(iii) The other complex conjugations satisfy $c_{v_2}(u_{[\mathfrak{a}]}) = \cdots = c_{v_{2g-1}}(u_{[\mathfrak{a}]}) = u_{[\mathfrak{a}]}^{-1}$

(iv) For any two classes $[\mathfrak{a}], [\mathfrak{b}] \in \mathrm{Cl}_\varpi^+(K)$ the following equation will be satisfied $\operatorname{rec}(\mathfrak{b})(u_{[\mathfrak{a}]}) = u_{[\mathfrak{a}\mathfrak{b}^{-1}]}$.

(v) For any class $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$, the e_H root of $u_{[\mathfrak{a}]}$ generates an abelian extension of K.

Given a class ideal $[\mathfrak{a}] \in \mathrm{Cl}_f^+(K)$, the unit defined in this conjecture $u_{[\mathfrak{a}]} \in \mathcal{O}_H$ will be called the Stark Number associated to the zeta function $Z_{\mathfrak{a}}(a,b;s)$.

The conjecture, on the non p-adic case, has only been proved in the cases where there is a real extension $K/\mathbb{Q}$ and the extension of the ray class field is of degree 2. Therefore, it is interesting to compute numerical examples to test the conjecture in differing cases.
In the cases where the ray class field is totally real (i.e. all its places are real) is relatively easy to compute numerical examples since we only have to compute the value of $Z_{\mathfrak{n}^*}(-b,a;s)$ because of equation 12.

$$\widetilde{v}_j(u_{[\mathfrak{n}]}) = \exp\left(-\frac{\pi}{2} e_H \, \mathrm{i}^{2(g-1)} \, |d_K|^{1/2} Z_{\mathfrak{n}^*}(-b,0;1)\right)$$

It is worth mentioning that various mathematicians have developed different algorithms to compute Stark Numbers in this case concrete case. For example, Takuro Shintani developed analytic expressions using special functions related to the double gamma function [Shi77a] [Shi77b] (he was able to compute 20 decimals of Stark Numbers 46 years ago). For the time being, expressions are being developed using general modular forms and Eisenstein Series (developed and coded in 2022 summer by H. Chapdelaine and me using a trick proposed by Pierre Colmez [Car22a], the reader can see the codes for the Riemann zeta function here [Car22b]).
In contraposition, if the ray class field is not totally positive the Stark Unit $u_{[\mathfrak{n}]}$ might be embedded into a strictly complex number by the place $\widetilde{v}_j$ of H. Therefore, the unit can not be induced by computing the right side of the following equation

$$\left|\widetilde{v}_j(u_{[\mathfrak{n}]})\right|^2 = \exp\left(-\frac{\pi}{2}e_H\,\mathrm{i}^{n_1}\,|d_K|^{1/2}Z_{\mathfrak{n}^*}(-b,0;1)\right)$$

This problem has been an obstacle in obtaining analytic expressions of the Stark Units in the case where H has complex places. One of the first results proposed for ATR fields was given by Samit Dasgupta in his senior thesis [Das99] where he computed the Stark number of a cubic ATR field. However, the more general advanced for complex Stark numbers is the main object of study: The Charollois-Darmon Conjecture.

1.3 Eisenstein Series

This subsection aims to define the lattice Eisenstein series over the number field F and their main properties. To define them and, more generally any modular form, we need to define a subgroup of the Hilbert group $\Gamma = \mathrm{SL}_2(\mathcal{O}_F)$ of F using the modularity respect to the level ϖ.

$$\Gamma^1(\varpi) := \left\{ S \in \Gamma \;:\; S \equiv \begin{pmatrix} 1 & * \\ 0 & 1 \end{pmatrix} \pmod{\varpi\mathcal{O}_F} \right\} \leq \Gamma$$

The modular forms of weight two respect of the subgroup $\Gamma^1(\varpi)$ are functions f that go from $\mathrm{SL}_2(F\otimes_{\mathbb{Q}}\mathbb{R})\big/\mathrm{SO}(2)^g$, which is isomorphic to g copies of the upper half plane $\mathfrak{h}^g$ of $\mathbb{C}$ satisfying

$$f(Sz) = (c^{(1)}z_1 + d^{(1)})^2 \cdots (c^{(g)}z_g + d^{(g)})^2 f(z),\ z \in \mathfrak{h}^g,\ S \in \Gamma^1(\varpi)$$

For the scope of this thesis, we are only interested in the lattice Eisenstein series and not the general modular forms. To give the definition of such series we shall introduce, for any $g \in \mathbb{N}$, over any vector $z = (z_1, ..., z_g)$ in $\mathfrak{h}^g \subset \mathbb{C}^g$, their norm $\mathbf{N}(z) = z_1 \cdots z_g$ and their trace $\mathbf{Tr}(z) = z_1 + \cdots + z_g$. During this chapter when we talk about the summation $z + v$ and the product zv of vectors is understanding the operations over their components $(z_1 + v_1, ..., z_n + v_n)$ and $(z_1v_1, ..., z_nv_n)$, respectively. After the introduction of this notation, we are in position to define the Eisenstein Series

Definition 1.16 (Holomorphic Eisenstein Series of parallel weight 2)**.** *Considering the totally positive generator $\delta \in K$ of $\mathcal{O}_F$, the holomorphic Eisenstein series of parallel weight two is formally defined (the expression we give does not converge but can be extrapolate to the Fourier series of the general Eisenstein series) through the nonzero summation over $\mathcal{O}_F \times \mathcal{O}_F/\mathcal{V}^+_{b,0,\mathcal{O}_F}$ of the following terms*

$$E_2^*(b;z) = \sum_{\substack{(m,n)\in\left(\mathcal{O}_F \times \mathcal{O}_F\big/\mathcal{V}^+_{0,b,\mathcal{O}_F}\right) \\ m,n\neq 0}} \frac{e^{2\pi\,\mathrm{i}\,\mathbf{Tr}_{F/\mathbb{Q}}(\delta b n)}}{\mathbf{N}\left(\delta(mz+n)\right)^2}$$

This function is periodic, a common property of modular forms, which implies it has a Fourier series expansion. We have to define two subsets to give a general expression of this series. The first one collects the products of two lattices

$$\mathcal{D} := \left\{ d \in K^\times \;:\; \exists(\xi_1,\xi_2) \in \left(\mathcal{O}_F^* - b\right) \times \mathcal{O}_F \text{ s.t. } d = \xi_1\xi_2 \right\}$$

The second one concretes the divisors of both lattices given an element of $d \in \mathcal{D}$

$$R_d := \left\{ (\xi_1,\xi_2) \in \left(\mathcal{O}_F^* - b\right) \times \mathcal{O}_F \;:\; \xi_1\xi_2 = d \right\}$$

Having defined both sets, we can give a general expression of the Fourier series expansion of $E_2^*(b;z)$ using the formulas developed by H. Chapdelaine [Cha16, p. 162]

$$E_2^*(b;z) = Z_{\mathcal{O}_F^*}(b,0;\bar{0},2) + (-4\pi^2)^g \sum_{d\in\mathcal{D}} \sigma_1(d) e^{2\pi i Tr(dz)}$$

where $\bar{0}$ represents the trivial sign (the one that sens all the elements to 1) and the function σ_1 is defined as

$$\sigma_1(d) = \sum_{(\xi_1,\xi_2)\in R_d//\mathcal{V}^+_{b,0,\mathcal{O}_F}} |N(\xi_2)|$$

where the double quotient is defined over the following action of $\mathcal{V}^+_{b,0,\mathcal{O}_F}$ over R_d

$$\epsilon \bullet (\xi_1,\xi_2) = (\epsilon\xi_1, \epsilon^{-1}\xi_2),\ \epsilon\in\mathcal{V}^+_{b,0,\mathcal{O}_F},\ (\xi_1,\xi_2)\in R_d$$

It is remarkable that if we assume $b=0$, the Fourier series expansion matches the one that P. Charollois and H. Darmon give in their paper up to some multiplicative factor. In the general definition of lattice Eisenstein series of H. Chapdelaine gives an expression with more variables that $a,b\in K$ with the formal definition

$$E_2^*(U,Z) = \sum_{(m,n)\in\left(\mathcal{O}_F\times\mathcal{O}_F\big/\mathcal{V}^+_{0,b,\mathcal{O}_F}\right)} \frac{e^{-2\pi\,\mathrm{i}\,\mathbf{Tr}(u_1(\delta m+v_1)+u_2(\delta n+v_2))}}{\mathbf{N}\left((\delta m+v_1)z+(\delta n+v_2)\right)^2},\quad U=\begin{pmatrix}u_1 & v_1\\ u_2 & v_2\end{pmatrix}\in\mathcal{M}_2(K)$$

This expression is useful because with it is possible to introduce an equation for the Mobius Transformation of the Eisenstein series

$$E_2(U;\gamma z) = \mathbf{N}(cz+d)^2 E_2((\mathcal{O}_F^*\times\mathcal{O}_F^*)\gamma, U^\gamma; z),\quad \forall\gamma=\begin{pmatrix}a' & b'\\ c & d\end{pmatrix}\in\Gamma^1(\varpi)$$

where

$$U^\gamma = \left(\gamma^{-1}\begin{pmatrix}u_1\\ u_2\end{pmatrix}, \gamma^t\begin{pmatrix}v_1\\ v_2\end{pmatrix}\right)$$

We will consider the following form, which will be essential for the rest of the paper

$$\Omega_{E_2^*}(b;z) := E_2^*(b;z)\,dz_1\wedge\cdots\wedge dz_g$$

We shall also define the following form, which depends on the dimension of the number field F and is based on the form $\Omega_{E_2^*}$

$$\Omega_{Eis}(b;z) := \begin{cases} -2\Omega_{E_2^*}(b;z) + \frac{R_F}{2}\left(\frac{dz_1\wedge d\bar{z}_1}{y_1^2} - \frac{dz_2\wedge d\bar{z}_2}{y_2^2}\right) & \text{if } g=2\\ (-2\,\mathrm{i})^g\Omega_{E_2^*}(b;z) & \text{if } g>2\end{cases}$$

For the rest of the subsection, we are going to define a function whose integral, over a cycle that we will define in the following sections, is equal to this last form. We start introducing the analytic Eisenstein series of weight zero from $\mathfrak{h}^g$ to $\mathbb{C}$ through the lattice of $\mathcal{O}_F^*\times\mathcal{O}_F^*$, which will be defined, for $\mathfrak{R}(s)>1$, as

$$E_0^*(b;z,s) = \sum_{(m,n)\in\left(\mathcal{O}_F\times\mathcal{O}_F\big/\mathcal{V}^+_{0,b,\mathcal{O}_F}\right)} \frac{e^{2\pi\,\mathrm{i}\,\mathbf{Tr}_{F/\mathbb{Q}}(\delta bn)}\mathbf{N}(y)^s}{|\mathbf{N}\left(\delta(mz+n)\right)|^{2s}}$$

This function, similar to the weight two case, has a meromorphic extension to the complex plane which is normally called the Analytic Eisenstein Series. It is remarkable that this function has a pole at $s=1$, consequently, using the Kronecker limit formula, we can deduce the existence of an analytic function $h:\mathfrak{h}^g\to\mathbb{C}$ such that the Taylor expansion of E_0^* is

$$E^*(b;z,s) = 2^{g-2}R_F\left(\frac{1}{s-1}+\gamma_F-\log\mathbf{N}(y)+h(b;z)\right)+O(s-1)\text{ as } s\to 0 \tag{13}$$

To simplify the notation on the proofs we will do in this document, we consider a multiple of this last function and denote it $\widetilde{h}$. This will be the key function that appears on the corollary of Section 6 which is fundamental to prove the main Theorem 2.1 and Theorem 3.2

$$\lambda_F = 4^{g-1} R_F, \ \ \widetilde{h}(b; z) = \lambda_F h(b; s)$$

To give some interesting properties of the functions h and $\widetilde{h}$, we shall introduce an essential operator for this paper. For every place $v_j \in S_F^\infty$, we define the Hecke Operator at infinity of a given place v_j over g copies of the upper half plane as

$$T_{v_j}(z) := \left(\epsilon_1^{(j)} z_{v_1}, ..., \epsilon_{v_j}^{(j)} \bar{z}_j, ..., \epsilon_{v_g}^{(j)} z_g\right)$$

where $\epsilon_j \in \mathcal{O}_F^\times$ is the unit that satisfies

$$\operatorname{sign}(v_j(\epsilon_j)) = -1 \text{ and } \operatorname{sign}(v_i(\epsilon_j)) = 1 \text{ for all } i \neq j$$

recall that the narrow class group of F is trivial and consequently, these units will always exist. We denote the involution of T_{v_j} over $H^g(\mathcal{X}, \mathbb{C})$ (recall that $\mathcal{X}$ was defined in the introduction and we will define it again in Section 3), defined with the pullback over the differential forms, as $T_{v_j}^*$. After defining this operator we give prove some properties of h and $\widetilde{h}$.

Proposition 1.17. *The functions $h(b; z)$ and $\widetilde{h}(b; z)$ satisfy the following properties*

(i) Both functions are harmonic with respect to all the variables $z_j \in \mathfrak{h}$.

(ii) They satisfy the following equations for $\begin{pmatrix} a' & b' \\ c & d \end{pmatrix} \in \Gamma^2(\varpi)$

$$h(b; Az) = h(b; z) - 2 \log |\mathbf{N}(cz + d)|$$
$$\widetilde{h}(b; Az) = \widetilde{h}(b; z) - 2\lambda_F \log |\mathbf{N}(cz + d)|$$

(iii) The derivative of $\widetilde{h}$ with respect all the complex variables satisfies

$$\frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial z_g} dz_1 \wedge \cdots \wedge dz_g = (-2\,\mathrm{i})^g \Omega_{E_2^*}(b; z)$$

(iv) The derivative of $\widetilde{h}$ with respect to all the complex variables and the conjugation of one satisfies

$$\frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_n} dz_1 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_n = T_{v_j}^* \left((-2\,\mathrm{i})^g \Omega_{E_2^*}(\epsilon_j^{-1} b; z)\right)$$

Proof. The first condition is a direct implication of the definition of the functions h and $\widetilde{h}$, and their relation with E_0^*. We will prove the three following properties separately

(ii) From the functional equation of the function E_0^* over the action of $\Gamma^1(\varpi)$, we get

$$E_0^*(b; Az, s) = E_0^*(b; z, s) \Longleftrightarrow h(b; Az) = h(b; z) - \log\left(\frac{\mathbf{N}(y)}{\mathbf{N}(Ay)}\right)$$

Developing the expression of Az for a general matrix $A \in \Gamma^1(f)$, we get

$$Az = \frac{az + b}{cz + d} = \frac{(az + b)(c\bar{z} + d)}{|cz + d|^2} = \frac{ac|z|^2 + adz + bc\bar{z} + bd}{|cz + d|^2} \Longrightarrow Ay = \frac{(ad - bc)y}{|cz + d|^2} = y|cd + d|^{-2}$$

Using this expression on the first one, we conclude

$$h(b; Az) = h(a, b; z) - \log\left(\frac{\mathbf{N}(y)}{\mathbf{N}(Ay)}\right) = h(b; z) - \log |\mathbf{N}(cz + b)|^2$$

The other expression is trivial after proving the first one using the definition of $\widetilde{h}$.

(iii) By deriving the individual components of E_0^* when $\mathfrak{R}(s) > 1$ and, remembering that E_2^* is defined with the same analytic extension, we have

$$\frac{\partial^g E_0^*(b; z, s)}{\partial z_1 \cdots \partial z_g} = \left(\frac{s}{2i}\right)^g E_2^*(b; z, s-1)$$

On the other hand, by deriving the expression that defines h, we get

$$\frac{\partial^g E_0^*(b; z, s)}{\partial z_1 \cdots \partial z_g} = 2^{g-2} R_F \left(\frac{\partial^g h(b; z)}{\partial z_1 \cdots \partial z_g}\right) + O(s-1) = 2^{-g} \frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial z_g} + O(s-1)$$

Specializing at $s = 0$ we get the equation of the statement.

(iv) By deriving the individual components of E_0^* for $\mathfrak{R}(s) > 1$ and, since E_2^* is defined with the same analytic extension, we have

$$\frac{\partial^g E_0^*(b; z, s)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_g} dz_1 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_g = \left(\frac{s}{2i}\right)^g T_{v_j}^* \left(E_2^*(\epsilon_j^{-1} b; z, s-1) dz_1 \wedge \cdots \wedge dz_g\right)$$

On the other hand, by deriving the expression that defines h, we get

$$\frac{\partial^g E^*(b; z, s)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_g} = 2^{g-2} R_F \left(\frac{\partial^g h(b; z)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_g}\right) + O(s-1) = 2^{-g} \frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_g} + O(s-1)$$

Specializing at $s = 0$ we get the equation of the statement.

□

For the following lemma, we shall consider the function U_{ϵ_j} that applies the matrix $\begin{pmatrix} \epsilon_j & 0 \\ 0 & \epsilon_j^{-1} \end{pmatrix}$ to a modular form. In concrete, using the formula specified before, we have that the image of $\Omega_{E_2^*}$ is

$$U_{v_j}(\Omega_{E_2^*}(b; z)) = \Omega_{E_2^*}(\epsilon_j b; z)$$

The involution T_{v_j} added with this last map will be important for the main theorem of Section 3. Now, we specify a lemma that will be useful for this theorem.

Lemma 1.18. *The form $(U_{v_j} + T_{v_j}^*)\Omega_{Eis}$ is exact.*

Proof. Since our $\widetilde{h}$ function has similar properties to the one presented by P. Charollois and H. Darmon we will reproduce their proof [CD08, pp. 664–665] but instead of assuming $j = 1$, we will work out the general case where $j \in \{1, ..., g\}$. We start defining the $(g-1)$-differential form of $\mathfrak{h}^g$

$$\eta = \frac{\partial^{g-1} \widetilde{h}(b; z)}{\partial z_1 \cdots \partial z_{j-1} \partial z_{j+1} \partial z_g} dz_1 \wedge \cdots \wedge dz_{j-1} \wedge dz_{j+1} \wedge \cdots \wedge dz_g$$

When $g > 2$, by the second property of Proposition 1.17, we have that the form η will be invariant through the Mobius action of $\Gamma^1(\varpi)$ and, consequently, will be a $(g-1)$ differential form of the fundamental domain. Since $\widetilde{h}$ is harmonic, this form is holomorphic through all the variables $z_1, ..., z_{j-1}, z_{j+1}, ..., z_g$, which helps us deduce the following formula

$$d\eta = \left(\frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial z_g} dz_1 \wedge \cdots \wedge dz_g + \frac{\partial^g \widetilde{h}(b; z)}{\partial z_1 \cdots \partial \bar{z}_j \cdots \partial z_g} dz_1 \wedge \cdots \wedge d\bar{z}_j \wedge \cdots \wedge dz_g\right)$$

The properties three and four of the function $\widetilde{h}$ specified in Proposition 1.17 directly imply that $d\eta = (U_{v_j} + T_{v_j}^*)\Omega_{\text{Eis}}$ and, equivalently, that the form $(U_{v_j} + T_{v_j}^*)\Omega_{\text{Eis}}$ is exact.

In the case where $j = 2$ the form η will not be invariant through the action of $\Gamma^1(\varpi)$, consequently, we make a slight modification to the definition of this form in order to make it invariant

$$\eta' := \left(\frac{\widetilde{h}(a,b;z)}{\partial z_j} + \frac{2R_F}{\mathrm{i}\,\widetilde{y}_t} \right) dz_j \text{ where } \widetilde{y}_t := \begin{cases} -y_2 & \text{if } t = 1 \\ y_1 & \text{if } t = 2 \end{cases}$$

The formula of $d\eta$ is adaptable to this case taking under consideration the two following equations

$$d\left(\frac{2R_F}{\mathrm{i}\,y_1} dz_1 \right) = R_F \frac{dz_1 \wedge d\bar{z}_1}{y_1^2} \text{ and } d\left(\frac{-2R_F}{\mathrm{i}\,y_2} dz_2 \right) = -R_F \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}$$

Since the respective terms $dz_1 \wedge d\bar{z}_1/y_1^2$ and $dz_2 \wedge d\bar{z}/y_2^2$ (in the cases $j = 2$ and $j = 1$, respectively) will be invariant through $T^*_{v_j}$ and consequently will not appear on $(U_{v_j} + T^*_{v_j})\Omega_{\mathrm{Eis}}$, we can conclude

$$(U_{v_j} + T^*_{v_j})\Omega_{\mathrm{Eis}} = d\eta'$$

which proves that $(U_{v_j} + T^*_{v_j})\Omega_{\mathrm{Eis}}$ is also exact on the case $g = 2$. □

For every $q \geq 0$, we define $C^0_m(\mathcal{X})$ as the group formed by all the combinations of linear forms with coefficients in $\mathbb{Z}$ of differentially closed cycles of dimension m over $\mathcal{X}$. With this last concept, we can define the following group of integrals

$$\Lambda_{\mathrm{Eis}} := \left\{ \int_C \Omega_{\mathrm{Eis}} \text{ for all } C \in C^0_g(\mathcal{X}) \right\} \subset \mathbb{C}$$

It is remarkable that the group Λ_{Eis} satisfies the following statement that will imply (proven in Section 2), that for any totally real quadratic extension K/F, the values of the lattice zeta functions are always rationals.

Proposition 1.19. *Assuming that the determinant of a certain square matrix $\mathcal{A}$ (see (23) below) is non-vanishing, the group Λ_{Eis} is a lattice of rank one inside $\pi^{2g} \cdot \mathbb{Q} \subseteq \mathbb{C}$.*

We warn the reader that this proof has a level of complexity slightly higher than the rest of the document and we do not introduce from scratch all the concepts we use. However, the author considers that this is an interesting result that should be proven.

Proof. In this proof we will denote $\Gamma = \Gamma^1(\widetilde{\varpi})$ and $\mathcal{X} = \Gamma \backslash \mathfrak{H}^g$. This group acts on the left of $\mathbb{P}^1(F) = \{\frac{a}{c}\}$ by Mobius transformations and its orbit corresponds to the so-called cusps of $\mathcal{X}$. Let $\mathcal{K} := \{1, 2, \dots, p^* := \frac{p-1}{2}\}$. A set of representatives of cusps of Γ is given by

$$C := \left\{ \left[\frac{k}{1}\right], \left[\frac{\widetilde{\varpi}}{k}\right] : k \in \mathcal{K} \right\}, \tag{14}$$

which is a set of cardinality $p - 1$. To each of these cusp we shall associate a holomorphic Eisenstein series of parallel weight 2 relative to Γ. If $c = \left[\frac{\widetilde{\varpi}}{k}\right] \in C$ we associate the lattice Eisenstein series $E^*_2(k;z)$ given by the expression (4) in the introduction. If $c = \left[\frac{k}{1}\right]$ we associate the lattice Eisenstein series

$$G_2(k;z) := E^*_2\left(k; \frac{-\widetilde{\varpi}}{z}\right) = \text{“} \sum_{\substack{(m,n) \in \left(\mathcal{O}_F \times \widetilde{\varpi}\mathcal{O}_F \big/ \mathcal{V}^+_{k,0,\mathcal{O}_F} \right) \\ (m,n) \neq (0,0)}} \frac{e^{2\pi \mathrm{i}\,\mathrm{Tr}(-\delta k n)}}{\mathbf{N}\left(\delta(mz+n)^2\right.} \text{”} \tag{15}$$

where the second equality comes from a direct computation or from the transformation formulas given in [Cha19] for the lattice Eisenstein series. Note that $E^*_2(k;z) = E^*_2(-k;z)$ and $G_2(k;z) = G_2(-k;z)$.

For each $1 \leq k \leq p-1$, we choose a matrix $\alpha_k = \begin{pmatrix} * & b_k \\ * & d_k \end{pmatrix} \in \Gamma_0(\widetilde{\varpi})$ such that $d_k \equiv k \pmod{\widetilde{\varpi}}$ (recall $b_k \equiv 0 \pmod{\varpi}$) and we let $S := \begin{pmatrix} 0 & \widetilde{\varpi} \\ -1 & 0 \end{pmatrix} \in \Gamma_0(\widetilde{\varpi})$ which is an involutive automorphism of $\mathfrak{h}^g$. Note that conjugation by S normalizes the set $\{\alpha_k\}_{1 \leq k \leq p-1}$ modulo Γ. Now note that the subgroup

$$\mathcal{G} := \langle \{\alpha_k \Gamma, S\Gamma \,:\, 1 \leq k \leq p-1\} \rangle / \Gamma \leq N(\Gamma)/\Gamma$$

is of size $2(p-1)$ and that it normalizes the group Γ. Here $N(\Gamma)$ is the normalizer of Γ inside $SL_2(\mathbb{R})^g$. It is clear that the action of α_k on $\mathcal{X}$ depends only on k and not on the special choice of the matrix α_k itself. In particular, the group $\mathcal{G}$ acts on the modular variety $\mathcal{X} = \Gamma \backslash \mathfrak{h}^g$ and therefore on the homology groups, cohomology groups and also on differential forms associated to $\mathcal{X}$. Note that by construction $\mathcal{G}$ acts transitively on the set of cusps $\mathcal{C}$.
Now let

$$\Omega_{Eis}[k] := E_2^*(k;z)dz_1 \wedge \ldots \wedge dz_g \quad \text{and} \quad \Omega_{Eis}^*[k] := G_2(k;z)dz_1 \wedge \ldots \wedge dz_g \tag{16}$$

Since these differential forms are holomorphic they are closed and they thus give rise to cohomology classes in $H^g(\mathcal{X}, \mathbb{C})$. However, these cohomology classes could be potentially trivial. In order to ensure their non-triviality we need to make sure that there exists at least one period over a non-trivial g-cycle which is nonzero.
We now view $\mathcal{X}$ as a Riemannian manifold using the Poincare metric which makes X a Kähler manifold. Let us fix g positive real numbers $Y_1, Y_2, \ldots Y_g \in \mathbb{R}_{>0}$ and consider the horizontal line $L_j \subseteq \mathfrak{h}_j$ ($1 \leq j \leq g$) which is placed at the imaginary part of height Y_j. Consider now the following real analytic manifold (which is not totally geodesic) of real dimension g inside $\mathfrak{h}^g$:

$$R_\infty = L_1 \times \ldots L_g.$$

If we let $\Gamma_{\infty,1} := \left\{ \begin{pmatrix} 1 & x \\ 0 & 1 \end{pmatrix} : x \in \widetilde{\varpi}\mathcal{O}_F \right\} \leq \Gamma$ we see that R_∞ is stable, as a set, under $\Gamma_{\infty,1}$. The quotient $R_\infty/\Gamma_{\infty,1}$ is a g-cycle which is homeomorphic to $(S^1)^g$. Moreover one can check that this quotient injects inside $\mathcal{X}$; let us denote this image by D_∞. However, this g-cycle D_∞ inside $\mathcal{X}$ could potentially be trivial (i.e. a g-boundary). In order to make sure that it is no trivial we look at the periods generated by this g-cycle when integrated against the lattice Eisenstein series. Note that the Riemannian volume of D_∞ is $p\sqrt{d_F}$ and therefore

$$\int_{D_\infty} \Omega_{Eis}[r] = p\sqrt{d_F} \cdot Z_{\delta\mathcal{O}_F}(0,0,\mathbb{1};2) \tag{17}$$

where $\mathbb{1} : F^\times \to \{\pm 1\}$ is the trivial sign character (i.e. for all $x \in F^\times$, $\mathbb{1}(x) = 1$). In particular, note that the right-hand side of (17) does not depend on r. It follows from the Euler product (or the mere fact that the underlying Dirichlet series of the value $Z_{\delta\mathcal{O}_F}(0,0,\mathbb{1};2)$ is a sum of strictly positive numbers) that $Z_{\delta\mathcal{O}_F}(0,0,\mathbb{1};2) \neq 0$ which simultaneously shows that the closed differential form $\Omega_{Eis}[r]$ is not exact and that the cycle D_∞ is not a boundary in $\mathcal{X}$.
Using the Poincare duality and the well-known fact that the universal and cuspidal part of the cohomology of $\mathcal{X}$ are orthogonal to the Eisenstein part (under the Riemannian metric) we find that

$$\Lambda_{Eis} \subseteq \left\langle \left\{ \int_C \omega_{Eis} \,:\, [C] \in H_{g,Eis}(\mathcal{X}, \mathbb{Z}) \right\} \right\rangle_{\mathbb{Z}} \tag{18}$$

where ω_{Eis} is the differential form which appears in the statement of the proposition. However, it is not clear if one can generate $H_{g,Eis}(\mathcal{X}, \mathbb{Z})$, up to a finite index, using only D_∞ and the action of $\mathcal{G}$. In order to guarantee that this is indeed the case we shall impose that a certain period matrix has a non-zero determinant.
Let c be a cusp either of the form $c = \left[\frac{\widetilde{\varpi}}{k}\right]$ or $c = \left[\frac{k}{1}\right]$, for some $k \in \mathcal{K}$, and choose matrices $\gamma_k = \begin{pmatrix} a_k & b_k \\ c_k & d_k \end{pmatrix}, \eta_k = \begin{pmatrix} e_k & f_k \\ g_k & h_k \end{pmatrix} \in SL_2(\mathcal{O}_F)$ such that $\gamma_k\infty = \left[\frac{\widetilde{\varpi}}{k}\right]$ or $\eta_k\infty = \left[\frac{k}{1}\right]$. Let us fix an $r \in \mathcal{K}$.

Using the modularity property of $E_2^*(r;z)$ one finds that that

$$\int_{(\gamma_k)_* D_\infty} \Omega_{Eis}[r] = p\sqrt{d_F} \cdot Z_{\delta\mathcal{O}_F}(0, kr, \mathbb{1}; 2), \tag{19}$$

and

$$\int_{(\eta_k)_* D_\infty} \Omega_{Eis}[r] = p\sqrt{d_F} \cdot Z_{\delta\mathcal{O}_F}(0, r, \mathbb{1}; 2) \tag{20}$$

where we see that the last integral is independent of $k \in \mathcal{K}$. In a similar way a direct computation shows that

$$\int_{(\gamma_k)_* D_\infty} \Omega^*_{Eis}[r] = p\sqrt{d_F} \cdot Z_{\delta\mathcal{O}_F}\left(0, -\frac{d_k r}{\varpi}, \mathbb{1}; 2\right) \tag{21}$$

and

$$\int_{(\eta_k)_* D_\infty} \Omega^*_{Eis}[r] = p\sqrt{d_F} \cdot Z_{\delta\mathcal{O}_F}\left(0, -\frac{h_k r}{\varpi}, \mathbb{1}; 2\right). \tag{22}$$

Consider now the period matrix of size $(p-1)\times(p-1)$ given by

$$\mathcal{A} := \begin{pmatrix} \int_{(\gamma_k)_* D_\infty} \Omega_{Eis}[r] & \int_{(\eta_k)_* D_\infty} \Omega_{Eis}[r] \\ \int_{(\gamma_k)_* D_\infty} \Omega^*_{Eis}[r] & \int_{(\eta_k)_* D_\infty} \Omega^*_{Eis}[r] \end{pmatrix}_{r,k\in\mathcal{K}} \tag{23}$$

It follows from the the functional equation of $Z_{\delta\mathcal{O}_F}(a, b, \mathbb{1}; s)$ (which relates in particular the value $Z_{\delta\mathcal{O}_F}(0, b, \mathbb{1}; 2)$ to $Z_{\mathcal{O}_F}(-b, 0, \mathbb{1}; -1)$) and the main result proved in [Shi76] that the entries of $\mathcal{A}$ has are in $\pi^{2g}\cdot\mathbb{Q}$. We now make the following key assumption: $\det(\mathcal{A}) \neq 0$.
Under this assumption, one can show the following

Lemma 1.20. *The $\mathbb{C}$-span of the cohomology classes generated by the differential forms $\{\Omega_{Eis}[k], \Omega^*_{Eis}[k] : k \in \mathcal{K}\}$ is of dimension $p-1$. Moreover, the $\mathbb{Q}$-span generated by the p-cycles $\{(\alpha_k)_* D_\infty, (\alpha_k S)_* D_\infty : k \in \mathcal{K}\}$ is of dimension $p-1$.*

In particular, it follows from this lemma that the set of p-cycles $\{(\alpha_k)_* D_\infty, (S\alpha_k)_* D_\infty : k \in \mathcal{K}\}$ are $\mathbb{Q}$-linearly independent homology classes which generate the "Eisenstein part" of the homology group $H_g(\mathcal{X}, \mathbb{Q})$. Since $\Omega_{Eis} = \Omega_{Eis}[r]$ for some $r \in \mathcal{K}$ it follows that

$$\Lambda_{Eis} \subseteq \left\langle \left\{ \int_C \Omega_{Eis}[k], \int_C \Omega^*_{Eis}[k] : k \in \mathcal{K}, [C] \in H_{g,Eis}(\mathcal{X}, \mathbb{Z}) \right\} \right\rangle_{\mathbb{Z}} \subseteq \frac{1}{N} \cdot \pi^{2g}\mathbb{Z}. \tag{24}$$

for a large enough integer N. The second inclusion follows from the already proven fact that the $\mathbb{Q}$-span generated by the p-cycles

$$\{(\alpha_k)_* D_\infty, (\alpha_k S)_* D_\infty : k \in \mathcal{K}\}$$

is of dimension $p-1$ and that the period matrix $\mathcal{A}$ has entries in $\pi^{2g}\cdot\mathbb{Q}$. All of this shows that Λ_{Eis} has a $\mathbb{Z}$-rank at most one. Finally, the fact that Ω_{Eis} is of rank one follows from the already proven fact that $\int_{D_\infty} \Omega_{Eis} \neq 0$. □

2 Totally Real Quadratic Extensions

Before proving the theorem that supports the Charollois-Darmon conjecture, which gives an analytic expression of the Stark Numbers for Almost Totally Real extension K, we will consider an easier example where K is a totally real number field (i.e. all its places are real).
Since K is a totally real quadratic relative extension of F all these places will split into two real places in K that we will denote as v_j and v'_j. Given an ideal class $[\mathfrak{a}] \in \mathrm{Cl}^+_\varpi(K)$, since the narrow class field of F is trivial, we

can find a representative of $\left[\left(\mathfrak{a}^{-1}\right)^{*}\right]$ of the form $\nu(\mathcal{O}_F + \tau\mathcal{O}_F)$, with $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$, as a basis. We will denote the different images of τ through the places of K as

$$(\tau_j, \tau_j') := \left(v_j(\tau), v_j'(\tau)\right) \in \mathbb{R}^2, \quad v_j \in S_\infty^F$$

We denote Υ_j as the totally geodesic real analytic submanifold of $\mathfrak{h}^g$ that goes through the point τ_j and τ_j' with orientation from τ_j' to τ_j. These curves generate the following subset of $\mathfrak{h}^g$

$$R_\tau = \Upsilon_1 \times \cdots \times \Upsilon_g \subseteq \mathfrak{h}^g$$

that is isomorphic to a subspace of $\mathbb{R}^g$ with the natural orientation induced by the one of Υ_j. The subgroup $\Gamma_\tau \leq \Gamma^1(\varpi) \leq \Gamma$ is defined as the collection of matrices that fix τ through the Moebius transformation. This subgroup induces an action, by applying the Mobius transformation in each coordinate, in R_τ. The quotient $\mathcal{X} := \Gamma_\tau \backslash R_\tau$ is compact and isomorphic to a real torus of dimension n. Given a fundamental domain for the action of Γ_τ in R_τ, we identify its image in the quotient as Δ_τ, which will be a closed cycle. The cycle Δ_τ induces an analytic expression for the value of the zeta function, for a given class $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$, in $s = 0$.

Theorem 2.1. *Given a class $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$ and its associated element $\tau \in K$, the following equations is satisfied*

$$\int_{\Delta_\tau} \Omega_{\mathrm{Eis}} = \frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F\sqrt{d_F}} Z_{\mathfrak{a}}^{\varpi}(b;0)$$

Proof. The statement is a direct consequence of the Corollary in section 7 of this document. Since in this case, we are considering a totally real extension K/F, and therefore all the places are real ($r = n$), of a totally real number field F, the first condition of the corollary implies

$$\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F\sqrt{d_F}} Z_{\mathfrak{a}}^{\varpi}(b;0) = \frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F\sqrt{d_F}} Z_{\varpi\mathfrak{a}^{-1}}(0,b,0) = \int_{\Delta_\tau} \frac{\partial^g \widetilde{h}}{\partial z_1 \cdots \partial z_g} dz_1 \wedge \cdots \wedge dz_g$$

Using one of the properties of the function $\widetilde{h}$ specified in Proposition 1.3. we have

$$\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F\sqrt{d_F}} Z_{\mathfrak{a}}^{\varpi}(b;0) = \int_{\Delta_\tau} (-\mathrm{i})^g \Omega_{E_2^*}(b/\nu; z)$$

For the case where $g \geq 3$ the statement will already be proven with this last equation. In the case $g = 2$, we will also have the equation with ω_{Eis} because the integrals over $dz_1 \wedge d\bar{z}_1$ and $dz_2 \wedge d\bar{z}_2$ are zero

$$\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F\sqrt{d_F}} Z_{\mathfrak{a}}^{\varpi}(b;0) = \int_{\Delta_\tau} \left[-\omega_{E_2} + \frac{R_F}{2}\left(\frac{dz_1 \wedge d\bar{z}_1}{y_1^2} - \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}\right)\right] = \int_{\Delta_\tau} \omega_{\mathrm{Eis}}$$

□

In the cases where the zeta function does not vanish at $s = 0$, we will have an analytic expression for the first value of the Taylor expansion at $s = 0$. The main idea of the following chapter is to generalize a similar analytic expression when K is an Almost Totally Real Field. Before, however, we can introduce a corollary to this theorem that brings up an interesting fact.

Corollary 2.2. *Given any class $[\mathfrak{a}] \in \mathrm{Cl}_\varpi^+(K)$ the value $\pi^{-2g} Z_{\mathfrak{a}}^{\varpi}(b;0)$ of the lattice zeta function is rational. Furthermore, there exists an integer $p_F \in \mathbb{Z}$ only depending of the field F such that*

$$p_F \pi^{-2g} Z_{\mathfrak{a}}(a,b;0) \in \mathbb{Z}$$

Proof. Using the last theorem and the fact, proven in Proposition 1.3.4., that $\Lambda_{\mathrm{Eis}} \subseteq (2\,\mathrm{i}\,\pi)^g\mathbb{Q}$ and Λ_{Eis} has rank one. These last two facts directly imply the statement of the corollary. □

3 ATR Quadratic Extensions

As we pointed out at the end of the last section, we are going to prove a similar statement to Theorem 2.1 for Almost Totally Rael extensions in order to give some heuristic argument to the generalisation of the Charollois-Darmon Conjecture specified in the following section.
In this case, all the places in F will split into two real places except for the first one will ramify into a complex place. Furthermore, since the narrow class field of F is trivial, given any class $[\mathfrak{a}] \in \mathrm{Cl}^+_{\varpi}(K)$ we have a representative of the form $\nu(\mathcal{O}_F + \tau\mathcal{O}_F)$ for the class $\left[\left(\mathfrak{a}^{-1}\right)\right]$, with $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$. We denote the different images of $\tau \in K$ through the different places of K as

$$\begin{aligned} \tau_1 &:= v_1(\tau) \in \mathfrak{h} \\ (\tau_j, \tau_j') &:= \left(v_j(\tau), v_j'(\tau)\right) \in \mathbb{R}^2 \text{ for } j = 2, ..., g \end{aligned}$$

For $2 \le j \le g$, we can define as Υ_j the totally geodesics real analytic submanifold of $\mathfrak{h}^g$ that go through τ_j and τ_j' with orientation from τ_j to τ_j'. These curves, together with the value τ_1 induce the subspace

$$R_\tau = \{\tau_1\} \times \Upsilon_2 \times \cdots \times \Upsilon_g \subset \mathfrak{h}^g$$

which is isomorphic to a subspace of $\mathbb{R}^{g-1}$ with an orientation induced by the ones in Υ_j. The subgroup Γ_τ, defined in the last section, acts on R_τ by applying the Mobius transformation in all coordinates. Furthermore, the quotient $\mathcal{X} := \Gamma_\tau \backslash R_\tau$ is isomorphic to a real torus of dimension $g - 1$. Giving a fundamental domain of the action Γ_τ over R_τ, we identify Δ_τ as its image on the quotient which will be a closed cycle of dimension $g - 1$. We introduce the following lemma that will be essential for the proof of the main result of this section.

Lemma 3.1. *The class of Δ_τ in $H^{g-1}(\mathcal{X}, \mathbb{Z})$ is of torsion. In particular, there exists an g-differentiable chain C_τ of coefficients inside of $\mathbb{Q}$ such that*

$$\partial C_\tau = \Delta_\tau$$

Proof. Considering the natural map $H^{g-1}(\mathcal{X}, \mathbb{Z}) \to H_{g-1}(\mathcal{X}, \mathbb{Q})$ and the Theorem 6.3. in [Fre90, p. 185], we have that if g is even the homology group is trivial

$$H^{g-1}(\mathcal{X}, \mathbb{Z}) \subseteq H^{g-1}(\mathcal{X}, \mathbb{Q}) = 0$$

On the other hand, when g is odd the rational homology group is $H^{g-1}(\mathcal{X}, \mathbb{Q}) = H^{g-1}_{\mathrm{univ}}(\mathcal{X}, \mathbb{Q})$. Since Δ_τ has dimensions one or two in the projection of R_τ, consequently, the image of Δ_τ in $H^{g-1}_{\mathrm{univ}}(\mathcal{X}, \mathbb{Q})$ is zero. □

Using the form $\Omega^+_{\mathrm{Eis}} = (U_{v_j} + T^*_{v_1})\Omega_{\mathrm{Eis}}/2$, which is the projection of the differential form Ω_{Eis} to the space of $T^*_{v_1}$ associated to the eigenvalue 1 (i.e. can be thought as the real part of the form Ω_{Eis}), we can introduce a similar Theorem to the one stated in the last section.

Theorem 3.2. *Given a class $[\mathfrak{a}] \in \mathrm{Cl}^+_f(K)$ associated to a $\tau \in K$, the following equation is satisfied*

$$\int_{C_\tau} \Omega^+_{Eis} = \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\frac{\partial}{\partial s}Z^{\varpi}_{[\mathfrak{a}]}(b; 0)$$

Proof. This statement is a consequence of the Corollary proven at the end of this document. Remembering that in Lemma 1.3. we proved that ω^+_{Eis} is exact and that is equal to $d\eta/2$ and using the Stokes theorem, we have

$$\int_{C_\tau} \Omega^+_{\mathrm{Eis}} = \frac{1}{2}\int_{\Delta_\tau} \eta = \frac{1}{2}\int_{\Delta_\tau} \frac{\partial^{n-1}\widetilde{h}(z_1, ..., z_g)}{\partial z_2 \cdots \partial z_n} dz_2 \wedge \cdots \wedge dz_n$$

If $g > 2$, we can directly apply the first part of Corollary 7. since we have $c = 1$ and $r = g - 1 \ge 2$, and conclude

$$\int_{C_\tau} \Omega^+_{\mathrm{Eis}} = \frac{1}{2}\int_{\Delta_\tau} \frac{\partial^{n-1}\widetilde{h}(z_1, ..., z_g)}{\partial z_2 \cdots \partial z_n} dz_2 \wedge \cdots \wedge dz_n = \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\frac{\partial}{\partial c}Z_{[\mathfrak{a}]}(b; 0)$$

In the case where $g = 2$ we have one real and one complex places, therefore, we have to use the second part of the Corollary 7. Using the fact that the second-term integral of the equation given by the corollary is zero, we can conclude

$$\int_{C_\tau} \Omega^+_{\mathrm{Eis}} = \frac{1}{2}\int_{\Delta_\tau} \frac{\partial \widetilde{h}(z_1,...,z_g)}{\partial z_2} dz_2 = \frac{1}{2}\int_{\Delta_\tau}\left(\frac{\partial \widetilde{h}(z_1,...,z_g)}{\partial z_2} - \frac{4R_F}{z_2 - \bar{z}_2}\right) dz_2 = \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\frac{\partial}{\partial s} Z^{\varpi}_{[\mathfrak{a}]}(b;0)$$

□

As we pointed out after the enunciation of the Stark Conjecture, with this theorem we are giving an analytic expression of the first derivative of the zeta function in $s = 0$ but does not help in computing the Stark Number, since it can be complex and therefore we lose information by computing its norm.

4 Abel-Jacobi Map and the Charollois-Darmon Conjecture

Assuming that the Stark Conjecture is true, the Theorem 3.0.2. gives us an analytic expression of the norm associated with the element of a given element $[\mathfrak{a}] \in \mathrm{Cl}^+_F(K)$ and the elements $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$ such that $\left[\nu(\mathcal{O}_F + \tau\mathcal{O}_F)\right] = \left[\left(\mathfrak{a}^{-1}\right)^*\right]$

$$e_H \int_{C_\tau} \Omega^+_{Eis} = -2\left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\log|\widetilde{v}_1(u_{[\mathfrak{a}]})|$$

where e_H denotes the number of roots of units in the field H. By definition, we have that $\log|\widetilde{v}_1(u_{[\mathfrak{a}]})| = \mathfrak{R}\left(\log\left(\widetilde{v}_1(u_\tau)\right)\right)$. The conjecture that we are going to present in this section proposes an analytic expression of the complex value $\widetilde{v}(u_{[\mathfrak{a}]})$ using the integral of Ω_{Eis} that its real part the integral of ω^+_{Eis} and, therefore, will satisfy the last equation.

For all $m \geq 0$, we denote the group formed by linear combinations of differential chains of real dimension m in $\mathcal{X}$ with coefficients in $\mathbb{Z}$, as $C_m(\mathcal{X})$. We define the subgroups $C^0_m(\mathcal{X})$ and $C^{00}_m(\mathcal{X})$ of $C_m(\mathcal{X})$ generated by the closed cycles and homologous to zero, respectively

$$C^0_m(\mathcal{X}) := \left\{\Delta \in C_m(\mathcal{X}) \text{ s.t. } \partial\Delta = 0\right\}$$

$$C^{00}_m(\mathcal{X}) := \left\{\Delta \in C_m(\mathcal{X}) \text{ s.t. } \exists C \in C_{m+1}(\mathcal{X}) \text{ with } \partial C = \Delta\right\}$$

After defining these subgroups, we can introduce the Abel-Jacobi application which will be essential for the generalised statement of the Charollois-Darmon Conjectures specified at the end of this section.

Definition 4.1 (Abel-Jacobi application)**.** *Using the differential form Ω_{Eis}, we can define the application of Abel-Jacobi*

$$\Phi : C^{00}_{g-1} \to \mathbb{C}\big/\Lambda_{\mathrm{Eis}}$$

as

$$\Phi_{\mathrm{Eis}}(\Delta) = \int_{\partial C = \Delta} \Omega_{\mathrm{Eis}} \pmod{\Lambda_{\mathrm{Eis}}}$$

It is remarkable that this last integral is well defined because of Proposition 1.19. We also define the subgroup

$$\widetilde{C}^{00}_m(\mathcal{X}) := \left\{\Delta \in C_m(\mathcal{X}) \text{ s.t. } \exists C \in C_{m+1}(X) \otimes \mathbb{Q} \text{ with } \partial C = \Delta\right\}$$

It is known that the quotient $C^0_{g-1}(\mathcal{X})/C^{00}_{g-1} = H_{g-1}(\mathcal{X},\mathbb{Z})$ is a finite group, which has a torsion subgroup defined with $\widetilde{C}^0_{g-1}(\mathcal{X})/C^{00}_{g-1}(\mathcal{X}) =$. Let n_F the exponent of this finite group, and let

$$\Lambda'_{\mathrm{Eis}} := n_F^{-1}\Lambda_{\mathrm{Eis}}$$

If we change the residue of the definition in the Abel-Jacobi application from Λ_{Eis} to Λ'_{Eis}, we can understand Φ_{Eis} over the subgroup $\widetilde{C}^{00}_{m}(X)$ and its expression as

$$\Phi_{\text{Eis}}(\Delta) = n_F^{-1} \int_{\partial C = n_F \Delta} \omega_{\text{Eis}} \qquad (\text{mod } \Lambda'_{\text{Eis}})$$

Since in the proof of the Lemma 3.1 we proved that the g-differentiable chains C of X that satisfy $\partial C = \Delta_\tau$, for a $\tau \in K$, will be equivalent mod Λ'_{Eis}, we can understand

$$J_\tau := e_H \Phi_{\text{Eis}}(\Delta_\tau)$$

as an element in $\mathbb{C}/\Lambda'_{\text{Eis}}$.
Defining $\Lambda_{\text{Eis}''}$ as the reminder of $(\mathrm{i})^{-g}\mathbb{R}$ given by $\Lambda_{\text{Eis}'}$ and $(\mathrm{i})^{-g}\mathbb{Z}$. After fixing one of the splits $\widetilde{v}_1 \in S^H_\infty$ of $v_1 \in S^K_\infty$, we are in the position of giving the generalization of the Charoillois-Darmon Conjecture for Zeta Functions associated to a Ray Class Field.

Conjecture 4.2 (Generalization Charollois-Darmon Conjecture)**.** *For all $\mathcal{O}_K$-module $\mathfrak{n}$ with associated elements $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$, that satisfy $\left[\nu(\mathcal{O}_F + \tau\mathcal{O}_F)\right] = \left[\left(\mathfrak{n}^{-1}\right)^*\right]$, the Stark number $u_\tau \in \mathcal{O}_H$ of $Z^{\varpi}_{[\mathfrak{n}]}(b; s)$ satisfies the equation*

$$J_\tau = -2\left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\log\left(\widetilde{v}_1(u_\tau)\right) \qquad (\text{mod } \Lambda''_{Eis})$$

5 Charollois-Darmon Algorithm

In this section, we will describe the algorithm proposed by P. Charollois and H. Darmon [CD08, pp. 673–677] to give computational examples, for $g = 2$, of their conjecture for the zeta functions when $L(\mathfrak{a}, I, ; s)$ described at the beginning of this document. In their paper, instead of using the form Ω_{Eis}, they use the following form

$$\Omega^{\text{CD}}_{\text{Eis}} := \begin{cases} \frac{(2\pi\,\mathrm{i})^2}{\sqrt{d_F}}\Omega_{E_2^{\text{CD}}} + \frac{R_F}{2}\left(\frac{dz_1\wedge d\bar{z}_1}{y_1^2} - \frac{dz_2\wedge d\bar{z}_2}{y_2^2}\right) & \text{if } g = 2 \\ \frac{(2\pi\,\mathrm{i})^g}{\sqrt{d_F}}\Omega_{E_2^{\text{CD}}} & \text{if } g > 2 \end{cases}$$

where

$$E_2^{\text{CD}}(z) = \zeta_F(-1) + 2^g \sum_{\mu\in\mathcal{O}_F(\infty)} \sigma_1(\mu) e^{2\pi\,\mathrm{i}\,\mathbf{Tr}\left(\frac{\mu}{\delta}z\right)} \text{ and } \sigma_k(\mu) = \sum_{(\nu)|(\mu)} |\mathbf{N}(\nu)|^k$$

In order to compute the invariant J_τ of the Charollois-Darmon conjecture, in the case where $g = 2$, we need to compute the image of the Abel-Jacobi map for Δ_τ mod Λ'_{Eis} where

$$\Lambda''_{\text{Eis}} := \frac{1}{n_F}\left\{\int_C \Omega_{E_2^{\text{CD}}} \text{ for } C \in C^0_g(X)\right\}$$

where X is constructed similarly to our case but taking the Hiblert Group Γ instead of $\Gamma^1(f)$. Given a matrix $A \in \Gamma$ and a point $P \in \mathfrak{h}^2$, we can define the image of the geodesic joining P and AP in X as $S_P(A)$. From Lemma 3.1. in [CD08, p. 670] we know that $H^1(X, \mathbb{Q}) = 0$ and, consequently, there exists a differentiable 2-chain $D_P(A)$ in X with rational coefficients satisfying

$$\partial D_P(A) = n_F S_P(A)$$

It is useful for this section to introduce a notation for the integral of $\Omega^{\text{CD}}_{\text{Eis}}$ over $D_P(A)$ modulus Λ'_{Eis}

$$\rho_P(A) := \frac{1}{n_F}\int_{D_P(A)} \Omega^{\text{CD}}_{\text{Eis}} \ (\text{mod } \Lambda'_{\text{Eis}})$$

P. Charollois and H. Darmon prove in Proposition 5.2. [CD08, p. 674] that this last function of a matrix $\gamma_\tau \in \Gamma$, that fixes τ, is equivalent to $\Phi_{\text{Eis}}(\Delta_\tau)$ for any given point $P \in \mathfrak{h}^2$

$$\Phi_{\text{Eis}}(\Delta_\tau) = \rho_P(\gamma_\tau) \ (\text{mod } \Lambda'_{\text{Eis}})$$

In the following subsection, we will specify how to compute the value $\rho_P(\gamma_\tau)$.

5.1 Computation of the Rho Function

In this subsection, we will specify how to compute the image of the function ρ of a given $\tau \in K$ and its associated matrix $\gamma_\tau \in \Gamma$. From the definition of $d\rho$ and the expression at the end in Lemma 5.1. [CD08, p. 674], we have that for all $A, B \in \Gamma$ and $P \in \mathfrak{h}^2$

$$\kappa_P(A,B) = d\rho_P(A,B) = \rho_P(A) - \rho_P(AB) + \rho_P(B) \Longleftrightarrow \rho_P(AB) = \rho_P(A) + \rho_P(B) - \kappa_P(A,B), \pmod{\Lambda'_{\text{Eis}}}$$

where the function $\kappa_P(A, B)$, for a given $A, B \in \Gamma$ and a fixed point $P \in \mathfrak{h}^2$, is defined as

$$\kappa_P(A,B) := \int_{\triangle_P(A,B)} \Omega^{\text{CD}}_{\text{Eis}} \text{ where } \triangle_P(A,B) \text{ is the triangle between the points } \{P, AP, ABP\}$$

Since the image of the identity matrix is zero [CD08, p. 677], using the last expression, we have that for any $h \in \Gamma$ the image of ρ can be related to the κ function

$$\rho_P(h) + \rho_P(h^{-1}) - \kappa_P(h, h^{-1}) = \rho_P(Id_2) = 0 \Longleftrightarrow \rho_P(h) + \rho_P(h^{-1}) = \kappa_P(h, h^{-1}), \pmod{\Lambda'_{\text{Eis}}}$$

If we consider a commutator $[h, k] = hkh^{-1}k^{-1}$ of elements $h, k \in \gamma$, we can use these two last equations to give a general expression of the image of the ρ function of this commutator

$$\rho_P([h,k]) = \rho_P(h) + \rho_P(h^{-1}) + \rho_P(k) + \rho_P(k^{-1}) - \kappa_P(h, kh^{-1}k^{-1}) - \kappa_P(k, h^{-1}k^{-1}) - \kappa_P(h^{-1}, k^{-1}) =$$

$$= \kappa_P(h, h^{-1}) + \kappa_P(k, k^{-1}) - \kappa_P(h, kh^{-1}k^{-1}) - \kappa_P(k, h^{-1}k^{-1}) - \kappa_P(h^{-1}, k^{-1}), \pmod{\Lambda'_{\text{Eis}}}$$

As P. Charollois and H. Darmon point out in [CD08, p. 677], since we are assuming that the narrow class group of F is trivial, the Hilbert Group Γ will be generated by the fundamental matrices

$$S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}, \quad U = \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^{-1} \end{pmatrix}, \quad T_\mu = \begin{pmatrix} 1 & \mu \\ 0 & 1 \end{pmatrix}, \ \mu \in \mathcal{O}_F$$

We also have that for all matrix $A \in \Gamma$ its power γ^p with $p = 4\mathbf{N}(\varepsilon^2 - 1)$, where ε generates the subgroup of integer units of F, can be written as a product of commutator [CD08, p. 677]. Therefore, for the matrix γ_τ^r, which can be expressed as a product of the fundamental matrices, one can use the following expressions (that come from a trivial computation) and the Euclidean Algorithm to find an expression of commutators of γ_τ^r

$$UT_\mu U^{-1} T_\mu^{-1} = T_{\mu(\varepsilon^2-1)}, \quad SUS^{-1}U^{-1} = U^{-2}, \quad SU^{-1}S^{-1}U = U^2$$

Using the first property of ρ_τ we have shown in this subsection and the fact that γ_τ^r can be expressed as a product of commutator, that we will denote as $[h_j, k_j]$, we can give a general expression for $\rho_P(\gamma_\tau^r)$

$$\rho_P(\gamma_\tau^r) = \sum \left(\rho_P([h_j, k_j]) - \kappa_P\left([h_j, k_j], \prod_{t>j} [h_t, k_t] \right) \right), \pmod{\Lambda'_{\text{Eis}}}$$

Since γ_τ^r is also a product of matrices, we can use the same property to give a general expression of $\rho_P(\gamma_\tau)$ with $\rho_P(\gamma_\tau^r)$ and values of the κ_P function

$$\rho_P(\gamma_\tau^r) = r\rho_P(\gamma_\tau) - \sum_{j=1}^{p-1} \kappa_P(\gamma_\tau, \gamma_\tau^j) \Longleftrightarrow \rho_P(\gamma_\tau) = \frac{1}{r}\left(\rho_P(\gamma_\tau^r) + \sum_{j=1}^{p-1} \kappa_P(\gamma_\tau, \gamma_\tau^j) \right), \pmod{\Lambda'_{\text{Eis}}}$$

The function κ_P is defined as an integral of ω_{Eis} over a triangle [CD08, p. 673]. Using the expressions of $\rho_P(\gamma_\tau^r)$ and $\rho_P(\gamma_\tau)$, we can obtain an expression of $\rho_P(\gamma_\tau)$ with images of commutators through the ρ_P function, which we know how to compute, and the κ_τ function

$$\rho_P(\gamma_\tau) = \frac{1}{r} \sum \left(\rho_P([h_j, k_j]) - \kappa_P\left([h_j, k_j], \prod_{t>j} [h_t, k_t] \right) \right) + \sum_{j=1}^{p-1} \kappa_P(\gamma_\tau, \gamma_\tau^j), \pmod{\Lambda'_{\text{Eis}}}$$

5.2 Computation of the Kappa Function

The algorithm is based on the ability to compute the integral of the differential form ω_{Eis} over a triangle defined over a point $P \in \mathfrak{h}^2$ and two matrices $A, B \in \Gamma$ as $\triangle_P(A, B) = \triangle(P, AP, ABP)$. We will denote this integral as a function, which we will denote as $\kappa_P(A, B)$

$$\kappa_P(A, B) := \int_{\triangle_P(A,B)} \Omega_{\text{Eis}}^{\text{CD}}$$

The proposition 5.3. proven on the paper from P. Charollois and H. Darmon [CD08, p. 675] assures that if we assume $P \in \{\tau_1\} \times \mathfrak{h}$ then

$$\kappa_P(A, B) = \widetilde{\kappa}_P(A, B) := \kappa_P^{\square}(A, B) + \mathrm{i}\,\pi R_F \left(\text{area}(A_2, B_2) - \text{area}(A_1, B_1)\right), \quad (\text{mod } \Lambda'_{\text{Eis}})$$

where the area is defined as $\text{area}(M, N) := -\,\text{sign}(m_{2,1}n_{2,1}(mn)_{2,1})$ and $\kappa_P^{\square}(A, B)$ is the integral of the differential form ω_{Eis} over the rectangle $\square_P(A, B) = \Upsilon[P1, A_1P_1] \times \Upsilon[A_2P_2, A_2B_2P_2]$

$$\kappa_P^{\square}(A, B) = \int_{\square_P(A,B)} \Omega_{\text{Eis}}^{\text{CD}}$$

The computation of the function $\kappa_P^{\square}(A, B)$ will depend on the integration of the Fourier expansion terms of the Eisenstein series

$$\kappa_P^{\square}(A, B) = \int_{\square_P(A,B)} \left[\frac{(2i\pi)^2}{\sqrt{d_F}}\omega_{E_2} + \frac{R_F}{2}\left(\frac{dz_1 \wedge d\bar{z}_1}{y_1^2} - \frac{dz_2 \wedge d\bar{z}_2}{y_2^2}\right)\right]$$

Since we are computing the integral over two geodesics, we can immediately conclude that the second and third terms of the integral will be zero. We have that the integral of the first coefficient, using the Fourier expansion described before, is

$$\kappa_P^{\square}(A, B) = \frac{-4\pi^2}{\sqrt{d_F}}\zeta(-1)A_{\square_P(A,B)} + 4\sqrt{d_F} \sum_{\mu \in \mathcal{O}_F,\ \mu >> 0} \frac{\sigma_1(\mu)}{\mathbf{N}(\mu)} \left(e^{2\pi\mathrm{i}\frac{\mu_1}{\delta_1}A_1P_1} - e^{2\pi i\frac{\mu_1}{\delta_1}P_1}\right)\left(e^{2\pi\mathrm{i}\frac{\mu_2}{\delta_2}A_2B_2P_2} - e^{2\pi\mathrm{i}\frac{\mu_2}{\delta_2}A_2P_2}\right)$$

where the coefficient $A_{\square_P(A,B)}$ is described as

$$A_{\square_P(A,B)} = \left(A_1z_1 - z_1\right)\left(A_2B_2z_2 - A_2z_2\right)$$

This last expression is not the best one to compute numerical approximation since the apportion of the coefficients associated to a $\mu \in \mathcal{O}_F$ will not be related to its norm or trace. Instead, we will use a method proposed by Henri Darmon and Adam Logan in [DL04, p. 2176] where we compute the last summation over all the ideals $\mathcal{J} \in \mathcal{I}(K)$, where $\mathcal{I}(K)$ is the collection of all the ideals in K and $\mu_{\mathcal{J}}$ is a totally positive generator of a given ideal $\mathcal{J}$ (the narrow class field is trivial)

$$\frac{-4\pi^2}{\sqrt{d_F}}\zeta(-1)A_{\square_P(A,B)} + 4\sqrt{d_F} \sum_{\mathcal{J} \in \mathcal{I}(F)} \frac{\sigma_1(\mathcal{J})}{|\mathbf{N}(\mathcal{J})|} \sum_{j \in \mathbb{Z}} \left(e^{2\pi\mathrm{i}\frac{\mu_1^{\mathcal{J}}}{\delta_1}\varepsilon^{2j}A_1P_1} - e^{2\pi i\frac{\mu_1^{\mathcal{J}}}{\delta_1}\varepsilon^{2j}P_1}\right)\left(e^{2\pi\mathrm{i}\frac{\mu_2^{\mathcal{J}}}{\delta_2}\varepsilon^{2j}A_2B_2P_2} - e^{2\pi\mathrm{i}\frac{\mu_2^{\mathcal{J}}}{\delta_2}\varepsilon^{2j}A_2P_2}\right)$$

With this last expression and the definitions of the area, we can compute the value of the kappa function modulus of the lattice Λ'_{Eis}. H. Darmon and A. Logan point out in their paper that the minimum norm we have to take to compute M digits of $\kappa_P^{\square}(A, B)$ is

$$M \Big/ \min\left(\left|\Im(P_1A_1P_1)\right|, \left|\Im((A_2B_2P_2)(A_2P_2))\right|\right)$$

5.3 Computation of Narrow Ray Class Groups

To compute the narrow class group we will use the code language Magma. Magma will give us integer representatives of all the classes of K and, to find an associated element $\tau \in K$ of a given $[\mathfrak{a}] \in \mathrm{Cl}^+(K)$, we will consider its $\mathcal{O}_F$-basis $\{\omega_1, ..., \omega_n\}$

$$\mathfrak{a} = \omega_1\mathcal{O}_F + \omega_2\mathcal{O}_F + \cdots + \omega_n\mathcal{O}_F$$

Since the integer ring of K is a 2-module of the integer rings of F, we know that given two elements in $\mathcal{O}_K$ there exists a linear combination that lies in F. Using this facts we can find $\omega_2', ..., \omega_n' \in F$ such that $\{\omega_1, \omega_2', ..., \omega_n'\}$ is a basis of $\mathfrak{a}$

$$\mathfrak{a} = \omega_1\mathcal{O}_F + \omega_2'\mathcal{O}_F + \cdots + \omega_n'\mathcal{O}_F$$

The lattice $\omega_2'\mathbb{Z} + \cdots + \omega_{2g}'\mathbb{Z}$ is a fractional ideal of $\mathcal{O}_F$. Since we are supposing that the narrow class field of F is trivial, this last fractional ideal is a fully positive principal ideal of $\mathcal{O}_F$ and, consequently, there exists a fully positive element $\delta >> 0$ such that

$$\mathfrak{a} = \omega_1\mathcal{O}_F + \delta\mathcal{O}_F$$

Since δ is a fully positive element, we know that the lattice $\mathcal{O}_F + \tau\mathcal{O}_F$, where $\tau = \omega_1/\delta$, represents the same class as $\mathfrak{a}$.

5.4 Computation of Fixed Matrices

After computing the associated τ of every class in $\mathrm{Cl}^+(K)$ we have to find a matrix $\gamma_\tau \in \Gamma$ that fixes the element τ through the Mobius transformation. To obtain this matrix we will consider a non-trivial two-degree in $\mathcal{O}_F[x]$ that has τ as a root

$$Ax^2 + Bx + C, \quad A, B, C \in \mathcal{O}_F$$

The coefficients of all the Hilbert group matrices that fix τ through the Mobius transformation will satisfy the following equations

$$\frac{a\tau + b}{c\tau + d} = \tau \iff c\tau^2 + (d-a)\tau - b$$

This polynomial must be a multiple of the minimum polynomial expressed at the beginning of this subsection, we have that there exists a $\lambda \in F$ such that

$$c = \lambda A, \quad b = -\lambda C, \quad d - a = \lambda B$$

Using this equation and the determinant condition that all the matrices in Γ satisfy, we have

$$\begin{cases} ad - bc = 1 \\ c = \lambda A \\ b = -\lambda C \\ d = a + \lambda B \end{cases} \iff \begin{cases} a^2 + a\lambda B + \lambda^2 CA - 1 = 0 \\ c = \lambda A \\ b = -\lambda C \\ d = a + \lambda B \end{cases}$$

By choosing $\lambda \in F$ such that the two-degree equation has a solution in $\mathcal{O}_F$, we would have found all the coefficients in a matrix of Γ that fixes τ through the Mobius transformation.

5.5 Generalization for Lattice Zeta Functions

We shall discuss the generalization of the algorithms expressed in the last subsection to the zeta functions associated to a ray class field case. The general algorithm proposed by P. Charollois and H. Darmon can be easily generalized, however, the difficulty rise from the basic algorithms that we use along the way.
Using the same process as in Section 5.3, given a class $[\mathfrak{a}] \in \mathrm{Cl}^+_\varpi(K)$ we can find a representative of the dual class of the form

$$\nu\left(\mathcal{O}_F + \tau\mathcal{O}_F\right), \quad \nu \in \mathcal{O}_F(\infty), \ \tau \in K$$

We need to find an efficient algorithm to compute a matrix $\gamma_\tau \in \Gamma^1(\varpi)$ that fixes $\tau \in K$. We could work out when the expression given in Section 5.4 lies in the subgroup $\Gamma^1(\varpi) \subsetneq \Gamma$, however, it might be more useful using the fact that the subgroup of matrices that fix τ in Γ and $\Gamma^1(\varpi)$ have dimension one and consequently, given the generator of $\langle\gamma\rangle = \Gamma_\tau$ (the subgroup of matrices that fix τ) there exists $n \in \mathbb{Z}$ such that

$$\langle\gamma^n\rangle = \Gamma_\tau \cap \Gamma^1(\varpi)$$

Another problem we face is that we do not know the generators of $\Gamma^1(\varpi)$. We shall generalize the work of Leonid Vasertin [Vas72] to find the generators of such group and also generalize the algorithms presented by Xavier Guitart and Marc Masdeu [GM12, p. 2] to compute the decomposition of a given matrix in $\Gamma^1(\mathcal{O}_K)$. Both papers shall be generalized for any ideal $I \subset \mathcal{O}_K$, immediately giving us the generators of $\Gamma^1(\varpi)$.
Finally, we shall also generalize the work of H. Darmon and A. Logan to the the integration of Eisenstein series. To generalize such work it will probably be necessary to find expressions similar to the ones proportionated by [Cha16] that take under consideration ideals instead of values of summations over quotients of elements which will be more efficient computationally.

6 Periods of Eisenstein Series

In this section, we will prove special formulas between the Eisenstein series and the lattice zeta functions at $s = 0$ essential for the main theorems in sections 2 and 3. P. Charollois worked out similar expressions in his PhD thesis [Cha04] for the Hecke L-functions, we will give similar proofs as the ones expressed in [CD08, pp. 683–687] but adapted to our zeta functions.
Giving a totally real number field F (all its places are real) with a trivial narrow class group and degree $g = [F : \mathbb{Q}]$, we denote its collection of places as

$$S_\infty^F = \{v_1, ..., v_g\}$$

We define, for any $n \in \mathbb{N}$, over any vector $z = (z_1, ..., z_n)$ over $\mathfrak{h}^n \subset \mathbb{C}^n$, their imaginary part $y = (\Im z_1, ..., \Im z_n)$, their norm $\mathbf{N}(z) = z_1 \cdots z_n$ and their trace $\mathbf{Tr}(z) = z_1 + \cdots + z_n$. During this chapter when we talk about the summation $z + v$ and the product zv of vectors over their components $(z_1 + v_1, ..., z_n + v_n)$ and $(z_1 v_1, ..., z_n v_n)$, respectively. Considering the elements of $\mathcal{O}_F$ as vectors in $\mathbb{R}^g$ through the places of F, we can define the Analytic Eisenstein Series, for $\Re(s) > 1$, as

$$E(b; z, s) = \sum_{(m,n)\in\left(\mathcal{V}^+_{b,0,\mathcal{O}_K}\middle\backslash \mathcal{O}_F \times \mathcal{O}_F\right)} \frac{\mathbf{N}(y)^s}{|\mathbf{N}(mz + (n-b))|^{2s}}, \quad b \in K, \quad z \in \mathfrak{h}^g$$

Given a relative quadratic extension K/F, we can define the variables $g = [F : \mathbb{Q}] = r + c$ where r is the number of places in F that split into two real places in K and c is the number of places of F that ramify in K. We will suppose that the first c places of S_∞^F are the ones that will ramify to complex places to K, which will also be denoted as v_j, $j = 1, .., c$, the rest o the places will split into to real places which will be denoted as v_j and v'_j for $j = c + 1, ..., n$.
For any class ideal $[\mathfrak{a}] \in \mathrm{Cl}^+_\varpi(K)$, since the narrow class field of K is trivial, we know there exist $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$ such that $\nu\left(\mathcal{O}_F + \tau\mathcal{O}_F\right)$ is a representative of the class $[\mathfrak{a}]$. We define the following notations for the images of τ through the different places of K

$$\begin{aligned} \tau_j &:= v_j(\tau) \in \mathfrak{h} && \text{for } j = 1, ..., c \\ (\tau_j, \tau'_j) &:= \left(v_j(\tau), v'_j(\tau)\right) \in \mathbb{R}^2 && \text{for } j = c + 1, ..., n \end{aligned}$$

For all $j \in \{c + 1, ..., n\}$, we denote Υ_j as the totally geodesic real analytic submainfold of $\mathfrak{h}^g$ linking the points τ_j and τ'_j, with orientation from τ_j to τ'_j. We define the region R_τ as

$$R_\tau = \{\tau_1\} \times \cdots \times \{\tau_c\} \times \Upsilon_{c+1} \times \cdots \times \Upsilon_n \subset \mathfrak{h}^g$$

which is an orientated subspace congruent to a subspace in $\mathbb{R}^{c+2r}$. We define the abelian subgroup $\Gamma^1_\tau(f)$ of range r inside $\Gamma^1(\varpi)$ that identifies with the subgroup $\mathcal{V}_1$ of $\mathcal{V}^+$ the F units of relative norm 1. This subgroup acts over R_τ through homographies (applying the Moebius action in each coordinate), and the quotient $\mathcal{X} := \Gamma^1_\tau(f)\backslash R_\tau$ is compact. Considering a fundamental domain over the action of $\Gamma^1_\tau(f)$ over R_τ, we identify Δ_τ as the image of such domain inside $\mathcal{X}$, which is a closed cycle of dimension r inside the quotient $\Gamma^1_\tau(f)\backslash R_\tau$.

Theorem 6.1. *Given any class $[\mathfrak{a}] \in \mathrm{Cl}^+_f(K)$, we consider the elements $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$ such that the dual of the class $[\mathfrak{a}]$ can be represented as $\nu\left(\mathcal{O}_F + \tau\mathcal{O}_F\right)$ and an element $b \in \left(\mathfrak{N}^{-1}\right)^*$, where $\mathfrak{N}$ is the integer representation of the class, and we have*

$$\int_{\Delta_\tau} \frac{\partial^r E(b/\nu; z, s)}{\partial z_{c+1}\cdots\partial z_g} dz_{c+1}\wedge\cdots\wedge dz_g = \left(-\frac{\Gamma(\frac{s+1}{2})^2}{2\Gamma(s)}\right)^r \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^s \frac{Z_{\mathfrak{a}}(0, b; 1-s)}{\varphi(1-s)}$$

Proof. We associate, for a complex number $s \in \mathbb{C}$ such that $\mathfrak{R}(s) > 1$, the differentiable r-form $\Gamma^1_f(F)$-invariant over g copies of the upper half plane $\mathfrak{h}^g \subseteq \mathbb{C}^g$

$$\Omega^r_{\mathrm{Eis}}(s) := \frac{\partial^r E(b/\nu; z, s)}{\partial z_{c+1}\cdots\partial z_g} dz_{c+1}\wedge\cdots\wedge dz_g$$

We explicitly compute the derivative with respect to the variable z_j of the term defined by these variables in the products defining $\mathbf{N}(y)$ and $\mathbf{N}(mz + (n-b))$.

$$\frac{\partial}{\partial z_j}\left(\frac{y_j^s}{\left|m^{(j)}z_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right|^{2s}}\right) = \frac{s}{2i}\frac{y_j^{s-1}\left(m^{(j)}z_j' + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right)^2}{\left|m^{(j)}z_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right|^{2s+2}}$$

We can use this last derivative to give a general expression of the $\Omega^r_{\mathrm{Eis}}(s)$ integral with respect to Δ_τ

$$\int_{\Delta_\tau}\Omega^r_{\mathrm{Eis}}(s) = \left(\frac{s}{2\mathrm{i}}\right)^r \int_{\Delta_\tau} \sum_{(m,n)\in\left(\mathcal{V}^+_{-b/\nu,0,\mathcal{O}_K}\backslash\mathcal{O}_F^2\right)} \left(\prod_{j=1}^{c}\frac{(\mathfrak{I}\tau_j)^s}{\left|m^{(j)}\tau_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right|^{2s}}\right)$$

$$\left(\prod_{j=c+1}^{g}\frac{y_j^{s-1}\left(m^{(j)}\bar{z}_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right)^2}{\left|m^{(j)}z_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right|^{2(s+1)}}\right) dz_{c+1}\wedge\cdots\wedge dz_g$$

We remember that the norm of $\mathfrak{a} = \mathcal{O}_F + \tau\mathcal{O}_F$ can we expressed as

$$\mathbf{N}(\mathfrak{a}) = d_F\prod_{j=1}^{c}\mathfrak{I}\tau_j\prod_{j=c+1}^{g}(\tau_j' - \tau_j)$$

We define the natural action of $K^\times$ over $(\mathbb{R}^+)^r$ as

$$\alpha\bullet(t_{c+1},...,t_g) := \left(\left|\frac{\alpha_{c+1}}{\alpha'_{c+1}}\right|t_{c+1},...,\left|\frac{\alpha_g}{\alpha'_g}\right|t_g\right)$$

We state the measure canonic of Haar in the compact set $\Gamma^1_\tau(f)\backslash R_\tau$

$$d^\times t = \frac{dt_{c+1}}{t_{c+1}}\wedge\cdots\wedge\frac{dt_g}{t_g}$$

We make the change of variable $t_j = -\mathrm{i}(z_j - \tau_j')/(z_j - \tau_j)$ to obtain a parametrization $t_j \in \mathbb{R}^+$ of the geodesic Υ_j which allow us to identify the quotient $\Gamma_\tau^1(f) \backslash R_\tau$ with a collection of r real torus T^r. Considering the variable $\beta = m\tau + (n - b/\nu)$, we have

$$g_\beta(\beta \bullet t) = \prod_{j=c+1}^{g} \frac{(\beta \bullet t)_j^s \left(-i(\beta \bullet t)_j + \operatorname{sign}(\beta_j \beta_j')\right)^2}{\left((\beta \bullet t)_j^2 + 1\right)^{s+1}} =$$

$$= \prod_{j=c+1}^{g} \frac{i^{s-1}(z_j - \tau_j')y_j^{s-1}\left(m^{(j)}\bar{z}_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right)^2 |\beta_j\beta_j'|^{2s}}{(z_j - \tau_j)(\tau_j' - \tau_j)\left|m^{(j)}z_j + \left(n^{(j)} - b^{(j)}/\nu^{(j)}\right)\right|^{2(s+1)}}$$

Which, since $\mathfrak{a}^*/\nu$ is generated by $\{1, \tau\}$ and consequently β will go over the quotient $\mathcal{V}^+_{-b/\nu,0,\mathcal{O}_K} \Big\backslash \left(\frac{\mathfrak{a}^*-b}{\nu}\right)$, implies that the first expression is equivalent to

$$\int_{\Delta_\tau} \Omega^r_{\mathrm{Eis}}(s) = \left(\frac{s}{2}\right)^r \left(\frac{\mathbf{N}(\mathfrak{a})}{d_F}\right)^s \sum_{\beta \in \left(\mathcal{V}^+_{-b/\nu,0,\mathcal{O}_K} \Big\backslash \left(\frac{\mathfrak{a}^*-b}{\nu}\right)\right)} \frac{1}{\left|\mathbf{N}_{K/\mathbb{Q}}(\beta)\right|^s} \int_{T^r} g_\beta(\beta \bullet t) d^\times t$$

We can identify all the non-zero classes of $\mathcal{V}^+_{-b/\nu,0,\mathcal{O}_K} \Big\backslash \left(\frac{\mathfrak{a}^*-b}{\nu}\right)$ as $\{\epsilon\beta\}$ where $\beta \in \left(\mathcal{V}_{-b/\nu,0,\mathfrak{a}^*/\nu} \Big\backslash \left(\frac{\mathfrak{a}-b}{\nu}\right)\right)$ and $\epsilon \in \mathcal{V}_1/\{\pm 1\}$. Consequently,

$$\int_{\Delta_\tau} \Omega^r_{\mathrm{Eis}}(s) = \left(\frac{s}{2}\right)^r \left(\frac{\mathbf{N}(\mathfrak{a})}{d_F}\right)^s \sum_{\beta \in \left(\mathcal{V}^+_{-b/\nu,0,\mathfrak{a}^*/\nu} \Big\backslash \left(\frac{\mathfrak{a}^*-b}{\nu}\right)\right)} \left|\mathbf{N}_{K/\mathbb{Q}}(\beta)\right|^{-s} \int_{\mathcal{V}^+ \backslash (\mathbb{R}^+)^r} \sum_{\mathcal{V}_1/\{\pm 1\}} g_{\epsilon\beta}(\epsilon\beta \bullet t) d^\times t$$

$$= \left(\frac{s}{2}\right)^r \left(\frac{\mathbf{N}(\mathfrak{a})}{d_F}\right)^s \sum_{\beta \in \left(\mathcal{V}^+_{-b/\nu,0,\mathfrak{a}^*/\nu} \Big\backslash \left(\frac{\mathfrak{a}^*-b}{\nu}\right)\right)} \left|\mathbf{N}_{K/\mathbb{Q}}(\beta)\right|^{-s} \int_{(\mathbb{R}^+)^r} g_\beta(\beta \bullet t) d^\times t$$

The change of variable $u = \beta \bullet t$ allows us to compute an expression for the r integrals of this last equation

$$\int_0^\infty \frac{u_j^s \left(-\mathrm{i}\, u_j + \operatorname{sign}(\beta_j\beta_j')\right)^2}{\left(u_j^2 + 1\right)^{s+1}} \frac{du_j}{u_j} = -\mathrm{i}\operatorname{sign}(\beta_j\beta_j') \frac{\Gamma\left(\frac{s+1}{2}\right)^2}{\Gamma(s+1)}$$

With this last integral, we can conclude the equation asked by the statement of the theorem

$$\int_{\Delta_\tau} \Omega^r_{\mathrm{Eis}}(s) = \left(\frac{s\Gamma\left(\frac{s+1}{2}\right)^2}{2\,\mathrm{i}\,\Gamma(s+1)}\right)^r \left(\frac{\mathbf{N}(\mathfrak{a}^*)\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^s \sum_{\beta' \in \left(\mathcal{V}^+_{-b,0,\mathfrak{a}^*} \backslash \mathfrak{a}^* - b\right)} \frac{\omega(\beta')}{\left|\mathbf{N}_{K/\mathbb{Q}}(\beta')\right|^s}$$

$$= \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^s \left(\frac{s\Gamma\left(\frac{s+1}{2}\right)^2}{2\,\mathrm{i}\,\Gamma(s+1)}\right)^r Z_{\mathfrak{a}}(-b, 0; s)$$

Using the functional equation of the lattice zeta function, we can conclude the statement

$$\int_{\Delta_\tau} \Omega^r_{\mathrm{Eis}}(s) = d_F^{-s} \left(\frac{s\Gamma\left(\frac{s+1}{2}\right)^2}{2\,\mathrm{i}\,\Gamma(s+1)}\right)^r Z_{\mathfrak{a}^*}(-b, 0; s) = \left(\frac{\Gamma(\frac{s+1}{2})^2}{2\Gamma(s)}\right)^r \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^s \varphi(1-s) Z_{\mathfrak{a}}(0, b; 1-s)$$

$\square$

We will finish this section by proving a corollary of this theorem, that relates the function $\widetilde{h}$ defined in the equation 13 and will be the main source for Theorem 2.1 and Theorem 3.2.

Corollary 6.2. *Let K be a quadratic extension of F with the associated variables $g = [F : \mathbb{Q}] = r + c$ defined before. Given a class $[\mathfrak{a}] \in \mathrm{Cl}^{+}_{\varpi}(K)$ with associated elements $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$ (such that $\nu\left(\mathcal{O}_F + \tau\mathcal{O}_F\right)$ is a representative of the dual class), the lattice zeta function has a zero of order $c \geq 0$ at $s = 0$ and the following formulas will be satisfied*

(i) If $r \geq 2$, then

$$\frac{Z^{(c)}_{\mathfrak{a}}(0, b; 0)}{c!} = \frac{d_F}{\mathbf{N}_{K/\mathbb{Q}}(\nu)}\sqrt{\frac{d_F}{\pi^{g-r}}}\int_{\Delta_\tau}\frac{\partial^r\widetilde{h}(b/\nu, z)}{\partial z_{c+1}\cdots\partial z_g}dz_{c+1}\wedge\cdots\wedge dz_g$$

(ii) If the field K does not have more than one real place, then

$$\frac{Z^{(g-1)}_{\mathfrak{a}}(0, b; 0)}{(g-1)!} = \frac{d_F}{\mathbf{N}_{K/\mathbb{Q}}(\nu)}\sqrt{\frac{d_F}{\pi^{g-1}}}\int_{\Delta_\tau}\left(\frac{\partial\widetilde{h}(\vec{z}, s)}{\partial z_g} - \frac{2^{2n-2}R_F}{z_g - \bar{z}_g}\right)dz_g$$

Proof. The Eisenstein series $E(b/\nu; z, s)$ have an analytic continuation to $\mathbb{C}$, as shown on the paper [Cha16], and satisfies the following equation

$$G(s, z)E(b/\nu; z, s) = e^{2\pi\,\mathrm{i}\,\mathbf{N}_{F/\mathbb{Q}}(ab)}G(1-s, z)E^*(b/\nu; z, 1-s)$$

where

$$G(s, z) = \frac{\Gamma^g(s)}{\pi^{2g}\,|\mathbf{N}(y)|^{2g-1}}$$

In the definition of the function h, present in section 1., we stated a relation with $E^*(b/\nu; z, 1)$. Using this equation, we can state the Taylor expansion of E^* around $s = 1$

$$E^*(b/\nu; z, s) = 2^{g-2}R_F\left(\frac{1}{s-1} + \gamma_F - \log\mathbf{N}(y) + h(b/\nu; z)\right) + O(s-1)$$

Using the functional equation of the Eisenstein series, this last statement and the Taylor expansion of the $1/\Gamma(s)$ function, we can give the Taylor expansion of the function E at $s = 0$

$$E(b/\nu; z, s) = -2^{g-2}R_F s^{g-1} + 2^{g-2}R_F\left(\gamma_F - \log\mathbf{N}(y) + h(b/\nu; z)\right)s^g + O(s^{g+1})$$

Remembering that $\widetilde{h} = 4^{g-1}R_F h$. For $r \geq 2$, deriving this last expression and Theorem 7. we conclude

$$\left(\frac{\Gamma(1-\frac{s}{2})^2}{2\Gamma(1-s)}\right)^r\left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^{1-s}\frac{Z_{\mathfrak{a}}(0, b; s)}{\varphi(s)} = \int_{\Delta_\tau}\frac{\partial^r E(b/\nu, z, 1-s)}{\partial z_{c+1}\cdots\partial z_g}dz_{c+1}\wedge\cdots\wedge dz_g =$$

$$= \frac{s^g}{2^g}\int_{\Delta_\tau}\left(\frac{\partial^r\widetilde{h}(b/\nu; z)}{\partial z_{c+1}\cdots\partial z_g} + O(s)\right)dz_{c+1}\wedge\cdots\wedge dz_g$$

Using the Taylor expansion of $\Gamma(\frac{s+1}{2})^2/\Gamma(s)$ around $s = 0$, we obtain an expression of the zeta function with a remaining of order $g - r + 1$.

$$Z_{[\mathfrak{a}]}(0, b; s) = \frac{s^{g-r}\sqrt{d_F^3}}{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi^{g-r}}}\int_{\Delta_\tau}\frac{\partial^r\widetilde{h}(b/\nu; z)}{\partial z_{c+1}\cdots\partial z_g}dz_{c+1}\wedge\cdots\wedge dz_g + O(s^{g-r+1})$$

The first expression of this corollary statement is a direct consequence of the last expression. For the case when $r = 1$, deriving the expression that relates the function E and h, together with the Theorem 7. we can conclude

$$\left(\frac{\Gamma(1-\frac{s}{2})^2}{2\Gamma(1-s)}\right)^r\left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)}{d_F}\right)^{1-s}\frac{Z_{\mathfrak{a}}(0, b; s)}{\varphi(s)} = \frac{s^g}{2^g}\int_{\Delta_\tau}\left(\frac{\partial\widetilde{h}(b/\nu, z)}{\partial z_g} + \frac{4^{g-1}R_F}{z_g - \bar{z}_g} + O(s)\right)dz_g$$

Using the Taylor expansion of the quotient of gamma functions computed on the last case, we can find an expression for the zeta function with a remaining of order

$$Z_{[\mathfrak{a}]}(a,b;s) = \frac{d_F}{\mathbf{N}_{K/\mathbb{Q}}(\nu)}\sqrt{\frac{d_F}{\pi^{g-1}}}s^{g-1}\int_{\Delta_\tau}\left(\frac{\partial\widetilde{h}(\vec{z},s)}{\partial z_g} + \frac{4^{g-1}R_F}{z_g-\bar{z}_g}\right)dz_g + O(s^g)$$

The statement of the corollary for the case $r = 1$ is a direct consequence of deriving the last expression $g - 1$ times with respect to s. □

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