Computation of values of zeta functions using Eisenstein series

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Lattice Eisenstein series

Lattice real analytic Eisenstein series

For $z, s \in \mathbb{C}$, with $Re(s) > 1$, we define the lattice real analytic Eisenstein series as

$$G(z = x + yi, s) := \sum_{(m,n)\in\mathbb{Z}^2\setminus\{(0,0)\}} \frac{y^s}{|mz+n|^{2s}}$$

This function satisfies the functional equation

$$G(z,s) = \frac{\pi^{-(1-s)}\Gamma(1-s)}{\pi^{-s}\Gamma(s)} G(z, 1-s)$$

Completed Lattice Eisenstein series

For $z, s \in \mathbb{C}$, with $s \neq 1$, we define the Completed Lattice Eisenstein series as

$$\hat{G}(z,s) := \pi^{-s}\Gamma(s)G(z,s)$$

Fourier expansion of $G(z,s)$

The function G is 1-periodic and, therefore, it admits a Fourier expansion

$$G(z=x+yi,s)=a_0(y,s)+\sum_{\xi_1\in\mathbb{Z}\setminus\{0\}}\sum_{\xi_2\in\mathbb{Z}\setminus\{0\}}\tau(s,s;\xi_2,\xi_1 y)e^{2\pi i\xi_1\xi_2 x}$$

(i)

$$a_0(y,s)=2\zeta(2s)y^s+\varphi(s)2\zeta(2s-1)y^{1-s}$$

$$\varphi(s):=\frac{\pi 2^{2-2s}\Gamma(2s-1)}{\Gamma(s)^2}=\frac{\pi^{1/2}\Gamma\left(s-\frac{1}{2}\right)}{\Gamma(s)}$$

(ii) For $z\in\mathbb{C}$ with $Re(z)>0$, we define the Tricomi's confluent hypergeometric function as

$$U(\beta,\alpha+\beta;z)=\frac{1}{\Gamma(\beta)}\int_0^\infty(u+1)^{\alpha-1}u^{\beta-1}e^{-zu}du$$

It can be proved that

$$\tau(s,s;\xi_2,\xi_1 y)=(2\pi)^{2s}|\xi_2|^{2s-1}e^{-2\pi|\xi_1\xi_2 y|}\Gamma(s)^{-1}U(s,2s;|4\pi\xi_1\xi_2 y|)$$

Defining the terms

$$a_n(y,s) = y^s \frac{(2\pi)^{2s}}{\Gamma(s)} \left(\sum_{0\neq d|n} d^{2s-1} \right) e^{-2\pi|n|y} U(s, 2s; 4\pi|n|y)$$

and regrouping the terms of the double sum in the G function, such that $\xi_1\xi_2 = n$, we have

$$G(z,s) = a_0(y,s) + \sum_{n\in\mathbb{Z}\setminus\{0\}} a_n(y,s) e^{2\pi i n x}$$

For $z \in \mathbb{C}$ we have, where $K_\mu(z)$ is the Bessel function of the second kind (K-Bessel function)

$$U(s, 2s; 2z) = \frac{(2z)^{\frac{1}{2}-s}}{\sqrt{\pi}} e^z K_{s-\frac{1}{2}}(z)$$

We redefine the expression of $a_n(y,s)$

$$a_n(y,s) = \frac{4|n|^{s-\frac{1}{2}}}{\pi^{-s}\Gamma(s)} \sigma_{1-2s}(|n|) y^{1/2} K_{s-\frac{1}{2}}(2\pi|n|y), \quad \sigma_\mu(|n|) = \sum_{0<d|n} d^\mu$$

Colmez's Trick

A direct calculation shows that

$$G(\gamma z, s) = G(z, s), \quad \gamma \in SL_2(\mathbb{Z})$$

From this last statement, and from defining $R(z, s) = G(z, s) - a_0(y, s)$, we have

$$G(z, s) - G\left(-\frac{1}{z}, s\right) = 0 \Longleftrightarrow$$

$$a_0(y, s) - a_0\left(\frac{y}{x^2 + y^2}, s\right) = -R(z, s) + R\left(-\frac{1}{z}, s\right) = F(z, s)$$

Colmez's Trick

Considering two elements $z_1, z_2 \in \mathbb{C}$ we have

$$2M \begin{pmatrix} \zeta(2s) \\ \varphi(s)\zeta(2s-1) \end{pmatrix} = \begin{pmatrix} F(z_1, s) \\ F(z_2, s) \end{pmatrix}$$

With,

$$M = \begin{pmatrix} (y_1^s - Im(-1/z_1)^s) & (y_1^{1-s} - Im(-1/z_1)^{1-s}) \\ (y_2^s - Im(-1/z_2)^s) & (y_2^{1-s} - Im(-1/z_2)^{1-s}) \end{pmatrix}$$

Taking

$$z_1 = \sqrt{a^{-1} - 1} + i, \quad z_2 = \sqrt{a^{-2} - 1} + i$$

It can be deduced that

$$\boxed{\zeta(2s) = \frac{1}{2} \frac{(1 + a^{1-s})F(z_1, s) - F(z_2, s)}{a^{2s} - a^s + a^{1-s} - a}}$$

Stark Conjecture

Conjecture

Giving an extension $K = \mathbb{Q}(\sqrt{d})/\mathbb{Q}$, there exists an algebraic number **a** such that

$$Z'(0) = -i\frac{|d_k|^{1/2}}{2\pi}\mathbf{N}\mathcal{O}_K Z_{\mathbf{n}}(0,1,\bar{p},2s) = -\frac{1}{2}\log|\mathbf{a}|$$

with

$$Z_{\mathbf{n}}(0,1,\bar{p},2s) = -\operatorname{signe}(f^{(2)}) \sum_{\mathcal{O}_K^{\times}(f\infty)\backslash\{0\neq\mu\in\mathcal{O}_K\}} \frac{\operatorname{signe}(\mu^{(2)})e^{2\pi i TR\left(\frac{\mu}{f\sqrt{d}}\right)}}{|\mathbf{N}\mu|^2 s}$$

Eisenstein series

Eisenstein Series

Giving two lattices **n**, **m**; a vector $p \in \mathbb{Z}^2$ and $U \in M_2(K)$ we define the following real analytic Eisenstein series

$$G(z,s)_{\mathcal{Q}} = G^0_{(\mathbf{m},\mathbf{n})}(U,p;z,s) =$$

$$= \sum_R \frac{\omega_p((m+v_1)z+(n+v_2))e^{2\pi i Tr(u_1(m+v_1)+u_2(n+v_2))}}{|\mathbf{N}((m+v_1)z+(n+v_2))|^{2s}} y^{\mathbf{1}\cdot s}$$

Since it is 1-periodic, we can consider its Fourier expansion

$$G_{\mathcal{Q}}(z=x+yi,s) = a_0(y,s) + \sum_{0 \neq \in D} a_d(y,s)e^{2\pi i Tr(dx)}$$

with

$$a_0(y,s) =$$

$$e_1\delta_{\mathbf{m}}(v_1)Z(v_2,u_2,\omega_{\bar{p}},2s)\mathbf{N}(y)^s + \Omega_p(s)\delta_{\mathbf{n}^*}(u_2)e_2 Z(v_1,u_2,\omega\bar{p},2s-1)\mathbf{N}(y)^{1-s}$$

Comez's Trick over a quadratic field

Since for $S = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \in SL_2(K)$, we have

$$G_{\mathcal{Q}}(Sz, s) = \omega_{-p}(z) G_{\mathcal{Q}^*}(z, s)$$

Then we can define a similar equation as the one find it on the last scenario

$$M \begin{pmatrix} Z_{\mathbf{n}}(0, 1, \bar{p}, 2s) \\ W \end{pmatrix} = \begin{pmatrix} T(z_1, s) \\ T(z_2, s) \end{pmatrix}$$