Algorithms to compute Stark Numbers

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This presentation is part of an ongoing project that started in the summer of 2022 with Professeur Hugo Chapdeline (Université Laval).

Strak Conjecture

Theorem. Class Number Formula

The function $\zeta_K(s)$ converges absolutely for $\Re(s) > 1$ and extends to a meromorphic function defined for all complex s with only one simple pole at $s = 1$ with residue

$$\lim_{s\to 1}(s-1)\zeta_K(s) = \frac{2^r(2\pi)^c R_K h_K}{\#\mu_K \sqrt{|d_K|}}$$

Using the functional equation we can get some intuition of the first coefficient of the Taylor expansion.

Strak Conjecture

Rank one Stark Conjecture

Let K be a number field. For all L function $L_K(s)$ there exists an algebraic integer $u \in \mathcal{O}_H$, where H is the associated ray class field, such that

$$L'_K(0) = \frac{\delta(v)}{\#\mu_K} \log|\widetilde{v}(u)|$$

where $\delta(v)$ is 1 if v is real and 2 if it is complex.

Let K be a Number Field and $\{v_1, ..., v_t\}$ the set of places of this field. Given any vector $p \in (\mathbb{F}_2)^t$, we define the sign map

$$\omega_p(x) := \prod_{j=1}^{t} (s_{v_j}(x))^{p_j} \text{ where } s_v(x) = \begin{cases} \text{sign}(v(x)) & \text{if } v \text{ real} \\ 1 & \text{if } v \text{ complex} \end{cases}$$

We can also define the fully positive integers

$$\mathcal{O}_K(\infty) := \{\alpha \in \mathcal{O}_K \backslash \{0\} \ : \ s_v(\alpha) = 1 \text{ for all places } v\}$$

Given a lattice $\mathfrak{n} \subseteq K$

$$\mathcal{V}^{+}_{a,b,\mathfrak{n}} := \left\{\varepsilon \in \mathcal{O}_K^{\times}(\infty) \ : \ (\varepsilon - 1)a \in \mathfrak{n}, \ (\varepsilon - 1)b \in \mathfrak{n}^*, \ (\varepsilon - 1)ab \in \mathcal{O}_K^*\right\}$$

Lattice Zeta Function

Definition. Lattice Zeta function

Given a lattice $\mathfrak{n} \subseteq K$ and $s \in \mathbb{C}$ such that $\Re(s) > 1$, the lattice zeta function of $\mathfrak{n}$ is defined as

$$Z_{\mathfrak{n}}(a, b, p; s) = \sum_{x \in \left(\mathcal{V}^{+}_{a,b,\mathfrak{n}} \backslash \mathfrak{n}\right)}{}' \frac{\omega_p(x+a) e^{2\pi \mathrm{i}\, \mathbf{Tr}_K(b(x+a))}}{|\mathbf{N}_K(x+a)|^s}$$

Hugo Chapdelaine proved that this function admits a meromorphic extension to $\mathbb{C}$.

Zeta function associated to a ray class field

Giving a level $f \in \mathcal{O}_K$ and a fractional ideal $\mathfrak{a} \subseteq K$, we define the Zeta function associated to the ray class field of level f as

$$Z_{f,\mathfrak{a}}(a,b,p;s) := Z_{f\mathfrak{a}^{-1}}(a,b,p;s)$$

Proposition.

All the representatives of a given class in $\mathrm{Cl}_f^p(K)$ generate the same zeta function. Consequently, we will compute the Stark Numbers associated to the classes of the ray class group.

Rank one Stark Conjecture

Let μ_H be the set of roots of units in H. For a given class $[\mathfrak{a}] \in \mathrm{Cl}_f^p(K)$, there exists an algebraic number $u_\mathfrak{a} \in \mathcal{O}_H$ such that

$$\frac{\partial}{\partial s} Z_{f,\mathfrak{a}}(a,b,p;0) = -\frac{\delta(\widetilde{v}_1)}{\#\mu_H} \log \left|\widetilde{v}_1(u_{[\mathfrak{a}]})\right|$$

We start assuming K is a totally real quadratic extension, that $\widetilde{v_1}$ is real and $Cl^+(K) = 0$. Chapdelaine proved the following equation for the first derivative of Lattice Zeta functions

$$\frac{\partial}{\partial s} Z_{\mathfrak{a}}(a, b, p; 0) = \left(\frac{\pi}{2}\right)^2 |d_K|^{1/2} Z_{\mathfrak{a}^*}(-b, a, p; 1)$$

Using basic algebra, we have the following expression

$$Z_{(f\mathfrak{a}^{-1})^*}(0, 1, p; 1) = 2\,\mathrm{i}\,\kappa(\mathbf{N}(f\mathfrak{a}^{-1})) \left(\sum_{\gamma} \frac{\sin(\alpha_\gamma)}{|\mathbf{N}\gamma|} - \sum_{\gamma} \frac{\sin(\alpha_\gamma)}{|\mathbf{N}\gamma|} \right)$$

During the Summer of 2022, we computed three decimals at Université Laval for the example $\mathbb{Q}(\sqrt{29})$, $f = (3 - \sqrt{29})/2$ and $p = (1, 0)$

$$4.624$$

Shintani was able to compute 33 decimals of this number 45 years ago!!!

Real analytic Eisenstein series

For $z \in \mathfrak{h}$, $s \in \mathbb{C}$, with $\Re(s) > 1$, we define the lattice real Eisenstein series as

$$G(z,s) := \sum_{(m,n)\in\mathbb{Z}^2}{}' \frac{\Im(y)^s}{|mz+n|^{2s}}$$

This function admits a functional equation

$$G(z,s) = \frac{\pi^{-(1-s}\Gamma(1-s)}{\pi^{-s}\Gamma(s)} G(z,1-s)$$

The Eisenstein series is 1-periodic and, therefore, admits a Fourier expansion

$$G(x + y\,\mathrm{i}) = a_0(y,s) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n(y,s)e^{2\pi nx\,\mathrm{i}}$$

where

$$a_0(y,s) = 2\zeta(2s)y^s + \varphi(s)2\zeta(2s-1)y^{1-s}$$

$$\varphi(s) := \frac{\pi^{1/2}\Gamma(s-\frac{1}{2})}{\Gamma(s)}$$

and

$$a_n(y,s) = \frac{4|n|^{s-1/2}}{\pi^{-s}\Gamma(s)}\sigma_{1-2s}(|n|)y^{1/2}K_{s-\frac{1}{2}}(2\pi|n|y), \quad \sigma_\mu(|n|) = \sum_{0<d|n} d^\mu$$

Colmez's Trick

It is easy to check that

$$G(\gamma z, s) = G(z, s),\ \gamma \in \mathrm{SL}_2(\mathbb{Z})$$

Defining $R(z, s) = G(z, s) - a_0(y, s)$, we have

$$G(z, s) - G(-z^{-1}, s) = 0 \iff$$

$$a_0(y, s) - a_0\left(\frac{y}{x^2 + y^2}, s\right) = -R(z, s) + R\left(-z^{-1}, s\right) = F(z, s)$$

Colmez's Trick

Taking two different elements $z_1, z_2 \in \mathfrak{h}$, we get the system of equations

$$2M \begin{pmatrix} \zeta(2s) \\ \varphi(s)\zeta(2s-1) \end{pmatrix} = \begin{pmatrix} F(z_1, s) \\ F(z_2, s) \end{pmatrix}$$

If we specialise to

$$z_1 = \sqrt{a^{-1} - 1} + \mathrm{i}, \quad z_2 = \sqrt{a^{-2} - 1} + \mathrm{i}, \quad a \in (0, 1)$$

We can give a general expression of the Riemann Zeta function

$$\zeta(2s) = \frac{1}{2} \frac{(1 + a^{1-s})F(z_1, s) - F(z_2, s)}{a^{2s} - a^s + a^{1-s} - a}$$

Computational Results

A classic example

$$\zeta(2) = 1.64493406684..., \qquad 9.63 \cdot 10^{-200}$$

A trivial example

$$\zeta(-6) = 0.0000000000..., \qquad 0.00 \cdot 10^{-200}$$

A non-trivial one

$$\zeta(1/2 + 14.1347251417...\,\mathrm{i}) = 1.406844906066... \cdot 10^{-200}$$

Problems that might arise

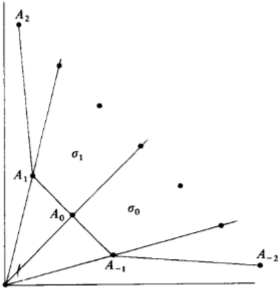
Which problems will we face when we generalize this algorithm to Zeta functions associated to a ray class field?

1. The computation of the quotient $\mathcal{V}^+_{a,b,f\mathfrak{a}^{-1}} \Big\backslash^{f\mathfrak{a}^{-1}}$
2. The generalization of the division algorithm.

For the second problem, simplifying $Cl^+(K) = 0$ is quite helpful.

$$(\lambda) = I_1 I_2 \cdots I_n$$

The extremount points



The extremount points satisfy the following equation $A_{k-1} + A_{k+1} = b_k A_k$ where b_k are the coefficients of the fraction expansion of τ.

The algorithm for Lattice Zeta functions

Lattice real analytic Eisenstein series

Given two lattices $\mathfrak{n}, \mathfrak{m} \subseteq K$, a vector $p \in \mathbb{F}_2^2$, $z \in \mathfrak{h}$ and $U \in M_2(K)$, we define the following

$$G_Q(z,s) = G_{(\mathfrak{m},\mathfrak{n})}(U,p;z,s) =$$

$$= \sum_R{}' \frac{\omega_p((m+v_1)z(n+v_2))e^{2\pi\,\mathrm{i}\,\mathbf{Tr}(u_1(m+v_1)+u_2(n+v_2)}}{|\mathbf{N}((m+v_1)z+(n+v_2))|^{2s}}\Im(z)^{\mathbf{1}s}$$

As in the last case, the function is 1-periodic and, therefore, admits a Fourier expansion

$$G_Q(x+y\,\mathrm{i},s) = a_0(y,s) + \sum_{d \in D\setminus\{0\}} a_d(y,s)e^{2\pi\,\mathrm{i}\,\mathbf{Tr}(dx)}$$

with the first coefficient equal to

$$e_1\delta_{\mathfrak{m}}(v_1)Z(v_2,u_2,p;2s)\mathbf{N}(y)^s + \Omega_p(s)\delta_{\mathfrak{n}^*}(u_2)e_2Z(v_1,u_2,p;2s-1)\mathbf{N}(y)^{1-s}$$

The algorithm for Lattice Zeta functions

The non-zero coefficients will be

$$a_d(y,s) = \mathbf{N}(y)^s \sqrt{d_K} |\mathbf{N} f| B_d(y,s) b_d(s)$$

with

$$b_d(s) := \sum_{(\xi_1,\xi_2)\in R_d//\mathcal{O}_K^\times(f\infty)}' (-1)^{W_p(\xi_2)} |\mathbf{N}(\xi_2)|^{2s-1}$$

The equation for the $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in \mathrm{SL}_2(\mathcal{O}_K)$ action is

$$G_Q(Sz,s) = \omega_{-p}(z) G_{Q^*}(z,s)$$

Computational results

Chapdelaine worked out an equation that relates the lattice invariants with Shintani's

$$\log X_{f\infty}(\mathfrak{a}) = \mathrm{i}\,\frac{|d_K|^{1/2}}{4\pi}\left(Z^*(\mathfrak{a}, f\infty, (1,0); 1) + Z^*(\mathfrak{a}^\sigma, f^\sigma\infty, (1,0); 1)\right)$$

Let $K = \mathbb{Q}(\sqrt{29})$, $f = (3-\sqrt{29})/2$ and $p = (1,0)$. We use magma to compute the ray class group in this case

$$\mathrm{Cl}_f^p(K) = \{[\mathcal{O}_K], [2\mathcal{O}_K]\} \cong \mathbb{Z}/2\mathbb{Z}$$

Using the algorithm we compute all the values present in the equation

$$Z^*(\mathcal{O}_K, f\infty, (1,0); 1) = -1.178\cdots 10^{-27} + 3.573\cdots \mathrm{i}$$

$$Z^*(\mathcal{O}_K^\sigma, f^\sigma\infty, (1,0); 1) = -2.984\cdots 10^{-28} + 1.051\cdots 10^{-27}\,\mathrm{i}$$

$$Z^*(2\mathcal{O}_K, f\infty, (1,0); 1) = 8.469\cdots 10^{-28} + 1.365\cdots \mathrm{i}$$

$$Z^*((2\mathcal{O}_K)^\sigma, f^\sigma\infty, (1,0); 1) = 7.688\cdots 10^{-28} - 3.495\cdots 10^{-27}\,\mathrm{i}$$

Computational results

With these values, we compute Shintani's invariant up to an error of 10^{-54}

$$X_f(\mathcal{O}_K) = 4.6242866227\cdots,\ X_f(2\mathcal{O}_K) = 1.7949175568\cdots$$

Using the LLL-algorithm we get the expected polynomial for these numbers

$$x^8 - 9x^7 + 29x^6 - 52x^5 + 63x^4 - 52x^3 + 29x^2 - 9x + 1$$

Magma even give us the decomposition of such polynomial over $\mathcal{O}_F[x]$

$$\left(x^4 - \frac{9+\sqrt{29}}{2}x^3 + (8+\sqrt{29})x^2 - \frac{9+\sqrt{29}}{2}x + 1\right)\cdot$$

$$\left(x^4 + \frac{-9+\sqrt{29}}{2}x^3 + (8-\sqrt{29})x^2 + \frac{-9+\sqrt{29}}{2}x + 1\right)$$

Charollois-Darmon Conjecture

Charollois and Darmon consider the L function

$$L(M,s) := (\mathbf{N}M)^s \delta_I \sum_{x\in M/V}{}' \frac{\operatorname{sign}(\mathbf{N}(x))}{|\mathbf{N}(x)|^s}$$

and the associated Eisenstein series

$$E(z_1,...,z_n) = \zeta_F(-1) + \sum_{\mu\in\mathcal{O}_F,\ \mu>>0} \left(\sum_{(\nu)|(\mu)} |\mathbf{N}(\nu)| \right) e^{2\,\mathrm{i}\,\pi\left(\frac{\mu_1}{\sigma_1}z_1+\cdots+\frac{\mu_n}{\sigma_n}z_n\right)}$$

Introducing the notation $\omega_E = E(z_1,...,z_n)dz_1 \wedge \cdots \wedge dz_n$, we can define the n-differential form

$$\omega_{\mathrm{Eis}} := \begin{cases} \frac{(2\pi\,\mathrm{i})^n}{\sqrt{d_F}}\omega_E + \frac{R_F}{2}\left(\frac{dz_1\wedge d\bar{z}_1}{y_1^2} - \frac{dz_2\wedge d\bar{z}_2}{y_2^2}\right) & \text{if } n=2 \\ \frac{(2\pi\,\mathrm{i})^n}{\sqrt{d_F}}\omega_E & \text{if } n>2 \end{cases}$$

Charollois-Darmon Conjecture

The form $\omega_{\mathrm{Eis}}^{+} = \frac{1}{2}(\mathrm{Id} + T_v^*)\omega_{\mathrm{Eis}}$ is "the complex real part" of the form ω_{Eis}.

Theorem (Charollois-Darmon, 2008)

Let M be an $\mathcal{O}_I$-module associated to a $\tau \in K$. The first derivative of the L function satisfies

$$(-2)^{n-1} \int_{C_\tau} \omega_{\mathrm{Eis}}^{+} = (2\,\mathrm{i}\,\pi)^{n-1} L'(M, 0)$$

Charollois-Darmon Conjectue (2008)

For all $\mathcal{O}_I$-module M associated to $\tau \in K$, there exists an algebraic number $u_\tau \in \mathcal{O}_H$ such that

$$e_I \frac{(-2)^{n-1}}{n_F} \int_{\partial C = n_F \Delta_\tau} \omega_{\mathrm{Eis}} = 2\delta_I (2\,\mathrm{i}\,\pi)^{n-1} \log(\widetilde{v}_1(u_\tau))$$

Generalization Charollois-Darmon Conjecture

For the generalization to lattice zeta functions, we define the lattice Eisenstein series of weight two

$$E_2^*(U,Z) = \sum_{(m,n)\in\left(\mathcal{O}_F^2\big/\mathcal{V}^+_{0,b,\mathcal{O}_F}\right)} \frac{e^{-2\pi\,\mathrm{i}\,\mathbf{Tr}(u_1(\delta m+v_1)+u_2(\delta n+v_2))}}{\mathbf{N}\left(((\delta m+v_1)z+(\delta n+v_2)\right)^2}$$

Using the same notation as the last case

$$\Omega_{E_2^*}(b;z) := E_2^*(b;z)\ dz_1\wedge\cdots\wedge dz_n$$

we define the n-differential form

$$\Omega_{Eis}(b;z) := \begin{cases} -2\Omega_{E_2^*}(b;z) + \frac{R_F}{2}\left(\frac{dz_1\wedge d\bar{z}_1}{y_1^2} - \frac{dz_2\wedge d\bar{z}_2}{y_2^2}\right) & \text{if } n=2 \\ (-2\,\mathrm{i})^n\Omega_{E_2^*}(b;z) & \text{if } n>2 \end{cases}$$

Generalization Charollois-Darmon Conjecture

Similarly to the last case the form $\Omega^+_{\text{Eis}} = \frac{1}{2}(U_{v_j} + T_{v_j})\Omega_{\text{Eis}}$ emulates the real part of Ω_{Eis}.

Theorem (Caralps, 2023)

Given a class $[\mathfrak{a}] \in \text{Cl}^+_f(K)$ associated to a $\tau \in K$, the following equation is satisfied

$$\int_{C_\tau} \Omega^+_{Eis} = \left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_f\sqrt{d_F}}\right)\frac{\partial}{\partial s} Z^{\varpi}_{[\mathfrak{a}]}(b;0)$$

Generalization Charollois-Darmon Conjecture (Caralps, 2023)

For all $\mathcal{O}_K$-module $\mathfrak{n}$ with associated elements $\nu \in \mathcal{O}_F(\infty)$ and $\tau \in K$, that satisfy $[\nu(\mathcal{O}_F + \tau\mathcal{O}_F)] = \left[\left(\mathfrak{n}^{-1}\right)^*\right]$, the Stark number $u_\tau \in \mathcal{O}_H$ of $Z^{\varpi}_{[\mathfrak{n}]}(b;s)$ satisfies the equation

$$\frac{\#\mu_H}{n_F}\int_{\partial C = n_F\Delta_\tau} \Omega_{\text{Eis}} = -2\left(\frac{\mathbf{N}_{K/\mathbb{Q}}(\nu)\sqrt{\pi}}{d_F\sqrt{d_F}}\right)\log\left(\widetilde{v}_1(u_\tau)\right)$$

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