Lattice zeta functions for number fields

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October 2024

Abstract

In this work we introduce a general class of Dirichlet series associated to lattices of number fields which we call *lattice zeta functions*. Lattice zeta functions are closely related to many of the classical Dirichlet series considered in algebraic number theory but no systematic treatment of them exists in the literature. This work aims at providing the first comprehensive reference for such Dirichlet series.

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1 Introduction

Let K be a number field of degree g over $\mathbb{Q}$. By a lattice $\mathfrak{n}$ of K we mean an additive subgroup $\mathfrak{n} \subseteq K$ which is a free $\mathbb{Z}$-module of rank g. In [2], the author studied a certain class of zeta functions that were associated to lattices of K. Let us call this class of zeta functions *signature lattice zeta functions* which we shall abbreviate by $\mathcal{SLZ}$ in the sequel, see Appendix A where their definition and basic properties are recalled. Recently in [5], for K totally real, we have found a way of proving a non-trivial conditional convergence result for the Dirichlet series $Z(s)$ of $\mathcal{SLZ}$ which have a *non-trivial additive oscillation on* Π_1 (see Definition 1.3 below). In a nutshell the result in [5] says that for such zeta functions, the sequence of "geometrical partial sums of $Z(s)$" converges when $\mathrm{Re}(s) > 1 - \frac{1}{g}$. Moreover, in a different direction, we also expect the special value at $s = 1$ of $\mathcal{SLZ}$ (for K totally real) to be closely related to the spectrum of a suitable Dirac operator and we hope to publish some work in that direction in the future. These two observations put together motivated us to revisit the topic initiated in [2] and developed further in [4] to the setting of GL_2-real analytic Eisenstein series (also under the assumption that K was totally real which simplified the presentation since the notation in [4] was already involved). For more than a decade, we have been aware that the objects considered in [2] and [4] are only special cases of a larger theory where one is allowed to replace the signature character considered in [2] by the infinite part of an arbitrary Größencharaktere of K. The main goal of this paper is to give a detailed presentation of this larger class of zeta functions which we call "*lattice zeta functions*" and that we shall abbreviate $\mathcal{LZ}$ in the following. In particular, the class of $\mathcal{LZ}$ provides an appropriate framework in which $\mathcal{SLZ}$ becomes a subclass. The zeta functions in $\mathcal{LZ}$ are kind of implicit in the fundamental work of Hecke on the meromorphic continuation of L-functions associated to Größencharaktere (see [17],[15] and [16]) but they are a bit buried by the many technical issues involved in the convoluted proof and therefore cannot be so easily extracted.

So on the one hand, the present paper can be viewed as an appendix to Hecke's work on Größencharaktere, but it is more than that since it provides a foundational reference on which some of our future work will rest. While introducing the larger class of $\mathcal{LZ}$ we shall also take the opportunity to introduce some well-adapted formalism to handle efficiently the infinite part of Größencharaktere. Also, it would have been possible to write this paper by restricting ourself to the maximal order $\mathcal{O}_K$ and thus work only with lattices which are $\mathcal{O}_K$-modules (so fractional $\mathcal{O}_K$-ideals) but doing so would obscure for example the various "distribution relations" which exist between $\mathcal{LZ}$ (such relations play for example a key role in the theory of modular units associated to an imaginary quadratic field, see for example [9]). Since we want to be able to associate $\mathcal{LZ}$ to arbitrary lattices of K, at a few places we shall use some basic results on lattices and orders in number fields which can be found for example in [6].

One stark contrast between $\mathcal{LZ}$ and the more classical L-functions associated to (abelian) Galois representations of number fields is that $\mathcal{LZ}$ **do not** admit an Euler product in general. For that reason, the use of the adelic language does not seem to confer much advantage over the more classical language of Hecke and in fact might even contribute to obscure some of the ideas. In fact, the author finds it easier to think of $\mathcal{LZ}$ in Hecke's classical language and for that reason this paper has been completely written in classic style. Also, in the spirit of the introduction given in [7], it is likely that the concept $\mathcal{LZ}$ can be further generalized to the setting of non-abelian Galois representations. It can be shown that a zeta function $Z(s)$ in $\mathcal{LZ}$ can be written as suitable finite linear combinations of L-functions of various Größencharaktere (of a fixed infinity type) and thus corresponds in that sense to abelian Galois representations (the GL_1 case so to speak). Since L-functions can also be associated to non-abelian Galois representations it is natural to ask for similar "*lattice automorphic objects*" which could be written as a suitable linear combination of L-functions of non-abelian Galois representations. In [4], we made a detailed study of what we now call "*signature* GL_2-*lattice real analytic Eisenstein series*" ($\mathcal{SLES}$). Such Eisenstein series solve an interpolation problem in the sense that it corresponds to a class of GL_2-Eisenstein series for which their 0-(parabolic)-Fourier coefficients is a simple 2-term linear combinations of $\mathcal{SLZ}$. The author has also worked out a few years ago the more general theory of "GL_2-*lattice real analytic Eisenstein series* for a general number field K where the general term of the Eisenstein series is allowed to be twisted by the infinite part of an arbitrary "complex Größencharaktere" but we did not publish such results yet. It follows essentially the same approach as in [4] except that in the case where K admits at least one complex embedding one is forced to work with *vector* valued Eisenstein series (in order to get modularity) since the irreducible unitary representations of $\mathrm{SU}(2)$ (the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{C})$) are no longer one dimensional as it was the case for $\mathrm{SO}(2)$ (the standard maximal compact subgroup of $\mathrm{SL}_2(\mathbb{R})$). So far our approach has been phenomenological by working out explicit examples, but in [7] some attempts to formalize are made in order to put under a common framework both the $\mathcal{SLZ}$ and $\mathcal{SLES}$. To the author's mind, his own understanding on this topic is judged to be naive and full of mysteries.

Let K be a number field of degree $g = r_1 + 2r_2$ over $\mathbb{Q}$. Here r_1 is the number of real embeddings and $2r_2$ the number of complex embeddings, and we say in that case that K has *signature* (r_1, r_2). Recall that by a *lattice* $\mathfrak{n}$ of K we mean a subset of K which is a free $\mathbb{Z}$-module of rank g. The most economical and precise way of defining what a lattice zeta function of K is, is simply as the Dirichlet series that one gets when one takes

(1.1) "*a weighted Mellin transform of a* **single** *number field (spherical) theta function*"

By a number field theta function we mean an average sum over the **full** lattice $\mathfrak{n}$ of a **single** shifted decorated Gaussian function multiplied by a *diagonal spherical polynomial.* The Gaussian function here can be viewed as a function on space which can be identified with $\mathfrak{h}^{r_1+r_2}$ (see Section 5.2 for the precise definition of the theta functions involved) where $\mathfrak{h} = \{x + \mathrm{i}\, y \in \mathbb{C} : y > 0\}$ is the Poincaré upper half-plane. By a weighted Mellin transform we mean a regular Mellin transform where the multi-dimensional $\mathfrak{s}$-parameter is shifted by the weight vector of the infinite part of the Größencharaktere considered.

In order to give a precise formula for the underlying generalized Dirichlet series describe by $\mathcal{LZ}$ and to state our main theorem we need to introduce some more notation. We let $K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R}$. It is an $\mathbb{R}$-algebra (non-canonically) isomorphic to the product ring $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and it is endowed with the usual maps $\mathrm{Tr} : K_{\mathbb{R}} \to \mathbb{C}$ (the trace: sum of the coordinates) and the $\mathbf{N} : K_{\mathbb{R}} \to \mathbb{C}$ (the norm: product of the coordinates). If $\mathfrak{n} \subseteq K$ is a lattice then $\mathfrak{n}^*$

denotes the *dual lattice* under the trace pairing $\mathrm{Tr}_{K/\mathbb{Q}} : K \times K \to \mathbb{Q}$ (see Section 1.1). We let

$$\mathcal{O}_{\mathfrak{n}} := \{\lambda \in K : \lambda \mathfrak{n} \subseteq \mathfrak{n}\} \tag{1.2}$$

be the *ring of multipliers* of $\mathfrak{n}$ (or the endomorphism ring of $\mathfrak{n}$). It is an order of K.

For a non-trivial real number $a \in \mathbb{R}$ we let

$$\Pi_a := \{s \in \mathbb{C} : \mathrm{Re}(s) > a\}$$

be the open right half-plane cut out by a. In particular, $-\Pi_a = \{s \in \mathbb{C} : \mathrm{Re}(s) < -a\}$ corresponds to the left open half plane cut out by $-a$.

A $\mathcal{LZ}$ of K will depend on some fixed choice of data namely on a triple

$$\mathfrak{T} = (\mathfrak{n}, (a, b); (\chi, \mathcal{V})) \tag{1.3}$$

where $\mathfrak{n} \subseteq K$ is a lattice, $a, b \in K$ and where

$$\chi : K_{\mathbb{R}}^{\times} \to S^1 \tag{1.4}$$

is a continuous character which factors through a subgroup $\mathcal{V}$ such that

(a) $\mathcal{V} \leq \mathcal{V}_{\mathfrak{n};a,b}$ has finite index,

and

(b) $\mathcal{V} \cap \mu_K = \{1\}$.

Here μ_K is the group of roots of unity in K and

$$\mathcal{V}_{\mathfrak{n};a,b} := \{\epsilon \in \mathcal{O}_{\mathfrak{n}}^{\times} : (\epsilon - 1)a \in \mathfrak{n}, (\epsilon - 1)b \in \mathfrak{n}^*, (\epsilon - 1)ab \in (\mathcal{O}_{\mathfrak{n}})^*\}. \tag{1.5}$$

We think of $\mathfrak{n}$, (a, b) and $(\chi, \mathcal{V})$ as the three elements which form the triple $\mathfrak{T}$. We think of the choices of $\mathfrak{n}$ and (a, b) as being free while the choice of $(\chi, \mathcal{V})$ as depending on $\mathfrak{n}$ and (a, b). We say that $(\chi, \mathcal{V})$ is an *admissible character* relative to the pair $(\mathfrak{n}, (a, b))$ if the above conditions are satisfied. Usually, we shall think of χ as the induced character on the quotient $K_{\mathbb{R}}^{\times}/\mathcal{V}$, i.e. as the map $\chi : K_{\mathbb{R}}^{\times}/\mathcal{V} \to S^1$, and therefore denote the triple simply as $\mathfrak{T} = (\mathfrak{n}, (a, b); \chi)$ while keeping in mind that implicitly the choice of χ determines such a choice of $\mathcal{V}$.

We call a triple $\mathfrak{T} = (\mathfrak{n}, (a, b); \chi)$ as above an **admissible triple**. To such an admissible triple one can then associate the **lattice zeta function** ($\mathcal{LZ}$)

(1.6)

$$Z_{\mathfrak{T}}(s) := Z_{\mathfrak{n}}(a, b; (\chi, \mathcal{V}); s) = Z_{\mathfrak{n}}(a, b; \chi; s) := (\mathbf{N}\,\mathfrak{n})^s \sum_{\substack{a+n \in \mathfrak{R} \\ a+n \neq 0}} \chi(a + n) \cdot \frac{e^{2\pi \mathrm{i}\, \mathrm{Tr}_{K/\mathbb{Q}}(b(a+n))}}{|\,\mathbf{N}_{K/\mathbb{Q}}(a + n)|^s}.$$

Here $\mathbf{N}\,\mathfrak{n}$ is the rational index of the lattice $\mathfrak{n}$ relative to $\mathcal{O}_K$ which coincides with the usual index when $\mathfrak{n} \subseteq \mathcal{O}_K$ (see Definition 2.8). The indexing set $\mathfrak{R} = \{n_i \in \mathfrak{n}\}_{i\in I}$ is a complete set of representatives of $\{a+\mathfrak{n}\}/\mathcal{V}$ in the sense that every element $0 \neq a+n \in a+\mathfrak{n}$ can be written uniquely as $\epsilon(a+n_i)$ for some $n_i \in \mathfrak{R}$ and $\epsilon \in \mathcal{V}$. It can be shown that $Z_{\mathfrak{T}}(s)$ does not not depend on such a choice of $\mathfrak{R}$ (see Proposition 1.13) It is a classical fact that such a series converges absolutely on Π_1 (see Proposition 1.16 where we included two distinct proofs for the convenience of the reader). The possibility for this series of having a conditional convergence for a suitable "geometrical summation order", within the strip $1-\epsilon \le \mathrm{Re}(s) \le 1$, for ϵ small enough, is very interesting problem and recently the author has obtained new results in that direction for $\mathcal{SLZ}$ associated to totally real fields, see [5].

Remark 1.1. So by definition the index of the sum which defines a lattice zeta function ($\mathcal{LZ}$) is not a (punctured) lattice but rather a (punctured) lattice *modulo* a finite index subgroup of $\mathcal{O}_K^{\times}$. This should be compared with the definition of a $\mathfrak{t}$-*decorated Epstein zeta function* which is given in Section 1.2 for which in that case the index of the sum is *really* a (punctured) lattice.

Remark 1.2. Note that the function $[s \mapsto Z_{\mathfrak{T}}(s)]$ could be identically equal to equal to zero. For example this is always the case when there exists a unit $\epsilon \in \mathcal{V}_{\mathfrak{n};a,b}$ such that $\chi(\epsilon) \neq 1$ (cf. with Question 7.3 in Section 7.1).

Definition 1.3. *We say that $Z_{\mathfrak{T}}(s)$ has a non-trivial additive oscillation on Π_1 if $b \notin \mathfrak{n}^*$ where $\mathfrak{n}^*$ is the dual lattice of $\mathfrak{n}$ (see Definition 1.13).*

The expression "finite additive" here refers to the following: on Π_1, the Dirichlet series in 1.6 computes the value of $Z_{\mathfrak{T}}(s)$ and the general term of its sum involves the map $n \mapsto e^{2\pi\,\mathrm{i}\,\mathrm{Tr}(bn)}$ which is a **non-trivial** finite order character of the **additive** group $(\mathfrak{n},+)$.

To each continuous character $\psi : K_{\mathbb{R}}^{\times} \to S^1$ one can associate a *unique* quadruple (the weight vector of ψ)

$$\mathcal{Q}_\psi = (\overline{m}_\psi, n_\psi, \gamma_\psi, \delta_\psi) \tag{1.7}$$

where $\overline{m}_\psi \in (\mathbb{Z}/2\mathbb{Z})^{r_1}$, $n_\psi \in \mathbb{Z}^{r_2}$, $\gamma_\psi \in \mathbb{R}^{r_1}$ and $\delta_\psi \in \mathbb{R}^{r_2}$. In particular, this applies to the character $\chi : K_{\mathbb{R}}^{\times}/\mathcal{V} \to S^1$. Let $\mathcal{Q}_\chi = (\overline{m}, n, \gamma, \delta)$ be the weight vector of χ. To this data one defines the following product

$$\begin{aligned}\mathrm{F}_\chi(s) := |d_K|^{\frac{s}{2}} \cdot \left(\pi^{-\frac{1}{2}\mathrm{Tr}(s\cdot\mathbb{1}_{r_1}+[\overline{m}]-\mathrm{i}\,\gamma)} \cdot \prod_{j=1}^{r_1} \Gamma\left(\frac{s}{2}+\frac{[m_j]-\mathrm{i}\,\gamma_j}{2}\right)\right)\cdot \\ \left(2^{r_2}\cdot(2\pi)^{-\mathrm{Tr}(s\cdot\mathbb{1}_{r_2}+\frac{|n|}{2}-\mathrm{i}\,\delta)} \cdot \prod_{j=1}^{r_2}\Gamma\left(s+\frac{|n_j|}{2}-\mathrm{i}\,\delta_j\right)\right)\end{aligned} \tag{1.8}$$

where $s \in \mathbb{C}$, $\mathrm{i} = \sqrt{-1}$ (while i will usually be reserved for an index),

(i) $\mathbb{1}_{r_1} = 1_{\mathbb{R}^{r_1}} := (1,\ldots,1)$ is 1 repeated r_1 times, $\mathbb{1}_{r_2} := 1_{\mathbb{R}^{r_2}} = (1,\ldots,1)$ is 1 repeated r_2 times;

(ii) $[\overline{m}] = ([\overline{m}_1], \ldots, [\overline{m}_{r_1}])$, where $[\overline{0}] = 0$, $[\overline{1}] = 1$ where 0 and 1 are viewed as elements in $\mathbb{Z}$;

(iii) $|n| = (|n_1|, \ldots, |n_{r_2}|) \in \mathbb{Z}_{\geq 0}$ is the absolute value vector of n;

(iv) Tr corresponds to the natural trace functions $\mathrm{Tr} : \mathbb{R}^{r_1} \to \mathbb{R}$ and $\mathrm{Tr} : \mathbb{R}^{r_2} \to \mathbb{R}$ where one takes the sum over all coordinates.

We call $\mathrm{F}_\chi(s)$ the *Euler factor at infinity of the character* χ. Note that $\mathrm{F}_\chi(s)$ does not depend on the data $(\mathfrak{n}, (a, b))$. We have preferred here to use the letter F (for "Factor") instead of the more classic notation L_∞ since, as was pointed out in the first paragraph, $\mathcal{LZ}$ do not admit an Euler product in general as it is the case for L-functions associated to Galois representations.

We let

$$\widehat{Z}_{\mathfrak{n}}(a, b, \chi; s) := \mathrm{F}_\chi(s) \cdot Z_{\mathfrak{n}}(a, b; \chi; s) \tag{1.9}$$

be the **completed lattice zeta function** associated to $Z_{\mathfrak{n}}(a, b; \chi; s)$. We are now ready to state our first main theorem.

Theorem 1.4. *Firstly, the function* $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)]$ *admits a holomorphic continuation to* $\mathbb{C} \setminus \{\mu_0, \mu_1\}$ *where*

(1.10)

$$\mu_0 := -\lambda_0, \;\; \mu_1 := -\lambda_0 + 1 + \frac{2\,\mathrm{Tr}([\overline{m}], |n|)}{g} \;\; \textit{where} \;\; \lambda_0 := \frac{\mathrm{Tr}\Big(([\overline{m}], |n|) - \mathrm{i}(\gamma, 2\delta)\Big)}{g},$$

and has at worst a pole of order one at μ_0 *and* μ_1*. Secondly,* $\widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)$ *satisfies the following functional equation:*

$$(\mathrm{i})^{\mathrm{Tr}([\overline{m}], |n|)} \cdot e^{-2\pi \mathrm{i}\, \mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s) = \widehat{Z}_{\mathfrak{n}^*}(-b, a, \overline{\chi}, 1 - s). \tag{1.11}$$

Recall here $\mathfrak{n}^*$ *is the dual lattice of* $\mathfrak{n}$ *(see Definition 1.13), and* $\overline{\chi}$ *is the complex conjugate character of* χ *which is viewed as a character on the quotient* $K_{\mathbb{R}}^\times / \mathcal{V} \to S^1$*; this makes sense since* $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n}^*;-b,a}$ *(see (1) of Proposition 1.12). Note that if* $(\overline{m}, n, \gamma, \delta)$ *is the quadruple associated to* χ *then* $(\overline{m}, -n, -\gamma, -\delta)$ *is the quadruple associated to* $\overline{\chi}$*. Thirdly, the function* $s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)$ *admits*

(a) a pole of order one at $s = \mu_1$*, if and only if,* $-b \in \mathfrak{n}^*$ *and* $(\overline{m}, n) = (\overline{\mathbb{0}}_{r_1}, \mathbb{0}_{r_2})$.

(b) a pole of order one at $s = \mu_0$*, if and only if,* $a \in \mathfrak{n}$ *and* $(\overline{m}, n) = (\overline{\mathbb{0}}_{r_1}, \mathbb{0}_{r_2})$.

As direct consequences of the above we may extract the following more precise statements:

(i) If $-b \notin \mathfrak{n}^*$ *and* $a \notin \mathfrak{n}$ *then* $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)]$ *is holomorphic on all of* $\mathbb{C}$.

(ii) If $(\overline{m}, n) \neq \mathbb{0}$ *then* $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)]$ *is holomorphic on all of* $\mathbb{C}$.

(iii) If $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)]$ *has a pole at* s_0 *then necessarily* $(\overline{m}, n) = (\overline{\mathbb{0}}_{r_1}, \mathbb{0}_{r_2})$ *and*

$$s_0 \in \left\{ \mu_0 = \frac{\mathrm{i}}{g} \operatorname{Tr}(\gamma, 2\delta), 1 - \mu_0 = 1 - \frac{\mathrm{i}}{g} \operatorname{Tr}(\gamma, 2\delta) \right\}.$$

The second main theorem of our paper is Theorem 7.8 which can be viewed as complementing Theorem 1.4. It provides some lower bound for the order of vanshing of $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi; s)]$ at non-positive integers and also an exact formula which relates a certain higher order derivative of a lattice zeta function, at a non-positive integer $\ell \in \mathbb{Z}_{\leq 0}$, to the special value at $1 - \ell$ of the dual lattice zeta function (see (7.17)).

As a surprising consequence of the functional equation (1.11) see the second part of Corollary 7.6.

Remark 1.5. In Section 3.5 it will be shown that any continuous character $\chi : K_{\mathbb{R}}^{\times}/\mathcal{V} \to S^1$ can be written *uniquely* as $\chi = \chi_0 \cdot (\mathbf{N})^{\mathrm{i}\, t_0}$ where

(1) $t_0 \in \mathbb{R}$,

(2) and if $\mathcal{Q}_{\chi_0} = (\overline{m}, n, \gamma, \delta)_{\chi_0}$ is the $\mathcal{Q}$-weight vector of χ_0 (see Definition 3.7) then $\operatorname{Tr}(\gamma, 2\delta) = 0$.

Since $\chi = \chi_0 \cdot (\mathbf{N})^{\mathrm{i}\, t_0}$ it follows that

$$Z_{\mathfrak{n}}(a, b, \chi; s) = Z_{\mathfrak{n}}(a, b, \chi_0; s - \mathrm{i}\, t_0). \tag{1.12}$$

So for a general character $\chi : K_{\mathbb{R}}^{\times} \to S^1$, if $\mathcal{Q}_{\chi} = (\overline{m}, n, \gamma, \delta)_{\chi}$ is its $\mathcal{Q}$-weight vector, there is not much loss of generality in assuming that $\operatorname{Tr}(\gamma, 2\delta) = 0$ and when this is the case it follows from (iii) of Theorem 1.4 that the possible poles of $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a, b, \chi_0; s)]$ can only be located in the "usual standard set" $\{0, 1\}$. So the a priori strange looking set $\{\mu_0, \mu_1\}$ essentially reduces to the set $\{0, 1\}$.

Remark 1.6. In the notation of [2], let K be a number field of degree $n = r_1 + 2r_2$ where r_1 is the number of real embeddings. Let $p \in \{0, 1\}^{r_1}$ be a choice of a real signature and let $\chi := \omega_p : K_{\mathbb{R}}^{\times} \to \{\pm 1\}$ be the *sign character* of signature p. In particular, using the above notation this means that $\gamma = \mathbb{0}_{r_1}, n = \mathbb{0}_{r_2}, \delta = \mathbb{0}_{r_2}$. Then, Theorem 1.4 above, when specialized to that case, agrees with Theorem 1.1 of [2] except for the factor $(-\,\mathrm{i})^{\operatorname{Tr}(p)}$ which appears in equation (1.3) of loc. cit. which should be read instead as $(\mathrm{i})^{\operatorname{Tr}(p)}$ because of a sign mistake which occurred in equation (4.12) of loc. cit. while retranscribing equation (4.11).

Remark 1.7. In [3], explicit relationships (which go in both ways) between classical L-functions twisted by finite order Hecke characters and signature lattice zeta functions are given. The method generalizes to $Z_{\mathfrak{n}}(a, b; \chi; s)$ (which was denoted in [3] by $\Psi_V(a, b, \omega, s)$, when $\chi = \omega$ is a sign character). The general principle of this method is in fact contained in Hecke's original approach; see [26] for a modern presentation of Hecke's approach given in terms of the so-called ideal numbers.

Let us now give a brief outline of the various sections of this paper. In Section 1, we first provide some background on lattices and orders in number fields and prove some basic results on the group of units $\mathcal{V}_{\mathfrak{n};a,b}$ on which lattice zeta functions depend. We then explain a connection between lattice zeta functions and Epstein zeta functions. In Section 2 we introduce some convenient labelling of the set of embeddings of a general number field K and recall Neurkich's notation that was introduced in Chapter VII of [26]. In particular in that section we introduce the space $\mathbf{R}$ ($\simeq K_{\mathbb{R}}$) with its canonical metric, the normalized multiplicative Haar measure on $\mathbf{R}_+$ and the multidimensional Gamma function. In Section 3 we introduce some useful notation to describe the set of quasi-characters of the Lie group $\mathbf{G} := K_{\mathbb{R}}^{\times}$; and given a fixed finite index subgroup $\mathcal{V}$ of $\mathcal{O}_K^{\times}$, we make a detailed study of the space $\mathfrak{X}_{\mathcal{V}}$ which corresponds to those quasi-characters on $\mathbf{G}$ which are trivial on $\mathcal{V}$. We also introduce some convenient way of parametrizing the quasi-characters in $\mathfrak{X}_{\mathcal{V}}$ in terms of quintuple and explain their relationship with Größencharaktere and ideal class group characters of K. In Section 4, we introduce the normalized weight $w_\chi = (p_\chi, q_\chi)$ associated to a character χ of $\mathbf{G}$ and the Euler factor associated to such a weight. In Section 5 we provide some minimal background on spherical polynomials and then define number field theta functions weighted by such spherical polynomials. In Section 6 we prove our main theorem. Most of the notation introduced before and the various lemmas proved in the previous sections appear in that proof. In Section 7 we record some of the basic properties of lattice zeta functions. Finally, for the convenience of the reader we added 3 appendices where each provides some complementary material on some parts of this work.

1.1 Lattices, orders and the finite index subgroup $\mathcal{V}_{\mathfrak{n},a,b}$

Definition 1.8. *Let $\mathfrak{n} \subseteq K$ be a lattice. We define*

$$\mathcal{O}_{\mathfrak{n}} := \{\lambda \in K : \lambda\mathfrak{n} \subseteq \mathfrak{n}\},$$

and call $\mathcal{O}_{\mathfrak{n}}$ the multiplier ring of $\mathfrak{n}$ (or the endomorphism ring of $\mathfrak{n}$). Given a lattice $\mathfrak{n} \subseteq K$ we let

$$\mathfrak{n}^* := \{x \in K : \mathrm{Tr}_{K/\mathbb{Q}}(x\ell) \in \mathbb{Z} \textit{ for all } \ell \in \mathfrak{n}\}. \tag{1.13}$$

One may check that $\mathcal{O}_{\mathfrak{n}}$ is an order of K, i.e. $\mathcal{O}_{\mathfrak{n}}$ is a subring of $\mathcal{O}_K$ such that $[\mathcal{O}_K : \mathcal{O}_{\mathfrak{n}}] < \infty$. Also $\mathfrak{n}^*$ is again a lattice that we call the *dual lattice* of $\mathfrak{n}$ (dual under the trace pairing). For a short account on the basic properties of lattices and orders in number field see [6].

We shall use the following elementary properties for which a proof can be found in [6] for example:

Proposition 1.9. *Let $\mathfrak{n} \subseteq K$ be a lattice. Then $\mathfrak{n}^{**} = \mathfrak{n}$, $\mathcal{O}_{\mathfrak{n}} = \mathcal{O}_{\mathfrak{n}^*}$ and $\mathfrak{n}\mathfrak{n}^* = (\mathcal{O}_{\mathfrak{n}})^*$.*

Proof The proof of the first equality is straightforward. For a proof of the two remaining equalities see Propositions 3.24 and 3.28 of [6]. □

Definition 1.10. *Let $\mathfrak{n}$ be a lattice of K. For a pair of elements $a, b \in K$, we define*

$$\mathcal{V}_{\mathfrak{n};a,b} := \{\epsilon \in \mathcal{O}_{\mathfrak{n}}^{\times} : (\epsilon-1)a \in \mathfrak{n}, (\epsilon-1)b \in \mathfrak{n}^*, (\epsilon-1)ab \in (\mathcal{O}_{\mathfrak{n}})^*\}. \tag{1.14}$$

Remark 1.11. In [2], a slightly different group than $\mathcal{V}_{\mathfrak{n};a,b}$ was used. It was denoted instead by $\Gamma_{a,b,\mathfrak{n}}$ and its definition was the same as in (1.14), except that the last condition in (1.14), which reads here "$(\epsilon-1)ab \in (\mathcal{O}_{\mathfrak{n}})^*$" was instead given by "$(\epsilon-1)ab \in \mathfrak{d}_K^{-1}$", where $\mathfrak{d}_K$ is the different ideal of K. Note that since $\mathfrak{d}_K^{-1} \subseteq (\mathcal{O}_{\mathfrak{n}})^*$, we always have that $\Gamma_{a,b,\mathfrak{n}} \leq \mathcal{V}_{\mathfrak{n};a,b}$. The reason why it is preferable to work with $\mathcal{V}_{\mathfrak{n};a,b}$ rather than $\Gamma_{a,b,\mathfrak{n}}$ is because we want property (3) below to hold true.

The next proposition makes precise the exact dependence of $\mathcal{V}_{\mathfrak{n};a,b}$ on the triple $(\mathfrak{n}; a, b)$.

Proposition 1.12. *We have*

(1) $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n};-a,b} = \mathcal{V}_{\mathfrak{n};a,-b} = \mathcal{V}_{\mathfrak{n};-a,-b}$ *and* $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n}^*;-b,a}$.

(2) For all $\lambda \in K\backslash\{0\}$, $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\lambda\mathfrak{n};\lambda a,\frac{b}{\lambda}}$

(3) if $a \equiv a' \pmod{\mathfrak{n}}$ *and* $b \equiv b' \pmod{\mathfrak{n}^*}$ *then* $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n};a',b'}$

(4) The subgroup $\mathcal{V}_{\mathfrak{n};a,b}$ *has finite index in* $\mathcal{O}_K^{\times}$.

Proof The proofs of the first sequence of equalities in (1) is clear. The last equality in (1) follows directly from the definition of $\mathcal{V}_{\mathfrak{n};a,b}$ and the facts that $\mathfrak{n}^{**} = \mathfrak{n}$ and $\mathcal{O}_{\mathfrak{n}} = \mathcal{O}_{\mathfrak{n}^*}$ (see Proposition 1.9). The proof of (2) follows from the observations that $(\lambda\mathfrak{n})^* = \frac{1}{\lambda}\mathfrak{n}^*$ and $\mathcal{O}_{\mathfrak{n}} = \mathcal{O}_{\lambda\mathfrak{n}}$. For proving (3), let $\mathcal{O} := \mathcal{O}_{\mathfrak{n}} = \mathcal{O}_{\mathfrak{n}^*}$, $a' = a + \ell$ and $b' = b + \ell^*$ for some $\ell \in \mathfrak{n}$ and $\ell^* \in \mathfrak{n}^*$. We have $\ell\mathfrak{n}^* \subseteq \mathcal{O}^*$, $\ell^*\mathfrak{n} \subseteq \mathcal{O}^*$ and $(\epsilon-1)\ell\ell^* \subseteq \mathcal{O}^*$. From this it follows that for each $\epsilon \in \mathcal{O}_{\mathfrak{n}}$, we get the following equivalence:

$$(\epsilon-1)ab \in \mathcal{O}^* \Longleftrightarrow (\epsilon-1)a'b' \in \mathcal{O}^*,$$

which proves that $\mathcal{V}_{\mathfrak{n};a,b} = \mathcal{V}_{\mathfrak{n};a',b'}$. It remains to prove (4). Let $\{\epsilon_i : i = 1, \ldots, g-1\} \subseteq \mathcal{O}_K^{\times}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K^{\times}/\mu_K$. The algebraic number a in the triple $(\mathfrak{n}; a, b)$ may be written as: $\frac{s}{t}$ for $s, t \in \mathcal{O}_K$ and $t \neq 0$, where s is chosen, so that $s\mathfrak{n} \subseteq \mathfrak{n}$. Choose $N \in \mathbb{Z}_{\geq 1}$, such that $N\mathcal{O}_K \subseteq \mathfrak{n}$. Let $m = \#(\mathcal{O}_K/tN\mathcal{O}_K)^{\times}$. Then for all $i \in \{1, \ldots, g-1\}$, we have

$$\epsilon_i^m \in 1 + tN\mathcal{O}_K \subseteq 1 + t\mathfrak{n}.$$

Therefore, $(\epsilon_i^m - 1)a = (\epsilon_i^m - 1)\frac{s}{t} \in s\mathfrak{n} \subseteq \mathfrak{n}$. Similarly, we may find an integer $m' \geq 1$, such that for each $i \in \{1, \ldots, g-1\}$, $(\epsilon_i^{m'} - 1)b \in \mathfrak{n}^*$. Let $m'' = \operatorname{lcm}(m, m')$. Finally, since $(\mathcal{O}_K^{\times})^{m'} \leq \mathcal{V}_{\mathfrak{n};a,b}$ it follows that $[\mathcal{O}_K^{\times} : \mathcal{V}_{\mathfrak{n};a,b}] \leq (m'')^{g-1}$. □

Proposition 1.13. *The summation in* (1.6) *is independent of a choice of representatives* $\mathfrak{R}$ *of the quotient set* $(a + \mathfrak{n})/\mathcal{V}$.

Proof Let $\epsilon \in \mathcal{V}_{\mathfrak{n};a,b}$ and let $n \in \mathfrak{n}$ be such that $a + n \neq 0$. Since $\epsilon \in \mathcal{V}_{\mathfrak{n};a,b}$ we have

(i) $\epsilon(a+n) = a + n'$ for (a unique) $n' \in \mathfrak{n}$,

(ii) $(\epsilon - 1)ab \in (\mathcal{O}_{\mathfrak{n}})^*$,

(iii) $(\epsilon - 1)b \in \mathfrak{n}^*$

Combining (i), (ii), (iii) with the fact that $\mathfrak{n}\mathfrak{n}^* \subseteq (\mathcal{O}_{\mathfrak{n}})^*$ (from Proposition 1.9 the last inclusion is in fact an equality) gives

$$e^{2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(b\cdot\epsilon(a+n))} = e^{2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(b\cdot(a+n'))}. \tag{1.15}$$

Since χ is $\mathcal{V}$-invariant combining (1.15) and (4.4) we find that

(1.16)

$$\chi(a+n)\cdot\frac{e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}(b\cdot(a+n))}}{|\,\mathbf{N}(a+n)|^{2s}} = \frac{\mathbf{N}((a+n)^p)}{|\,\mathbf{N}((a+n)^{p-\mathrm{i}\,q})|}\frac{e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}(b\cdot(a+n)}}{|\,\mathbf{N}(a+n)|^{2s}} = \chi(a+n')\cdot\frac{e^{2\pi i\,\mathrm{Tr}_{K/\mathbb{Q}}(b\cdot(a+n')}}{|\,\mathbf{N}(a+n')|^{2s}}.$$

This proves the claim. □

1.2 The relationship between lattice zeta functions and Epstein zeta functions

We follow the notation used in Section 1.4.2 and 1.4.3 of [29]. Let $\mathcal{P}_g$ be the symmetric space of positive definite symmetric $n\times n$ real matrices. Let $Y \in \mathcal{P}_g$. Given a column vector $x \in \mathbb{R}^g$ we let $Y[x] := x^tYx$ so that $[\,x \mapsto Y[x]\,]$ is a positive definite quadratic form on $\mathbb{R}^g$. The classical Epstein zeta function associated to Y is defined as

$$Z(Y,s) := \sum_{n\in\mathbb{Z}^g\setminus\{0\}} \frac{1}{Y[m]^s}, \tag{1.17}$$

where $s \in \Pi_{\frac{g}{2}}$. The fact that $Z(Y,s)$ converges absolutely when $s \in \Pi_{\frac{g}{2}}$ follows rather directly from the absolute convergence of the integral given in Section 1.3 (see (1.22)) further down below. More generally, given a triple $\mathfrak{t} = (a,b;\psi)$ where $a,b \in \mathbb{R}^g$ and where $\psi : (\mathbb{Z}^g + a) \to \mathbb{C}$ is a function, one may associate the following $\mathfrak{t}$-*decorated Epstein zeta function*:

$$Z_{\mathfrak{t}}(Y,s) = \sum_{\substack{n\in\mathbb{Z}^g\setminus\{(0,\ldots,0)\}\\ n+a\neq 0}} \psi(a+n)\cdot\frac{e^{2\pi\,\mathrm{i}\langle b,(a+n)\rangle}}{Y[a+n]^s} \tag{1.18}$$

where $\langle,\rangle$ is the standard inner product on $\mathbb{R}^g$ (i.e. the one corresponding to the Minkowski metric, Section 2.2). Under suitable assumptions, this Dirichlet series will again converge on a right half-plane. For example if ψ is bounded then obviously $[s \mapsto Z_{\mathfrak{t}}(Y,s)]$ converges absolutely on $\Pi_{\frac{g}{2}}$. Let us now assume that $\psi = 1$ until the end of this paragraph. At the very beginning of the 20th century, Epstein made a detailed study of such Dirichlet series (see [11] and [12]; see also §5 in Chapter 1 of [27]). In fact, in the introduction of

[11], one can read that Epstein attributes the discovery of the basic properties of these Dirichlet series to R. Lipschitz (who incidentally passed away in 1903): namely the proof of the analytic continuation to $\mathbb{C}$ and of the functional action in s following the classical approach of Riemann. See for example equation (7) of [11] or Theorem 3 of [27].

Let K be a number and let Σ be its set of embeddings into $\mathbb{C}$. We fix for the rest of this section an ordering of Σ. Following [29] (see Theorem 1.4.2), if one chooses an ordered basis $\mathcal{B}_1$ of a lattice $\mathfrak{n} \subseteq K$, a fixed ordered basis $\mathcal{B}_2$ of the free part of a finite index subgroup $\mathcal{V} \leq \mathcal{O}_{\mathfrak{n}}$ and a real vector $x \in \mathbb{R}^g$, one can associate a positive definite symmetric matrix

$$Y = Y_{\mathcal{B}_1,\mathcal{B}_2}(\mathfrak{n}, x) \in \mathcal{P}_g.$$

When $\mathfrak{t} = (0, 0; 1)$, it is straightforward to see that the Epstein zeta function $Z_{\mathfrak{t}}(Y, s)$ does not depend on the choices of the ordering of Σ and of the basis $\mathcal{B}_1$; it does depend however on the choice of $\mathcal{B}_2$. When $\mathcal{V} = \mathcal{O}_K^\times$ and $\mathfrak{n} = \mathcal{O}_K$, Hecke proved the following remarkable integral formula:

Theorem 1.14. *(Hecke, see [18])*

$$\frac{w}{2^{r_1} g R_K} \cdot \Lambda_K(s) = \pi^{-s}\Gamma(s) \sum_{[\mathfrak{a}] \in \mathcal{C}\ell_K} \int_{x \in [0,1]^r} Z_{\mathfrak{t}}\left(Y_{\mathcal{B}_1,\mathcal{B}_2}(\mathfrak{a}, x)^0, \frac{gs}{2}\right) dx \tag{1.19}$$

Here $\mathcal{C}\ell_K$ is the ideal class group of K, R_K is the regulator of K, $\mathfrak{t} = (0, 0, 1)$, $r = r_1 + r_2 - 1$, $Y^0 = |Y|^{-\frac{1}{g}} Y$ and $\Lambda_K(s)$ is the completed zeta function of $\zeta_K(s)$ so that $\Lambda_K(s) = \widehat{Z}_{\mathcal{O}_K}(0, 0, 1; s)$. In particular, the left hand side does not depend on the choice of $\mathcal{B}_2$.

A more modern and very detailed sketch of proof of Theorem 1.14 can be found on p. 77-79 of [29]. Of course Theorem 1.14 generalizes verbatim to arbitrary lattices $\mathfrak{n} \subseteq K$ and arbitrary finite index subgroups $\mathcal{V} \leq \mathcal{O}_{\mathfrak{n}}$ where again this time the left hand side of (1.14) depends only on $\mathfrak{n}$ and $\mathcal{V}$ and not on the choices of the ordering of Σ and of the bases $\mathcal{B}_1$ and $\mathcal{B}_2$.

Now, let $\mathfrak{T} = (\mathfrak{n}, (a, b); (\chi, \mathcal{V}))$ be a triple as in the introduction so that $\mathcal{V}$ satisfies the conditions (a) and (b) of the introduction. In particular $\mathcal{V} \cap \mu_K = \{1\}$ and $\chi \in \mathfrak{X}_{\mathcal{V}}^u$ (see Definition 3.12). Recall that Σ is endowed with a fix ordering, $\mathcal{B}_1$ is an ordered basis of $\mathfrak{n}$ and $\mathcal{B}_2$ is a ordered basis of $\mathcal{V} \leq \mathcal{O}_K^\times$. In particular, using the basis $\mathcal{B}_1$, we have that any element $a \in K$ corresponds to a unique vector in $\mathbb{Q}^g$ that we still denote by a. If one considers the special triple $\mathfrak{t} = (a = 0, b = 0; \chi)$ then a formula which generalizes (1.19) is proved in [19] (see their main Theorem 0.2). It seems very likely that such a formula holds true in complete generality, i.e., without the assumption that $a = b = 0 \in \mathbb{Q}^g$ but the author did not check all the details. Since the notation for the current paper is already quite involved, we refrain from introducing an additional layer of notation that would handle simultaneously Lattice zeta functions and $\mathfrak{t}$-decorated Epstein zeta functions. We might address this question somewhere else in the future.

At this point, what the reader should keep in mind is that a lattice zeta function $Z_{\mathfrak{T}}(s)$ can be written as a certain integral of a parametrized family of $\mathfrak{t}$-decorated Epstein zeta functions; and therefore the analytic continuation and functional equation of $Z_{\mathfrak{T}}(s)$, in principle, follows rather directly from the analytic continuation and functional equation of $\mathfrak{t}$-decorated Epstein zeta functions. Since the usual proof of the analytic continuation and functional equation of Epstein zeta functions follows Riemann's approach (it relies on the Poisson summation formula), ultimately, the proof of Theorem 1.4 which is given in Section 6 is fundamentally not so different from the one suggested here.

Remark 1.15. The study of signature lattice Eisenstein series that was made in [4], for K a totally real field, can also be obtained in an alternative way by using Epstein zeta functions. As was pointed out in the introduction of this paper, one can extend the results of [4] so that K is allowed to have complex embeddings and where the general term of a lattice Eisenstein series is allowed to be twisted by an admissible characters of $K_{\mathbb{C}}^{\times} = K \otimes_{\mathbb{Q}} \mathbb{C}$ (what we call a "complex Größencharaktere"). In fact, the writing of such an extended theory using the Epstein approach seems to allow a more uniform and smooth presentation than the more elementary approach followed in [4].

1.3 Absolute convergence on Π_1

Proposition 1.16. *We use the same notation as in the introduction so that K is an arbitrary number field of signature (r_1, r_2) and $Z_{\mathfrak{T}}(s)$ is the Dirichlet series defined in* (1.6) *associated to an admissible triple $\mathfrak{T} = (\mathfrak{n}, (a, b); \chi)$, so a lattice zeta function. Then the Dirichlet series $[s \mapsto Z_{\mathfrak{T}}(s)]$ converges absolutely for all $s \in \Pi_1$.*

(**1st Proof**) The simplest proof is to compare directly $Z_{\mathfrak{T}}(s)$ with the Dedekind zeta function $\zeta_K(s)$ and then use the fact that $\zeta_K(s)$ can be easily compared with $\zeta_{\mathbb{Q}}(s)$ (the Riemann zeta function) which is known to converge absolutely on Π_1. Let $Z(s) := Z_{\mathfrak{T}}(s)$ and $s = \sigma + \mathrm{i}\, t \in \Pi_1$. Since $|\chi(a+n)| = |e^{2\pi i \operatorname{Tr}_{K/\mathbb{Q}}(b\cdot(a+n))}| = 1$ we have

$$|Z(s)| \le \mathbf{N}(\mathfrak{n})^{\sigma} \cdot \sum_{\substack{a+n \in \mathfrak{N} \\ a+n \neq 0}} \frac{1}{|\,\mathbf{N}_{K/\mathbb{Q}}(a+n)|^{\sigma}}. \tag{1.20}$$

By rescaling we can assume without loss of generality that $\mathfrak{n}$ is a sublattice of $\mathcal{O}_K$ and that $a \in \mathcal{O}_K$ since this does not affect the convergence of (1.20). We have

$$\sum_{\substack{a+n \in \mathfrak{N} \\ a+n \neq 0}} \frac{1}{|\,\mathbf{N}_{K/\mathbb{Q}}(a+n)|^{\sigma}} \le [\mathcal{O}_K^{\times} : \mathcal{V}] \cdot \sum_{0 \neq x \in \mathcal{O}_K/\mathcal{O}_K^{\times}} \frac{1}{|\,\mathbf{N}\, x|^{\sigma}}. \tag{1.21}$$

Now the idea is to bound by above the right hand side of (1.21) by a Dirichlet series which can be related to the Riemann zeta function $\zeta_{\mathbb{Q}}(s)$ which is known to converge absolutely on Π_1. We let $\mathcal{C}\ell_K = \operatorname{Pic}(\mathcal{O}_K) = \bigsqcup_{i=1}^{h} [\mathfrak{a}_i]$ where $[\mathfrak{a}_i]$ is the $\mathcal{O}_K$-ideal class of $\mathfrak{a}_i$ where h is the class number of K. Now since $\mathcal{O}_K$ is a Dedekind domain, each non-zero integral $\mathcal{O}_K$-ideal $\mathfrak{b}$ can be written as $\mathfrak{b} = \lambda \mathfrak{a}_i^{-1}$ for a unique index $i \in \{1, \ldots, h\}$ and a $\lambda \in \mathfrak{a}_i$ which is uniquely determined modulo $\mathcal{O}_K^{\times}$. From this it follows that

$$[\mathcal{O}_K^\times : \mathcal{V}] \cdot \sum_{0 \neq x \in \mathcal{O}_K/\mathcal{O}_K^\times} \frac{1}{|\mathbf{N}\, x|^\sigma} \leq \sum_{\substack{(0)\neq \mathfrak{a} \subseteq \mathcal{O}_K \\ \mathfrak{a} \text{ is an } \mathcal{O}_K\text{-ideal}}} \frac{1}{(\mathbf{N}\, \mathfrak{a})^\sigma} = \zeta_K(\sigma) \leq \zeta_{\mathbb{Q}}(\sigma)^{[K:\mathbb{Q}]} < \infty.$$

Note that the first inequality is an equality only when $h = 1$ and the second equality uses the Euler product of $\zeta_K(s)$ and the fact that a prime $p \in \mathbb{Z}$ factors in a product of at most g prime ideals of $\mathcal{O}_K$.

The proof given above is not completely satisfactory since we are not proving directly the convergence of $Z(s)$ but bound it indirectly by a Dirichlet series which involves potentially more terms but which admits an Euler product and which allows it to be compared to $\zeta_{\mathbb{Q}}(s)$

(**2nd proof**) A potentially more enlightening and geometrical proof which does not use any Euler product is the following. As in the 1st proof we have the inequalities (1.20) and (1.21). For $s = \sigma + it \in \Pi_1$ we let

$$\mathcal{Z}(s) := \sum_{0 \neq x \in \mathcal{O}_K/\mathcal{O}_K^\times} \frac{1}{|\mathbf{N}\, x|^s}.$$

Recall that each matrix $Y \in \mathcal{P}_g$ is equivalent to the diagonal matrix I_g, under the congruence relation by matrices of $\mathrm{GL}_g(\mathbb{R})$. It follows from Theorem 1.14 and the compactness of $[0,1]^r$ ($r = r_1 + r_2 - 1$) that the absolute converge of $\mathcal{Z}(s)$ is equivalent to the absolute convergence of the standard Epstein zeta function $Z_{I_g}(\frac{gs}{2})$ where I_g is the $g \times g$ identity matrix. But the absolute convergence of $[s \mapsto Z_{I_g}(\frac{gs}{2})]$ on Π_1 follows directly from the simple observation that the integral

$$\int_{\substack{x \in \mathbb{R}^g \\ x^t \cdot x \geq 1}} (x^t \cdot x)^s dx \tag{1.22}$$

converges absolutely when $s \in \Pi_{\frac{g}{2}}$ (this can be seen most directly by using the usual spherical coordinates on $\mathbb{R}^g$). Here x is viewed as a column vector and x^t is the transpose of x. $\square$

Remark 1.17. The second proof described above is essentially the one found in Sections 6.11 and 6.12 of [10] where it is worked out in an elementary and detailed way for a number field K of signature $(0, r_2)$ (more precisely for K a cyclotomic field).

Remark 1.18. It is worth to emphasize that the number of lattice points of $\mathbb{Z}^g$ in a large centered balls of $\mathbb{R}^g$ and Weyl's law for the asymptotic number of eigenvalues of the euclidean Laplacian acting on the real torus $\mathbb{R}^g/\mathbb{Z}^g$ are two sides of the same coin, see for example Chapter 0 of [20] which deals with the case $g = 2$. In particular, any crude version of Weyl's asymptotic law implies readily the absolute convergence of (1.22) when $s \in \Pi_{\frac{g}{2}}$ (see exercise 1.4.5 of [29]).

1.4 The zeta function of an order $\mathcal{O}$

If $\mathcal{O} \subseteq \mathcal{O}_K$ is a non-maximal order, it seems natural to define the zeta function of $\mathcal{O}$ as

$$\zeta_{\mathcal{O}}(s) := \sum_{\substack{\mathfrak{a}\subseteq \mathrm{Inv}(\mathcal{O}) \\ \mathfrak{a} \subseteq \mathcal{O}}} \frac{1}{(\mathbf{N}\,\mathfrak{a})^s} \tag{1.23}$$

where $\mathrm{Inv}(\mathcal{O})$ is the group of $\mathcal{O}$-invertible fractional ideals and $\mathbf{N}(\mathfrak{a})$ corresponds to the absolute norm (see [6]). Since the group $\mathrm{Pic}(\mathcal{O})$ is finite, one can write $\zeta_{\mathcal{O}}(s)$ as a finite sum of partial zeta functions where the finite sum is over the ideal classes of $\mathrm{Pic}(\mathcal{O})$. In particular, it follows from Proposition 1.16 that $\zeta_{\mathcal{O}}(s)$ converges absolutely on Π_1. However, contrary to $\zeta_{\mathcal{O}_K}(s) = \zeta_K(s)$, in general $\zeta_{\mathcal{O}}(s)$ does not admit an Euler product. However, if $\mathfrak{f} = \mathfrak{c}_{\mathcal{O}} \subseteq \mathcal{O}$ is the conductor of $\mathcal{O}$ (the largest $\mathcal{O}_K$-ideal contained in $\mathcal{O}$, see Section 3.4 of [6]) then it follows from Proposition 3.39 of [6] that

$$\zeta_{\mathcal{O};\mathfrak{f}}(s) := \sum_{\substack{\mathfrak{a}\subseteq \mathrm{Inv}(\mathcal{O}) \\ \mathfrak{a} \subseteq \mathcal{O},\, (\mathfrak{a},\mathfrak{f}) = 1}} \frac{1}{(\mathbf{N}\,\mathfrak{a})^s} = [\mathcal{O}_K : \mathcal{O}]^{-s} \cdot \zeta_{\mathcal{O}_K;\mathfrak{f}}(s) \tag{1.24}$$

where

$$\zeta_{\mathcal{O}_K;\mathfrak{f}}(s) = \sum_{\substack{\mathfrak{a}\subseteq \mathcal{O}_K \\ \text{non-zero } \mathcal{O}_K\text{-ideal such that } (\mathfrak{a},\mathfrak{f}) = 1}} \frac{1}{(\mathbf{N}\,\mathfrak{a})^s} = \prod_{\substack{\mathfrak{p}\subseteq \mathcal{O}_K \\ \text{non-zero prime } \mathcal{O}_K\text{-ideal} \\ (\mathfrak{p},\mathfrak{f})=1}} \left(1 - \frac{1}{(\mathbf{N}\,\mathfrak{p})^s}\right)^{-1}.$$

2 Embeddings and a review of Neukirch's notation

The notation introduced in this section is largely based on the notation found on p. 444-445 of [26]. Let K be an number field of degree $g = r_1 + 2r_2$ over $\mathbb{Q}$ and let $\Sigma = \mathrm{Hom}(K, \mathbb{C})$ be a complete set of embeddings of K into $\mathbb{C}$. As in the introduction, r_1 is the set of real embeddings of K and $2r_2$ is the set of complex embeddings of K so that the number field K has signature (r_1, r_2). Let

$$c_\infty : \mathbb{C} \to \mathbb{C} \tag{2.1}$$

denote the complex conjugation. The group $\mathrm{Gal}(\mathbb{C}/\mathbb{R}) = \langle c_\infty \rangle$ acts on the left Σ. Usually for an element $a \in \mathbb{C}$ we denote its complex conjugate $c_\infty(a)$ by $\overline{a}$ and similarly if $\tau \in \Sigma$ then $\overline{\tau}$ means $c_\infty \circ \tau$. We choose to write Σ as

$$\Sigma_r = \{\tau_1 = \rho_1,\, \tau_2 = \rho_2,\, \ldots,\, \tau_{r_1} = \rho_{r_2}\} \tag{2.2}$$

for the set of real embeddings of K and

$$\Sigma_c = \{\tau_{r_1+1} := \sigma_1,\, \tau_{r_1+2} := \sigma_2,\, \ldots, \tau_{r_1+r_2} := \sigma_{r_2}, \tau_{r_1+r_2+1} = \overline{\sigma}_1,\, \ldots,\, \tau_{r_1+2r_2} := \overline{\sigma}_{r_2}\} \tag{2.3}$$

be the set of complex embeddings of K, so that $\Sigma = \Sigma_r \bigsqcup \Sigma_c$. It is also convenient to let

$$\Sigma_c' := \{\sigma_1, \dots, \sigma_{r_2}\}, \qquad \Sigma_c'' = \{\overline{\sigma}_1, \dots, \overline{\sigma}_{r_2}\}. \tag{2.4}$$

In particular, the Σ_r can be viewed as the $\mathrm{Gal}(\mathbb{C}/\mathbb{R})$ fix point set of Σ. We call Σ_c' and Σ_c'' *half set of complex embeddings.*

Definition 2.1. *A choice of writing of Σ as above is called an admissible labelling of Σ. Sometimes, the set Σ endowed with a choice of an admissible labelling will be denoted by ${}^{\ell}\Sigma$. A pair $(K, {}^{\ell}\Sigma)$ is called a framed number field.*

In particular, if Σ is endowed with an admissible labelling then the sets Σ_ρ, Σ_c, Σ_c', Σ_c'' and Σ come with a given ordering.

Recall that $K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R}$. The admissible labelling of Σ provides a priviledged identification

$$\begin{aligned} \iota : K_{\mathbb{R}} &\to \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \\ x \otimes \lambda &\mapsto (\rho_1(x)\lambda, \dots, \rho_{r_1}(x)\lambda; \sigma_1(x)\lambda, \dots, \sigma_{r_2}(x)\lambda) \end{aligned} \tag{2.5}$$

In fact, a choice of an admissible labelling of Σ

(1) provides an identification between the three sets $K_{\mathbb{R}}$, $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and $\mathbb{R}^{\Sigma_r} \times \mathbb{C}^{\Sigma_c'}$,

(2) and also an identification between the two sets $\mathbb{R}^{r_1+r_2} = \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$ and $\mathbb{R}^{\Sigma_r \cup \Sigma_c'}$.

We consider the framed g-dimensional $\mathbb{C}$-algebra attached to ${}^{\ell}\Sigma$

$$\mathbf{C}_\Sigma := \left(\mathbb{C}^\Sigma, {}^{\ell}\Sigma\right).$$

Using the labelling of Σ gives the following privileged isomorphism of $\mathbb{C}$-algebras

$$\begin{aligned} \psi : \mathbf{C}_\Sigma &\to \mathbb{C}^g \\ (z_{\tau_j})_{\tau_j} &\mapsto (z_{\tau_1}, \dots, z_{\tau_g}) \end{aligned} \tag{2.6}$$

In particular, note that the isomorphism class of $\mathbf{C}_\Sigma$ as a $\mathbb{C}$-algebra is only a dependence of the degree $[K : \mathbb{Q}] = g$. For the sequel, since ${}^{\ell}\Sigma$ is fixed we shall denote $\mathbf{C}_\Sigma$ simply by $\mathbf{C}$.

As for complex numbers, we have vector maps

$$\begin{aligned} \mathrm{Re} : \mathbf{C} &\to \mathbb{R}^\Sigma & \mathrm{Im} : \mathbf{C} &\to \mathbb{R}^\Sigma & |\_| : \mathbf{C} &\to \mathbb{R}^\Sigma_{\geq 0} \\ z &\mapsto x & z &\mapsto y & z &\mapsto |z| \end{aligned} \tag{2.7}$$

where $z_\tau = x_\tau + \mathrm{i}\, y_\tau$, $x = (x_\tau)_\tau$, $y = (y_\tau)_\tau$ and $|z| = (|z_\tau|)_\tau$. We call $|z|$ the absolute value vector of z.

The $\mathbb{C}$-algebra $\mathbf{C}$ is endowed with three $\mathbb{R}$-linear involutions. For every element $z = (z_\tau)_{\tau \in \Sigma} \in \mathbf{C}$ we define

(1) $c_1 : \mathbf{C} \to \mathbf{C}$, $(c_1(z))_\tau = z_{\overline{\tau}}$

(2) $c_2 : \mathbf{C} \to \mathbf{C}$, $(c_2(z))_\tau = \overline{(z_\tau)}$

(3) $c_3 = c_1 \circ c_2 : \mathbf{C} \to \mathbf{C}$, $(c_3(z))_\tau = \overline{(z_{\overline{\tau}})}$

Obviously all these involutions are $\mathbb{R}$-linear and commute with one another. In fact, considering the $\mathbb{C}$-linear structure of $\mathbf{C}$ we have that c_1 is $\mathbb{C}$-linear while c_2 and c_3 are $\overline{\mathbb{C}}$-linear. Note also that the involution $z \mapsto c_1(z)$ acts trivially on all the coordinates indexed by $\tau \in \Sigma_r$.

Definition 2.2. *Let W be a finite dimensional $\mathbb{C}$-vector space. A* **real structure on** *W is a choice of a $\overline{\mathbb{C}}$-linear involution (anti-linear involution) $c : W \to W$. For $w \in W$ we let $w^+ := \frac{1}{2}(w + \sigma w)$, $w^- := \frac{1}{2}(w - \sigma w)$, $W^+ = \{w^+ : w \in W\}$ and $W^- = \{w^- : w \in W\}$. We call W^+ the real part of W and W^- the purely imaginary of W. In this way we have $W = W^+ \oplus W^-$ where $W^+ = \{w \in W : \sigma w = w\}$ and $W^- = \{w \in W : \sigma w = -w\}$.*

If $z \in \mathbf{C}$ we let

$$\overline{z} := c_3(z) \tag{2.8}$$

and we call the involution c_3 the "complex conjugation" on $\mathbf{C}$. So the map $z \mapsto \overline{z}$ induces "simultaneously" a complex conjugation on the coordinates and an involutive permutation on the indices. We choose to endow $\mathbf{C}$ with the real structure coming from the $\overline{\mathbb{C}}$-involution c_3.

These involutions equip the $\mathbb{C}$-algebra $\mathbf{C}$ with certain distinguished subsets namely

(a) $\mathbf{R} := \mathbf{C}^+ = \{z \in \mathbf{C} : c_3(z) = z\}$,

(b) $\mathbf{R}_\pm := \{x \in \mathbf{R} : x = c_1(x)\}$,

(c) $\mathbf{R}_+ := \{x \in \mathbf{R}_\pm : x > 0\}$,

(d) $\mathbf{H} := \mathbf{R}_\pm + \mathrm{i}\,\mathbf{R}_+$.

Here $\mathrm{i} = \sqrt{-1}$; also if $x \in \mathbf{R}$ and $\delta \in \mathbb{R}$ then the notation $x > \delta$ means that $x_\tau > \delta$ for all $\tau \in \Sigma$. Note that the purely imaginary part of $\mathbf{C}$ is given by $\mathbf{C}^- = \mathrm{i}\,\mathbf{R}$ so that

$$\mathbf{C} = \mathbf{R} \oplus \mathrm{i}\,\mathbf{R}. \tag{2.9}$$

In particular, we must have $\mathbf{R} \cap \mathrm{i}\,\mathbf{R} = \{\mathbb{0}_g\}$ which can also be checked directly.

Remark 2.3. In [26] (see p. 445) and in [2] the set $\mathbf{R}_+$ above is denoted instead by $\mathbf{R}_+^*$.

More concretely one has the following explicit descriptions fo the $\mathbb{R}$-algebras $\mathbf{R}$ and $\mathbf{R}_\pm$: We let

$$\begin{aligned} \alpha_1 : K_{\mathbb{R}} = \mathbb{R}^{\Sigma_r} \times \mathbb{C}^{\Sigma_c'} = \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} &\xrightarrow{\simeq} \mathbf{R} \\ (z_{\rho_1}, \ldots, z_{\rho_{r_1}}; z_{\sigma_1}, \ldots, z_{\sigma_{r_2}}) &\mapsto (z_{\rho_1}, \ldots, z_{\rho_{r_1}}; z_{\sigma_1}, \ldots, z_{\sigma_{r_2}}; z_{\overline{\sigma}_1}, \ldots, z_{\overline{\sigma}_{r_2}}). \end{aligned} \tag{2.10}$$

where the last r_2 coordinates of the image are determined by the rule $z_{\overline{\sigma}_j} := \overline{(z_{\sigma_j})}$ so that image of α is indeed in $\mathbf{R}$. We let

$$\begin{aligned}(2.11)\qquad \alpha_2 : \mathbb{R}^{\Sigma_r \cup \Sigma'_c} = \mathbb{R}^{r_1+r_2} = \xrightarrow{\simeq} \mathbf{R}_{\pm} \\ (z_{\rho_1}, \dots, z_{\rho_{r_1}}; z_{\sigma_1}, \dots, z_{\sigma_{r_2}}) \mapsto (z_{\rho_1}, \dots, z_{\rho_{r_1}}; z_{\sigma_1}, \dots, z_{\sigma_{r_2}}; z_{\overline{\sigma}_1}, \dots, z_{\overline{\sigma}_{r_2}}).\end{aligned}$$

where the last r_2 coordinates of the image are determined by $z_{\overline{\sigma}_j} := z_{\sigma_j}$ so that the image of α_2 is indeed in $\mathbf{R}_{\pm}$. The descriptions of $\mathbf{R}_+$ and $\mathbf{H}$ are then clear. Note also that $\mathbf{H}$ admits also the following alternative description $\mathbf{H} = \{z \in \mathbf{C} : z = c_1(z), \mathrm{Im}(z) > 0\}$. If we let

$$(2.12)\qquad \mathfrak{h} := \{x + \mathrm{i}\, y \in \mathbb{C} : y > 0\}$$

be the Poincaré upper half-plane, the the space $\mathbf{H}$ can be naturally identified with $\mathfrak{h}^{r_1+r_2}$ in the following way:

$$\begin{aligned}(2.13)\qquad \alpha_3 : \mathfrak{h}^{r_1+r_2} \xrightarrow{\simeq} \mathbf{H} \\ (z_1, \dots, z_{r_1}; z_{r_1+1}, \dots, z_{r_1+r_2}) \mapsto (z_1, \dots, z_{r_1}; z_{r_1+1}, \dots, z_{r_2+1}; z_{r_1+1}, \dots, z_{r_2+1})\end{aligned}$$

Note that $\mathbf{H}$ is naturally a subset of $\mathfrak{h}^{r_1+2r_2}$. The maps α_1 and α_2 are compatible in the sense that they agree when evaluated on an element which is common to both of their domain i.e. in $\mathrm{dom}(\alpha_1) \cap \mathrm{dom}(\alpha_2) = \mathbb{R}^{r_1+r_2}$. However, when $r_1 = 0$ (so that $\mathrm{dom}(\alpha_1) \cap \mathrm{dom}(\alpha_3) = \mathfrak{h}^{r_1+r_2} \neq \emptyset$), the pair of maps (α_1, α_3) is not compatible. We also let $\beta_* := \alpha_*^{-1}$ for $* \in \{1, 2, 3\}$ be the reciprocal maps.

By definition we have the following diagram of inclusions:

$$(2.14)\qquad \mathbf{H} \subseteq \mathbf{C} \supseteq \mathbf{R} \supseteq \mathbf{R}_{\pm} \supseteq \mathbf{R}_+$$

For the additive group $\mathbf{C}$ and the multiplicative group $\mathbf{C}^\times$ we have the group homomorphisms

$$(2.15)\qquad \begin{aligned}\mathrm{Tr} : \mathbf{C} \to \mathbb{C} \\ z \mapsto \mathrm{Tr}(z) = \sum_\tau z_\tau\end{aligned} \qquad\qquad \begin{aligned}\mathbf{N} : \mathbf{C}^\times \to \mathbb{C}^\times \\ z \mapsto \mathbf{N}(z) = \prod_\tau z_\tau\end{aligned}$$

Definition 2.4. *We endow* $\mathbf{C}$ *with the hermitian scalar product*

$$\langle x, y \rangle_c := \sum_\tau x_\tau \overline{(y_\tau)} = \mathrm{Tr}(x \cdot c_2(y))$$

and call $\langle , \rangle_c$ *the* **canonical metric** *on* $\mathbf{C}$. *We let* $||z||_c = \sqrt{\langle z, z \rangle_c}$

By definition $\langle \_, \_ \rangle_c$ is $(\mathbb{C}, \overline{\mathbb{C}})$-bilinear and $\overline{(\langle x, y \rangle_c)} = c_\infty(\langle x, y \rangle_c) = \langle y, x \rangle_c$. Note that $\langle , \rangle_c$ is equivariant under "complex conjugation" in the sense that

$$(2.16)\qquad \overline{(\langle x, y \rangle_c)} = c_\infty(\langle x, y \rangle_c) = \langle c_3(x), c_3(y) \rangle_c = \langle \overline{x}, \overline{y} \rangle_c.$$

Also if $z \in \mathbf{C}$, then $c_2(z)$ is the adjoint element of z i.e.,

$$\langle xz, y\rangle_c = \langle x, c_2(z)y\rangle_c. \tag{2.17}$$

Restricting $\langle,\rangle_c$ to the $\mathbb{R}$-vector spaces $\mathbf{R}$ and $\mathbf{R}_\pm$ provide these with a euclidean metric which we denote again by $\langle,\rangle_c$ and call again the canonical metric.

Example 2.5.

(1) For $\rho_i \in \Sigma_r$, let $e_{\rho_i} \in \mathbf{R}$ be the vector where we place 0 in each coordinate $\tau \in \Sigma$ except for the position ρ_i where we place 1,

and

(2) for $\sigma_i \in \Sigma'$, let $e_{\sigma_i} \in \mathbf{R}$ be the vector where we place 0 in each coordinate $\tau \in \Sigma$ except for the positions σ_i and $\overline{\sigma}_i$ where we place $\frac{1}{\sqrt{2}}$.

Then the set $\{e_{\rho_i}, e_{\sigma_j}, \mathrm{i}\, e_{\overline{\sigma}_j} : 1 \leq i \leq r_1, 1 \leq j \leq r_2\}$ is an orthonormal basis of the euclidean space $(\mathbf{R}, \langle,\rangle_c)$

2.1 The power and the logarithm vector maps

For two tuples $z = (z_\tau)_\tau, p = (p_\tau)_\tau \in \mathbf{C}$ we define the power vector map

$$\begin{aligned}[p] : \mathbf{C} &\to \mathbf{C} \\ z &\mapsto z^p = (z_\tau^{p_\tau})\end{aligned} \tag{2.18}$$

where $z_\tau^{p_\tau} := e^{p_\tau \log z_\tau}$. It is well defined if we agree to take the principal branch of logarithm and assume that the coordinates z_τ's move only in the plane cut along the negative real axis. If we restrict to $\mathbf{R}^\times$ the absolute value vector map considered in (2.7) we obtain the map

$$|\;| : \mathbf{R}^\times \to \mathbf{R}_+, \qquad x = (x_\tau)_\tau \mapsto |x| = (|x_\tau|)_\tau. \tag{2.19}$$

There is also a logarithm vector map given by

$$\log : \mathbf{R}_+ \xrightarrow{\simeq} \mathbf{R}_\pm, \qquad x = (x_\tau)_\tau \mapsto \log x = (\log x_\tau)_\tau. \tag{2.20}$$

2.2 Canonical metric VS Minkowski metric

Recall that K is a number field of signature (r_1, r_2) so that $g = [K : \mathbb{Q}] = r_1 + 2r_2$. Recall from (2.10) that we have an isomorphism of $\mathbb{R}$-algebras

$$\beta_1 = \alpha_1^{-1} : \mathbf{R} \to \mathbb{R}^{\Sigma_r} \times \mathbb{C}^{\Sigma'_c}$$

We view $K \subseteq \mathbf{R}$ via $x \mapsto (\tau_j(x))_{j=1}^g$ which through the map β_1 corresponds to the natural inclusion $K \subseteq K_{\mathbb{R}} = \mathbb{R}^{\Sigma_r} \times \mathbb{C}^{\Sigma'_c}$. For a fixed $\lambda \in \mathbb{R}^\times$ let

$$\begin{aligned} (2.21) \qquad & f_\lambda : \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \to \mathbb{R}^g \\ & (x_1, \ldots, x_{r_1}; z_1, \ldots, z_{r_2}) \mapsto (x_1, \ldots, x_{r_1}; \lambda \operatorname{Re}(z_1), \ldots, \lambda \operatorname{Re}(z_{r_2}); \lambda \operatorname{Im}(z_1), \ldots, \lambda \operatorname{Im}(z_{r_2})) \end{aligned}$$

So f_λ provides an $\mathbb{R}$-vector space isomorphism between $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ and $\mathbb{R}^g$. In many textbooks on algebraic number theory, authors choose $\lambda = 1$ but the choice $\lambda = \sqrt{2}$ would also be natural (see Corollary 2.10). From now on we choose to identify $\mathbb{R}^{r_1} \times \mathbb{C}^{r_2}$ to $\mathbb{R}^g$ using the map f_1. This induces an identification between $\mathbf{R}$ and $\mathbb{R}^g$ as $\mathbb{R}$-vector spaces which is given by

$$\begin{aligned} (2.22) \qquad & \mathrm{f}_1 := f_1 \circ \beta_1 : \mathbf{R} \to \mathbb{R}^g \\ & (z_{\rho_1}, \ldots, z_{\rho_{r_1}}; z_{\sigma_1}, \ldots, z_{\sigma_{r_2}}; z_{\overline{\sigma}_1}, \ldots, z_{\overline{\sigma}_{r_2}}) \mapsto \\ & \qquad (z_{\rho_1}, \ldots, z_{\rho_{r_1}}; \operatorname{Re}(z_{\sigma_1}), \ldots, \operatorname{Re}(z_{\sigma_{r_2}}); \operatorname{Im}(z_{\sigma_1}), \ldots, \operatorname{Im}(z_{\sigma_{r_2}})) \end{aligned}$$

Definition 2.6. *Let $x, y \in \mathbb{R}^g$.*

(1) The Minkowski metric on $\mathbb{R}^g$ (the standard euclidean metric) is defined as

$$\langle x, y \rangle_{\mathcal{M}} := \sum_{i=1}^{g} x_i y_i.$$

(2) The canonical metric of type (r_1, r_2) on $\mathbb{R}^g$ (recall that $g = r_1 + 2r_2$) is defined as

$$\langle x, y \rangle_c := \sum_{i=1}^{r_1} x_i y_i + 2 \sum_{i=r_1+1}^{r_1+2r_2} x_i y_i.$$

Definition 2.7. *Let $(V, \langle, \rangle)$ be a euclidean space of dimension n. Let $\{v_1, \ldots, v_n\}$ be linearly independent vectors. The $\langle, \rangle$-volume of the "unit box" $\mathcal{B} = \{\sum_i t_i v_i : 0 \le t_i \le 1\}$ is defined as*

$$\operatorname{vol}_{\langle,\rangle}(\mathcal{B}) = |\det(\langle v_i, v_j \rangle)_{i,j}|^{1/2}. \tag{2.23}$$

The rule $\mathcal{B} \mapsto \operatorname{vol}_{\langle,\rangle}(\mathcal{B})$ gives rise to a (Borel) measure on V which we still denote by $\operatorname{vol}_{\langle,\rangle}$. Let $L \subseteq V$ be a lattice of maximal rank and $\{v_1, \ldots, v_n\}$ be a $\mathbb{Z}$-basis of L. Let $\mathcal{B}$ be the unit box generated by the v_i's. The $\langle, \rangle$-covolume of L is defined as

$$\operatorname{cov}_{\langle,\rangle}(L) = \operatorname{vol}_{\langle,\rangle}(\mathcal{B}). \tag{2.24}$$

We shall denote the volume measure on $(\mathbf{R}, \langle, \rangle_c)$ and $(\mathbb{R}^g, \langle, \rangle_c)$ by vol_c and the volume measure on $(\mathbb{R}^g, \langle, \rangle_{\mathcal{M}})$ by $\operatorname{vol}_{\mathcal{M}}$. Given a lattice $L \subseteq \mathbf{R}$ we define the **absolute value discriminant of** *L to be*

$$d_L := (\operatorname{cov}_{\langle,\rangle_c}(L))^2 \in \mathbb{R}_{>0} \tag{2.25}$$

Definition 2.8. *Let $L_1, L_2 \subseteq \mathbf{R}$ be two lattices (of maximal rank). We define the index*

$$[L_1 : L_2]$$

as the absolute value of the determinant of any g-by-g matrix with real entries which takes a $\mathbb{Z}$-basis of L_1 to a $\mathbb{Z}$-basis of L_2. When $\mathfrak{n} \subseteq K \subseteq \mathbf{R}$ we define the absolute norm of $\mathfrak{n}$ as

$$\mathbf{N}\,\mathfrak{n} := [\mathcal{O}_K : \mathfrak{n}] \in \mathbb{Q}_{>0}. \tag{2.26}$$

So we always have $[L_1 : L_2] \in \mathbb{R}_{>0}$. The rational index satisfies the transitivity formula $[L_1 : L_2][L_2 : L_3] = [L_1 : L_3]$ for all lattices $L_1, L_2, L_3 \subseteq \mathbf{R}$.

Straightforward calculations give the following:

Proposition 2.9. *For $z, w \in \mathbf{R}$ we have that*

$$\langle z, w\rangle_c = \langle \mathrm{f}_1(z), \mathrm{f}_1(w)\rangle_c. \tag{2.27}$$

Also, if $\mathfrak{n} \subseteq K \subseteq \mathbf{R}$ is a lattice then

$$\mathrm{cov}_c(\mathfrak{n}) = \sqrt{|d_K|} \cdot \mathbf{N}\,\mathfrak{n} \tag{2.28}$$

which is equivalent to

$$d_{\mathfrak{n}} = |d_K| \cdot (\mathbf{N}\,\mathfrak{n})^2, \tag{2.29}$$

where d_K is the discriminant of K.

Proof See p. 30-31 of [26]. $\square$

Corollary 2.10. *Let $\{e_{\rho_i}, e_{\sigma_j}, \mathrm{i}\, e_{\overline{\sigma}_j} : 1 \le i \le r_1, 1 \le j \le r_2\} \subseteq \mathbf{R}$ be the orthonormal basis, with respect to $\langle \_, \_\rangle_c$, which was introduced in Example 2.5. Then $\{\mathrm{f}_1(e_{\rho_i}), \mathrm{f}_1(e_{\sigma_j}), \mathrm{f}_1(\mathrm{i}\, e_{\overline{\sigma}_j}) : 1 \le i \le r_1, 1 \le j \le r_2\}$ is an orthogonal basis of $(\mathbb{R}^g, \langle,\rangle_{\mathcal{M}})$, such that for $1 \le i \le r_1$, $||\mathrm{f}_1(e_{\rho_i})|| = 1$, and for $1 \le j \le r_2$, $||\mathrm{f}_1(e_{\sigma_i})|| = ||\mathrm{f}_1(e_{\overline{\sigma}_i})|| = \frac{1}{\sqrt{2}}$. In particular, the map $\mathrm{f}_{\sqrt{2}} : (\mathbf{R}, \langle,\rangle_c) \to (\mathbb{R}^g, \langle,\rangle_{\mathcal{M}})$ is an isometry (here $\mathrm{f}_{\sqrt{2}} = f_{\sqrt{2}} \circ \beta_1$). Moreover, if $X \subseteq \mathbf{R}$ is a measurable set then $\mathrm{vol}_c(X) = 2^{r_2}\,\mathrm{vol}_{\mathcal{M}}(\mathrm{f}_1(X))$.*

Definition 2.11. *Let $L \subseteq \mathbf{R}$ a lattice. We define the dual lattice of L as*

$$L^* := \{\ell^* \in \mathbf{R} : \langle \ell, \ell^*\rangle_c \in \mathbb{Z} \text{ for all } \ell \in L\}. \tag{2.30}$$

Lemma 2.12. *Let $L \subseteq \mathbf{R}$ be a lattice. Then $d_L \cdot d_{L^*} = 1$.*

Proof See Lemma 3.32 of [6]. $\square$

Corollary 2.13. *For all lattices $\mathfrak{n} \subseteq K \subseteq \mathbf{R}$, one has*

$$[\mathcal{O}_K : \mathfrak{n}] \cdot [\mathcal{O}_K, \mathfrak{n}^*] = \mathbf{N}(\mathfrak{n}) \cdot \mathbf{N}(\mathfrak{n}^*) = \frac{1}{|d_K|}.$$

2.3 The normalized multiplicative Haar measure on $\mathbf{R}_+$

Consider the Lie group isomorphism

$$\kappa : \mathbf{R}_+ \longrightarrow \mathbb{R}_+^{r_1+r_2}$$

$$(2.31) \qquad (y_1, \ldots, y_{r_1}; y_{r_1+1}, \ldots, y_{r_1+r_2}; y_{r_1+1}, \ldots, y_{r_1+r_2}) \mapsto (y_1, \ldots, y_{r_1}; y_{r_1+1}^2, \ldots, y_{r_1+r_2}^2).$$

Note that the isomorphism κ agrees with the one at the bottom of p. 453 of [26] (recall also that $\mathbf{R}_+$ is denoted by $\mathbf{R}_+^*$ in [26]). On $\mathbb{R}_+$ we have the *standard Haar measure* $\frac{dt}{t}$. If $t = (t_1, \ldots, t_{r_1+r_2}) \in \mathbb{R}_+^{r_1+r_2}$ we define the measure on $\mathbf{R}_+$

$$(2.32) \qquad d^*y := \kappa^* \left(\prod_{i=1}^{r_1+r_2} \frac{dt_i}{t_i} \right),$$

We call d^*y the canonical Haar measure on $\mathbf{R}_+$ (this Haar measure is denoted by $\frac{dy}{y}$ in [26]). We consider the norm-one hypersurface

$$\mathbf{S} := \{x \in \mathbf{R}_+ : \mathbf{N}(x) = 1\}.$$

We can write uniquely every $y \in \mathbf{R}_+$ in the form

$$(2.33) \qquad y = x \cdot t^{1/g}, \quad \text{where } x = \frac{y}{\mathbf{N}(y)^{1/g}} \in \mathbf{S} \quad \text{and} \quad t = \mathbf{N}(y) \in \mathbb{R}_+.$$

Let

$$\widetilde{\mathbb{L}} := \{u \cdot \mathbb{1}_g : u \in \mathbb{R}_+\}$$

and $\widetilde{\ell} : \mathbb{R}_+ \to \widetilde{\mathbb{L}}$ be the isomorphism given by

$$(2.34) \qquad t \mapsto \widetilde{\ell}(t) = t^{1/g} \cdot \mathbb{1}_g = (t^{1/g}, \ldots, t^{1/g}) \in \widetilde{\mathbb{L}}.$$

It follows from (2.33) that we have an internal direct sum decomposition

$$\mathbf{R}_+ = \mathbf{S} \oplus \widetilde{\mathbb{L}}.$$

We let d^*x be the unique Haar measure on the multiplicative group $\mathbf{S}$ such that the canonical Haar measure d^*y on $\mathbf{R}_+$ becomes the product measure

$$(2.35) \qquad d^*y = d^*x \times d^*t,$$

where $(\widetilde{\ell})^*(d^*t) = \frac{dt}{t}$ where t is the parameter on $\mathbb{R}_+$ as in (2.34).

We choose to identify $\mathbb{R}_+$ with $\mathbb{R}$ using the log map and iterating this on each coordinate gives the Lie group identification

$$\log : \mathbb{R}_+^{r_1+r_2} \to V := \mathbb{R}^{r_1+r_2}.$$

We endow V with the Minkowski metric $\langle,\rangle_{\mathcal{M}}$ (see Section 2.2). In particular we have an isomorphism

(2.36)
$$\begin{aligned}\delta := \log\circ\kappa : \mathbf{R}_+ &\longrightarrow V\\ (x_{\rho_1},\dots,x_{\rho_{r_1}};y_{\sigma_1},\dots,y_{\sigma_{r_2}};y_{\sigma_1},\dots,y_{\sigma_{r_2}}) &\longmapsto (\log x_{\rho_1},\dots,\log x_{\rho_{r_1}};2\log y_{\sigma_1},\dots,2\log y_{\sigma_{r_2}})\end{aligned}$$

One may readily check that

$$\delta^*(\mathrm{vol}_{\mathcal{M}}) = \mathrm{vol}_{d^*y}\,. \tag{2.37}$$

If we restrict the map δ to $\mathbf{S}$, it provides a bijection between the norm-one hypersurface $\mathbf{S}$ and the hyperplane

$$H := \left\{(x_k)_k \in V : \sum_{k=1}^{r_1+r_2} x_k = 0\right\}.$$

In particular, the map δ maps $\mathcal{O}_K^\times$ into H. If we let $v_0 = \frac{1}{\sqrt{r_1+r_2}}\,\mathbb{1}_{r_1+r_2} \in V$ note that v_0 is a unit vector (with respect to $\langle,\rangle_{\mathcal{M}}$) which is perpendicular H.

Definition 2.14. *Let $L_1 = \delta(\mathcal{O}_K^\times)$. It is a lattice of H and let $L := L_1 + \mathbb{Z}v_0$ which is a lattice of V. We define the regulator R_K of K as*

$$R_K := \frac{1}{\sqrt{r_1+r_2}}\cdot \mathrm{cov}_{\langle,\rangle_{\mathcal{M}}}(L). \tag{2.38}$$

The definition of R_K above agrees with the more conventional one which is given in terms of the *absolute value* of the determinant of an $r\times r$ matrix where $r := r_1 + r_2 - 1$. Recall that this matrix is constructed out of a basis of the free part of $\mathcal{O}_K^\times$ and of an indexing set $(\Sigma_r \cup \Sigma_c')\backslash\{\tau\}$ where the removed embedding τ is arbitrarily chosen. For a proof that the two definitions agree see Proposition 7.5 on p. 43 of [26].

Proposition 2.15. *Let $\mathcal{V} \le \mathcal{O}_K^\times$ be a finite index subgroup and let*

$$e := [\langle(\mathcal{O}_K^\times)^2,\mu_K\rangle : \langle\mathcal{V}^2,\mu_K\rangle]$$

where $(\mathcal{O}_K^\times)^2$ corresponds to the subgroup of $\mathcal{O}_K^\times$ generated by the squares of units, and similarly for $\mathcal{V}^2$. The map $\epsilon\mapsto|\epsilon|$ sends $\mathcal{O}_K^\times$ into $\mathbf{S}$ and has finite kernel μ_K. Let $\mathcal{F}_{\mathcal{V}}$ be a fundamental domain for the action of $|\mathcal{V}^2|$ on $\mathbf{S}$. Then

$$\mathrm{vol}_{d^*x}(\mathcal{F}_{\mathcal{V}}) = e\cdot 2^r\cdot R_K \tag{2.39}$$

where $r = r_1 + r_2 - 1$.

Proof This is Lemma 5.6 on p. 461 of [26] which is stated for $\mathcal{V} = \mathcal{O}_K^\times$. Note that the quantity r in loc. cit. is defined as $r_1 + r_2$ which differs slightly from our definition of r given above. $\square$

and let $\mathbf{C} = \mathbf{C}_\Sigma$ (see Section 2).

Definition 2.16. *An element $p = (p_1, \dots, p_{r_1+2r_2}) \in \mathbb{Z}_{\geq 0}^{r_1+2r_2} \subseteq \mathbf{R}$ is called* **admissible** *if for $1 \leq i \leq r_1$, $p_i \in \{0,1\}$, and for $r_1+1 \leq i \leq r_1+r_2$, $p_i \cdot p_{i+r_2} = 0$. We define the weight space of $\mathbf{C}$ to be*

$$\mathbf{W}_\Sigma = \mathbf{W} := \mathbb{C} \cdot \mathbb{1}_g + \mathbb{Z}_{\geq 0}^{\Sigma} + \mathrm{i}\,\mathbf{R}_\pm \subseteq \mathbf{C}, \tag{2.40}$$

where $\mathbb{1} = 1_{\mathbf{C}}$ is the unit element of $\mathbf{C}$. Let

$$\mathfrak{s} = s \cdot \mathbb{1}_g + p + \mathrm{i}\, q \in \mathbf{W} \tag{2.41}$$

where $s \in \mathbb{C}$, $p \in (\mathbb{Z}_{\geq 0})^{r_1+2r_2}$ and $q \in \mathbf{R}_\pm$. We say that the writing on the RHS of (2.41) *is an admissible writing of $\mathfrak{s} \in \mathbf{W}$ if p is admissible.*

Lemma 2.17. *Let $\mathfrak{s} \in \mathbf{W}$ and assume that $\mathfrak{s} = s \cdot \mathbb{1}_g + p + \mathrm{i}\, q = s' \cdot \mathbb{1}_g + p' + \mathrm{i}\, q'$ are two admissible writings of $\mathfrak{s}$. Then either one of the following four outcomes is true:*

(1) $q = q'$, $p = p'$ *and* $s = s'$,

or

(2) $q = q'$, $r_2 = 0$, $s - s' = p' = \mathbb{1}_{r_1}$ *and* $p = \mathbb{0}_{r_1}$,

or

(3) $q = q'$, $r_2 = 0$, $s - s' = -p = -\mathbb{1}_{r_1}$ *and* $p' = \mathbb{0}_{r_1}$,

or

(4) $q \neq q'$, $s - s' \in \mathrm{i}\,\mathbb{R} \backslash \{0\}$ *(non-vanishing and purely imaginary),* $p = p'$ *and* $q' - q = -\mathrm{i}(s - s') \cdot \mathbb{1}$.

Proof By assumption we have

$$(s - s') \cdot \mathbb{1} + (p - p') = \mathrm{i}(q' - q). \tag{2.42}$$

<u>case $q = q'$</u>: So let us assume that $q = q'$ so that

$$(s - s') \cdot \mathbb{1}_{r_1+2r_2} = p' - p. \tag{2.43}$$

If $s = s'$ then $p = p'$. Let us now suppose that $s \neq s'$. In particular, in this case it follows from (2.43) that for all k, $p'_k \neq p_k$. We claim that the case $r_2 \geq 1$ is impossible. Indeed on one hand we have that $(p'_{r_1+1}, p'_{r_1+1+r_2}) - (p_{r_1+1}, p_{r_1+1+r_2}) = (s - s') \cdot (1,1)$ but on the other hand we must also have $p'_{r_1+1}, p'_{r_1+1+r_2}, p_{r_1+1}, p_{r_1+1+r_2} \in \mathbb{Z}_{\geq 0}$ and $p'_{r_1+1} p'_{r_1+1+r_2} = p_{r_1+1} p_{r_1+1+r_2} = 0$. If for example $p'_{r_1+1} > 0$ then we must have

$$p'_{r_1+1+r_2} = 0 \Rightarrow p_{r_1+1+r_2} > 0 \Rightarrow p_{r_1+1} = 0 \Rightarrow (p'_{r_1+1}, -p_{r_1+1+r_2}) = (s - s') \cdot (1,1) \tag{2.44}$$

which is absurd. The other cases are dealt similarly. So we may assume that $r_2 = 0$. Since $p, p' \in \mathbb{Z}_{\geq 0}^{r_1}$ are admissible it follows from (2.43) that $s - s' \in \{0, 1, -1\}$. If $s = s'$ then $p = p'$. If $s - s' = 1$ we must have $p' = \mathbb{1}_{r_1}$ and $p = \mathbb{0}_{r_1}$ and similarly if $s - s' = -1$ we must have $p' = \mathbb{0}_{r_1}$ and $p = \mathbb{1}_{r_1}$.

case $q \neq q'$: Assume now that $q \neq q'$ and let i_0 be the least index such that $q_{i_0} \neq q'_{i_0}$. It follows from (2.42) that $s - s' + (p_{i_0} - p'_{i_0}) = \mathrm{i}(q'_{i_0} - q_{i_0})$. In particular, since $p_{i_0} - p'_{i_0} \in \mathbb{R}$ (in fact this difference is even in $\mathbb{Z}$) and that $q_{i_0} \neq q'_{i_0}$, it follows that $s \neq s'$ and that $s - s'$ is uniquely determined by the i_0 coordinates of p, p', q and q'. We claim that $n := p - p' = \mathbb{0}_g$. Assume not, then we can find two indices i_1, i_2 such that $n_{i_1} \neq n_{i_2}$; but then $(s - s') + n_{i_1}$ and $(s - s') + n_{i_2}$ would be both purely imaginary which is absurd and therefore we must that $n = \mathbb{0}_g$. Finally, since $p = p'$ it follows from (2.42) that $s - s' \in \mathrm{i}\,\mathbb{R}\backslash\{0\}$ and we therefore obtain that $q' - q = \frac{1}{\mathrm{i}}(s - s')\,\mathbb{1}_g$. $\square$

Definition 2.18. *Let $\mathfrak{s} = (\mathfrak{s}_i)_i \in \mathbf{C}$ and assume that for all i, $\mathfrak{s}_i \notin \mathbb{Z}_{\leq 0}$. The multi-dimensional gamma function associated to $(K, {}^\ell\Sigma)$ and evaluated at $\mathfrak{s}$ is defined as*

$$\Gamma_\Sigma(\mathfrak{s}) := \left(\prod_{i=1}^{r_1} \Gamma(\mathfrak{s}_i)\right) \cdot \left(\prod_{i=r_1+1}^{r_1+r_2} 2^{1-(\mathfrak{s}_i + \mathfrak{s}_{i'})} \cdot \Gamma\left(\mathfrak{s}_i + \mathfrak{s}_{i'}\right)\right) \tag{2.45}$$

where for $u \in \mathbb{C}$, $\Gamma(u)$ is the usual gamma function evaluated at u, and where $i' = i + r_2$ for $r_1 + 1 \leq i \leq r_1 + r_2$.

Note that the definition of Γ_Σ only depends on the signature (r_1, r_2) of K.

Proposition 2.19. *For $\mathfrak{s} \in \mathbf{C}$ with $\mathrm{Re}(\mathfrak{s}) > 0$ we have*

$$\Gamma_\Sigma\left(\frac{\mathfrak{s}}{2}\right) = \int_{\mathbf{R}_+} \mathbf{N}(e^{-y} y^{\mathfrak{s}/2}) d^*y, \tag{2.46}$$

where d^*y is the normalized multiplicative Haar measure on $\mathbf{R}_+$ that was defined in (2.35).

Proof This follows rather directly from rewriting the RHS of (2.46) as an iterated integral. Note that some care must be taken to handle pairs of complex embeddings. For the details see for example §4 in Chapter 7 of [26]. $\square$

3 The group $\mathbf{G} = K_{\mathbb{R}}^\times$ and the space of quasi-characters $\mathfrak{X}_{\mathcal{V}}$

3.1 The abelian Lie group G

Recall that $\Sigma = \mathrm{Hom}(K, \mathbb{C})$ and that ${}^\ell\Sigma$ comes with a fixed choice of an admissible labelling (see Section 2). Since ${}^\ell\Sigma$ is fixed from the outset we shall write $\mathbf{C}$ instead of $\mathbf{C}_\Sigma$. Using this admissible labelling one gets an $\mathbb{R}$-algebra isomorphism

$$\iota : K_{\mathbb{R}} := K \otimes_{\mathbb{Q}} \mathbb{R} \longrightarrow \mathbb{R}^{r_1} \times \mathbb{C}^{r_2} \tag{3.1}$$

given by (2.5). Using ι we can view each element of $K_{\mathbb{R}}$ as an (r_1+r_2)-tuples $(x_1, \ldots, x_{r_1}; z_1, \ldots, z_{r_2})$ where x_i is real and z_i is complex. We let

$$\mathbf{G} := K_{\mathbb{R}}^{\times} \tag{3.2}$$

It is an abelian real Lie group. Let α_1 be the map in (2.10). It follows tautologically that

$$\alpha_1|_{\mathbf{G}} : \mathbf{G} \longrightarrow \mathbf{R}^{\times} \tag{3.3}$$

is a Lie group isomorphism.

The map ι in (3.1) induces the following Lie group decomposition

$$\mathbf{G} = \{\pm 1\}^{r_1} \times \mathbb{R}_+^{r_1} \times (\mathbb{C}^{\times})^{r_2}. \tag{3.4}$$

Let $S^1 := \{z \in \mathbb{C} : |z| = 1\}$ be the unit circle in $\mathbb{C}$. We shall always identify $\mathbb{R}_+ \times S^1$ and $\mathbb{C}^{\times}$ in the usual way namely $(r, u) \mapsto ru$, so that the one can refine the decomposition (3.4) to

$$\gamma : \mathbf{G} \longrightarrow \{\pm 1\}^{r_1} \times \mathbb{R}_+^{r_1+r_2} \times (S^1)^{r_2}, \tag{3.5}$$

which we simply call the *standard decomposition of* $\mathbf{G}$. If we let $\mathbf{G}_0 := \mathbf{G}^{\circ}$ denote the connected component of the identity we thus find that $\mathbf{G}_0 = \mathbb{R}_+^{r_1+r_2} \times (S^1)^{r_2}$. If $g \in \mathbf{G}$, it thus admits a unique writing of the form

$$g = (\epsilon_1, \ldots, \epsilon_{r_1}; x_1, \ldots, x_{r_1}; r_1, \ldots, r_{r_2}; u_1, \ldots, u_{r_2}) \tag{3.6}$$

where $\epsilon_j \in \{\pm 1\}$, $x_i, r_j \in \mathbb{R}_+$ and $u_j \in S^1$ so that $z_j = r_j u_j$ for $1 \leq j \leq r_2$. An element $g \in \mathbf{G}_0$ thus corresponds to a tuple of the form $(\underbrace{1, \ldots, 1}_{r_1 \text{ times}}; *; *; *)$. We let

$$\begin{aligned} \gamma_0 : \mathbf{G}_0 &\longrightarrow \mathbb{R}_+^{r_1+r_2} \times (S^1)^{r_2} \\ g &\mapsto \gamma_0(g) = (x_1, \ldots, x_{r_1}; r_1, \ldots, r_{r_2}; u_1, \ldots, u_{r_2}) \end{aligned} \tag{3.7}$$

(so one simply removes the string of 1's of length r_1 at the beginning of $\gamma(g)$).

At this point it is worth to introduce two distinguished subgroups of $\mathbf{G}$ namely

$$\mathbb{S} := \{(\pm 1, \ldots, \pm 1; 1, \ldots, 1; 1, \ldots, 1)\} \leq \mathbf{G} \tag{3.8}$$

and

$$\mathbb{L} := \{\ell(t) := (1, \ldots, 1; \underbrace{t^{1/g}, \ldots, t^{1/g}}_{r_1}; \underbrace{t^{1/2g}, \ldots, t^{1/2g}}_{r_2}, 1, \ldots, 1) \in \mathbf{G}_0 : t \in \mathbb{R}_+\} \leq \mathbf{G}_0. \tag{3.9}$$

We call $\mathbb{S}$ the *signature group* and $\mathbb{L}$ the *line group*. If $r_1 = 0$ then $\mathbb{S}$ is the trivial group. Note also that $\mathbb{L}$ comes with the natural parametrization $t \mapsto \ell(t)$ for $t \in \mathbb{R}_+$.

The norm map $\mathbf{N} : \mathbf{C}_\Sigma \to \mathbb{C}$ induces a norm map on $\mathbf{R}$. Using the isomorphism $\beta_3 = \alpha_3^{-1} : \mathbf{R} \to K_\mathbb{R}$ (see (2.13)) it induces a norm map as well on $\mathbf{G} = K_\mathbb{R}^\times$. Explicitly, we have

$$\begin{aligned} &\mathbf{N} : \mathbf{G} \to \mathbb{R}^\times \\ g = (\epsilon_i; x_j, z_k) = (\epsilon_i; x_j; r_k, u_\ell) &\mapsto \mathbf{N}\, g = \prod_i \epsilon_i \prod_j x_j \prod_k r_k^2 \end{aligned} \tag{3.10}$$

It is an onto Lie group homomorphism. Note that for $g \in \mathbf{G}$, if $\mathbf{N}(g) > 0$, we have the rule

$$\mathbf{N}(\ell(\mathbf{N}\, g)) = \mathbf{N}\, g.$$

The norm map in (3.10) induces also an onto Lie group homomorphism $\mathbf{N} : \mathbf{G}_0 \to \mathbb{R}_+$. We let

$$\begin{aligned} \mathbf{G}_1 := \ker|_{\mathbf{G}_0} &= \{g \in \mathbf{G}_0 : \mathbf{N}\, g = 1\} \\ &= \{(x_1, \dots, x_{r_1}; r_1, \dots, r_{r_2}; *) : \prod x_i \prod r_j^2 = 1\}. \end{aligned} \tag{3.11}$$

where $* = (u_1, \dots, u_{r_2}) \in (S^1)^{r_2}$ is arbitrary. The inclusion $\mathbf{G}_1 \leq \mathbf{G}_0$ admits the following retraction map (in the category of Lie groups)

$$\begin{aligned} r : \mathbf{G}_0 &\to \mathbf{G}_1 \\ g &\mapsto (\ell(\mathbf{N}\, g))^{-1} \cdot g \end{aligned} \tag{3.12}$$

which provides the direct product decomposition

$$\begin{aligned} \mathbb{L} \oplus \mathbf{G}_1 &\longrightarrow \mathbf{G}_0 \\ (\ell, g) &\mapsto \ell g \\ \big(\ell(\mathbf{N}\, g), r(g)\big) &\leftarrow\!\shortmid g \end{aligned} \tag{3.13}$$

It thus follows that we have the internal direct sum decompositions

$$\mathbf{G} = \mathbb{S} \oplus \mathbb{L} \oplus \mathbf{G}_1 \qquad \text{and} \qquad \mathbf{G}_0 = \mathbb{L} \oplus \mathbf{G}_1. \tag{3.14}$$

3.2 Standard identifications between abelian Lie groups

The building blocks for the connected abelian Lie groups are S^1 and $\mathbb{R}$. We choose the following normalized isomorphisms (identifications) of Lie groups

(1) $\log : \mathbb{R}_+ \to \mathbb{R}$, $t \mapsto \log t$, $\exp : \mathbb{R} \to \mathbb{R}_+$, $t \mapsto e^t$,

(2) $[\_] : \mathbb{Z} \longrightarrow \mathrm{Hom}(S^1, \mathbb{C}^\times)$ where for $u \in S^1$, $[n](u) = u^n$,

(3) $\alpha_+ : \mathbb{R} \to \mathrm{Hom}(\mathbb{R}, \mathbb{R}_+)$ where for $t, x \in \mathbb{R}$, $\alpha_+(t)(x) = e^{tx}$

(4) $\alpha_u : \mathbb{R} \to \mathrm{Hom}(\mathbb{R}, S^1)$ where for $t, x \in \mathbb{R}$, $\alpha_u(t)(x) = e^{\mathrm{i}\,tx}$,

(5) $\alpha_{+,u} := (\alpha_+, \alpha_u) : \mathbb{R} \times \mathbb{R} \to \mathrm{Hom}(\mathbb{R}, \mathbb{C}^\times)$ so that for $x, y \in \mathbb{R}$,

$$\alpha_{+,u}(x, y)(\_) = \alpha_+(x)(\_) \cdot \alpha_u(y)(\_).$$

(6) If we identify $\mathbb{C}$ with $\mathbb{R}^2$ using the map $a + \mathrm{i}\,b \mapsto (a, b)$ one may consider the following sequence of isomorphisms:

$$\mathbb{C} \longrightarrow \mathbb{R}^2 \xrightarrow{\alpha_{+,u}} \mathrm{Hom}(\mathbb{R}, \mathbb{C}^\times) \xrightarrow{\log^*} \mathrm{Hom}(\mathbb{R}_+, \mathbb{C}^\times) \tag{3.15}$$

which provides the identification

$$\begin{aligned} \beta : \mathbb{C} &\longrightarrow \mathrm{Hom}(\mathbb{R}_+, \mathbb{C}^\times) \\ a + \mathrm{i}\,b &\mapsto (\beta(a + \mathrm{i}\,b))(t) = (\alpha_{+,u}(a, b))(\log t) \\ &= t^{a+\mathrm{i}\,b} \end{aligned} \tag{3.16}$$

3.3 Some notation and basic results on topological groups

If A, B are LCA (Locally Compact Abelian groups) we let $\mathrm{Hom}(A, B) = \mathrm{Hom}_{\mathrm{cont}}(A, B)$ be the group of continuous homomorphisms from A to B endowed with the compact-open topology. Note that the compact-open topology coincides with the topology of compact convergence on the set $\mathrm{Hom}(A, B)$. Let us denote the category of LCA by $\underline{\mathrm{LCA}}$. The category $\underline{\mathrm{LCA}}$ is a quasi-abelian category but it is not abelian. For example the map $\mathbb{R}^{disc} \to \mathbb{R}$ is simultaneously a monomorphism and an epimorphism but it fails to be an isomorphism.

Remark 3.1. For an elementary introduction to $\underline{\mathrm{LCA}}$ and the statement of the structure theorem for LCA see [23]. For a more thorough presentation of LCA see the short book [22]. In general, if A, B are LCA then $\mathrm{Hom}(A, B)$ is not necessarily locally compact. For example if $\mathbb{Q}$ and $\mathbb{Q}/\mathbb{Z}$ are viewed as discrete groups then $\mathrm{Hom}(\mathbb{Q}, \mathbb{Q}/\mathbb{Z}) \simeq \prod_p \mathbb{Q}_p$ where $\mathbb{Q}_p$ is the p-adic field endowed with its p-adic topology and where $\prod_p \mathbb{Q}_p$ has the *product topology*; clearly $\prod_p \mathbb{Q}_p$ is not locally compact. Recall that a LCA B is said *to contain no small subgroup* if one can find an open neighborhood U of 0 which contains no non-trivial subgroup. When B is compact and contains no small subgroup then one can show that $\mathrm{Hom}(A, B)$ is again locally compact, see for example the corollary on the top of p. 377 of [24]. For the reader eager to learn more about usual homological algebra constructions in the setting of LCA we refer to [24].

For the convenience of the reader we list below some basic results on Lie groups that we shall apply later on.

Proposition 3.2. *(1) Let G be a connected abelian Lie group. Then $G \simeq \mathbb{R}^a \times (S^1)^b$ for some unique integers $a, b \in \mathbb{Z}_{\geq 0}$.*

(2) *Let G be an abelian Lie group. Then there exists a discrete Lie subgroup $D \leq G$ such that $G = G^\circ \oplus D$ (internal direct sum) where G° is the connected component of the identity and $D \leq G$. Here $D \simeq \pi_0(G) := G/G^\circ$ where $\pi_0(G)$ is called the component group of G, which is viewed as a discrete Lie group. In general the choice of D is not unique.*

(3) *Let G be a Lie group and $K \leq G$ a normal closed subgroup. Then G/K carries a natural structure of a Lie group. Let $\pi : G \to H$ be an onto Lie group homomorphism and let $K := \ker(\pi)$. Then K is a closed normal Lie subgroup of G, and the Lie group quotient and G/K is isomorphic as a Lie group to H via π.*

Proof The statement (1) is classical and follows rather directly from the surjectivity of the exponential map $\exp : T_eG \to G$ when G is a connected abelian Lie group. (2) follows from the observations that G° is a divisible open subgroup of G by (1); and then one can apply Corollary 4.22 of [28]. The first part of (3) is Theorem 3.64 of [30] and the second part is Corollary 5.3.10 of [8]. □

3.4 Quasi-characters and characters of G

Definition 3.3. *Let G be a locally compact abelian group. We let $\widehat{G} := \mathrm{Hom}(G, S^1)$ be the character group of G. It is again a locally compact abelian group. An element of $\chi \in \widehat{G}$ is called a (unitary) character.*

We shall now introduce some useful terminology and notation in order to organize the characters of the Lie group **G**. Recall that LCA is the quasi-abelian category of locally compact abelian groups. Let $\mathcal{C}$ be the subcategory of LCA which consists of abelian Lie groups with finitely many components and where the arrows are Lie group homomorphisms (so smooth maps). It follows for example from Theorem 5.3.9 [8] that $\mathcal{C}$ is a full subcategory of LCA. If $G \in \mathcal{C}$ it follows from (1) and (2) of Proposition 3.2 that $G \simeq G^\circ \oplus \pi_0(G)$ where $G^\circ \simeq \mathbb{R}^a \times (S^1)^b$ and where $\pi_0(G)$ is viewed a discrete (so finite) Lie group. In general, if $G_1, G_2 \in \mathcal{C}$ and $f : G_1 \to G_2$ is a Lie group homomorphism then $\mathrm{im}(f) \in \mathcal{C}$ but $\ker(f)$ may fail to be in $\mathcal{C}$. For example the map $\exp : \mathbb{R} \to S^1$, $z \mapsto e^{2\pi \mathrm{i}\, z}$ has $\mathbb{Z}$ as kernel which is not in $\mathcal{C}$. If $G \in \mathcal{C}$ and $H \leq G$ is a closed subgoup then H may not lie in $\mathcal{C}$ but the topological group quotient G/H always lie in $\mathcal{C}$. The fact that G/H carries a natural structure of a Lie group follows from (4) of Proposition 3.2 (but when G is an abelian the proof becomes rather straightforward). So to summarize $\mathcal{C}$ is a full subcategory of LCA which admits quotients but it is not abelian since in general the kernels of maps are not necessarily in $\mathcal{C}$.

Definition 3.4. *Let $G \in \mathcal{C}$. We let $\check{G} := \mathrm{Hom}(G, \mathbb{C}^\times)$ and $\overset{\frown}{G} := \mathrm{Hom}(G, \mathbb{R}_+)$. Note that $\overset{\frown}{G}$ is always in $\mathcal{C}$ and if G° does not contain any subgroup isomorphic to S^1 then $\check{G}$ is again in $\mathcal{C}$. We call $\check{G}$* **the group of quasi-characters** *of G and $\overset{\frown}{G}$ the* **group of positive characters** *of G. If $G = H_1 \oplus H_2$ is an internal direct sum decomposition of G we agree to write $\check{G} = \check{H}_1 \oplus \check{H}_2$ where a quasi-character χ on $\check{H}_1$ is viewed as a quasi-character on G by extending it trivially to H_2 (and similarly when $\chi \in \check{H}_2$).*

For $G \in \mathcal{C}$ we always have the internal direct sum decomposition within the category LCA

$$\check{G} = \widehat{G} \oplus \widehat{G}. \tag{3.17}$$

Definition 3.5. *Let $G_1, G_2 \in \mathcal{C}$ and let $H \leq G$ be a closed subgroup (not necessarily in $\mathcal{C}$). We define*

$$\mathrm{Hom}^H(G_1, G_2) := \{\rho \in \mathrm{Hom}(G_1, G_2) : \rho|_H = 1\}. \tag{3.18}$$

Using (4) of Proposition 3.2 we have the identification $\mathrm{Hom}^H(G_1, G_2) = \mathrm{Hom}(G_1/H, G_2)$. Now let $\mathbf{G} \in \mathcal{C}$ where $\mathbf{G} = K_{\mathbb{R}}^{\times}$. Recall that $\mathbf{G}$ decomposes as $\mathbf{G} = \mathbb{S} \oplus \mathbf{G}_0 = \mathbb{S} \oplus \mathbb{L} \oplus \mathbf{G}_1$.

(1) An element $\chi \in \check{G} = \mathrm{Hom}(\mathbf{G}, \mathbb{C}^{\times})$ is called a quasi-character.

(2) An element $\chi \in \widehat{G} = \mathrm{Hom}(\mathbf{G}, \mathbb{R}_+)$ is called a positive quasi-character.

(3) An element $\chi \in \widehat{\mathbf{G}} = \mathrm{Hom}(\mathbf{G}, S^1)$ is called a (unitary) character.

(4) We call $\widehat{\mathbb{S}} := \mathrm{Hom}(\mathbb{S}, \{\pm 1\}) = \mathrm{Hom}(\mathbf{G}/\mathbf{G}_0, \{\pm 1\})$ the signature group and an element $\chi \in \widehat{\mathbb{S}}$ is called a signature character.

(5) An element $\chi \in \mathrm{Hom}^{\mathbb{S}}(\mathbf{G}, S^1) = \widehat{\mathbf{G}_0}$ is called a connected character.

(6) Let $\Gamma \leq \mathbf{G}$ be a closed subgroup. An element $\chi \in \mathrm{Hom}^{\Gamma}(\mathbf{G}, S^1) = \mathrm{Hom}(\mathbf{G}/\Gamma, S^1)$ is called a Γ-invariant character of $\mathbf{G}$.

Remark 3.6. Of course the terminology above can be extended to any element $G \in \mathcal{C}$. For example in (4), one could replace $\mathbb{S}$ by $\pi_0(G)$, and $\{\pm 1\}$ by μ_n (the group of n-th roots of unity), where n is the exponent of $\pi_0(G)$; however perhaps in that case it would be better to talk of "μ_n-signature characters".

3.5 The factorization of $\mathcal{V}$-invariant quasi-characters of G

Recall the distinguished subgroup $\mathbb{L} \leq \mathbf{G}$ which we call the line group and which comes with the explicit isomorphism $\ell : \mathbb{R}_+ \to \mathbb{L}$ where ℓ was defined in (3.9). The distinguished isomorphism $\beta : \mathbb{C} \to \widecheck{\mathbb{R}_+}$ in (3.16) gives rise to the following isomorphism, which we still denote by β:

$$\beta : \mathbb{C} \longrightarrow \mathrm{Hom}(\mathbb{L}, \mathbb{C}^{\times}), \tag{3.19}$$

where for $a + \mathrm{i}\, b \in \mathbb{C}$ and $g = \ell(t) \in \mathbb{L}$,

$$\beta(g) = t^{a+\mathrm{i}\, b} = (\mathbf{N}\, g)^{a+\mathrm{i}\, b} \tag{3.20}$$

Since $\mathbf{G} \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{R}^{r_1+r_2} \times (S^1)^{r_2}$ it follows that

$$\check{G} \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{C}^{r_1+r_2}, \tag{3.21}$$

and that

$$\widehat{G} \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1+r_2}.$$

We now focus on the group of characters of $\mathbf{G}$ namely on $\widehat{\mathbf{G}} = \mathrm{Hom}(\mathbf{G}, S^1)$. We choose to decompose an element $\chi \in \widehat{G}$ in the following way:

$$\chi = \chi_{\overline{m}} \cdot \chi_n \cdot \chi_\gamma \cdot \chi_\delta, \tag{3.22}$$

where $\overline{m} \in (\mathbb{Z}/2\mathbb{Z})^{r_1}$, $\delta \in \mathbb{R}^{r_1}$, $\gamma \in \mathbb{R}^{r_2}$ and $n \in \mathbb{Z}^{r_2}$. Here, for

$$g = (\epsilon_1, \dots, \epsilon_{r_1}; x_1, \dots, x_{r_1}; r_1, \dots, r_{r_2}; u_1, \dots, u_{r_2}) = (*; *; *; *) \in \mathbf{G}, \tag{3.23}$$

(a) $\chi_{\overline{m}}(g) = \prod_j \epsilon_j^{[\overline{m}_j]}$ where $[\overline{m}_j] = 0$ if $\overline{m}_j = \overline{0}$ and $[\overline{m}_j] = 1$ if $\overline{m}_j = \overline{1}$.

(b) $\chi_n(g) = \prod_j u_j^{n_j}$.

(c) $\chi_\gamma(g) = \prod_j e^{\gamma_j x_j \mathrm{i}}$,

(d) $\chi_\delta(g) = \prod_j e^{2\delta_j r_j \mathrm{i}}$,

The quadruple $\mathcal{Q}_\chi := (\overline{m}, n, \gamma, \delta)$ associated to χ in (3.22) is unique since the map

$$\begin{aligned} \widehat{\mathbf{G}} &\to (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \\ \chi &\mapsto \mathcal{Q}_\chi = (\overline{m}, n, \gamma, \delta)_\chi \end{aligned} \tag{3.24}$$

is a group isomorphism.

Definition 3.7. *A character $\chi \in \widehat{\mathbf{G}}$ admits a unique writing given by (3.22) which is described by a unique quadruple $\mathcal{Q}_\chi = (\overline{m}, n, \gamma, \delta) \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$. We call $\mathcal{Q}_\chi$ the $\mathcal{Q}$-weight vector of χ. In particular, if $\chi \in \mathrm{Hom}(\mathbf{G}_0, S^1)$ then there exists a unique triple $\mathfrak{T}_\chi = (n, \gamma, \delta)_\chi \in \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$ such that $\chi = \chi_n \cdot \chi_\gamma \cdot \chi_\delta$. We call $\mathfrak{T}_\chi$ the $\mathfrak{T}$-weight of χ.*

When the character χ is clear from the context we may drop the subscript index χ on $\mathcal{Q}_\chi$ and write $\mathcal{Q}$ instead. Conversely, any quadruple $\mathcal{Q} = (\overline{m}, n, \gamma, \delta)$ as above gives rise to a unique character $\chi_{\mathcal{Q}} \in \widehat{\mathbf{G}}$.

Definition 3.8. *For $x \in \mathbb{R}^{r_1+r_2}$ we define the modified trace of x as*

$$\widetilde{\mathrm{Tr}}(x) := \sum_{j=1}^{r_1} x_j + 2 \sum_{j=r_1+1}^{r_1+r_2} x_j. \tag{3.25}$$

So if $x = (y, z) \in \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$, $\widetilde{\mathrm{Tr}}(x) = \mathrm{Tr}(y, 2z)$. Let $\chi \in \widehat{\mathbf{G}}$ be a character of parameters $(\overline{m}, n, \gamma, \delta)$. We define the trace of χ as

$$\mathrm{Tr}(\chi) := \widetilde{\mathrm{Tr}}(\gamma, \delta) = \mathrm{Tr}(\gamma, 2\delta) = \sum_{j=1}^{r_1} \gamma_j + 2 \sum_{j=1}^{r_2} \delta_j. \tag{3.26}$$

In particular, a character $\chi \in \widehat{\mathbf{G}}$ has a trivial trace if and only if χ is $\mathbb{L}$-invariant, i.e. $\chi \in \mathrm{Hom}^{\mathbb{L}}(\mathbf{G}, S^1)$.

Remark 3.9. Recall that $\alpha_2 : \mathbb{R}^{r_1+r_2} \to \mathbf{R}_{\pm}$ was the embedding defined in (2.11). One may check that the reduced trace on $\mathbb{R}^{r_1+r_2}$ is compatible with the trace on $\mathbf{R}_{\pm}$ in the sense that for all $x \in \mathbb{R}^{r_1+r_2}$ we have $\widetilde{\mathrm{Tr}}(x) = \mathrm{Tr}(\alpha_2(x))$.

3.6 $\mathbf{G}_1$ and Dirichlet's unit theorem

We know that $\mathbf{G}_1 \simeq \mathbb{R}_+^{r_1+r_2-1} \times (S^1)^{r_2}$. For convenience let us choose such an isomorphism γ_1 in the following way:

$$\begin{aligned} \gamma_1 : \mathbf{G}_1 &\longrightarrow \mathbb{R}^{r_1+r_2-1} \times (S^1)^{r_2} \\ g &\mapsto (x_1, \ldots, x_{r_1}; r_1, \ldots, r_{r_2-1}; u_1, \ldots, u_{r_2}) \end{aligned} \tag{3.27}$$

if $r_2 \geq 1$, and

$$\begin{aligned} \gamma_1 : \mathbf{G}_1 &\longrightarrow \mathbb{R}^{r_1-1} \\ g &\mapsto (r_1, r_2, \ldots, r_{r_1-1}) \end{aligned} \tag{3.28}$$

if $r_2 = 0$.

Consider the composition of maps

$$\underbrace{\mathcal{O}_K^\times \longrightarrow \mathbf{G}_1 \xrightarrow{\gamma_1} \mathbb{R}_+^{r_1+r_2-1} \times (S^1)^{r_2} \xrightarrow{p_1} \mathbb{R}_+^{r_1+r_2-1} \xrightarrow{\log} \mathbb{R}^{r_1+r_2-1}}_{\theta}$$

where $p_1 : \mathbb{R}_+^{r_1+r_2-1} \times (S^1)^{r_2} \to \mathbb{R}_+^{r_1+r_2-1}$ is the projection on the first factor.

Recall Dirichlet's unit theorem (see Theorem 7.4 on p. 42 of [26]):

Theorem 3.10. *(Dirichlet's unit theorem) We have* $\ker\theta = \mu_K$ *and* $\mathrm{im}(\theta)$ *is a complete lattice in* $\mathbb{R}^{r_1+r_2-1}$*. In particular,* $\mathcal{O}_K^\times/\mu_K$ *injects in* $\mathbb{R}^{r_1+r_2-1}$*.*

3.7 The spaces $\mathfrak{X}_{\mathcal{V}}$, $\mathfrak{X}_{\mathcal{V}}^u$ and $\mathfrak{X}_{\mathcal{V}}^{u,1}$

Let us fix a Lie group universal covering map

$$\pi_0 : \mathbb{R}^{r_1+2r_2-1} \to \mathbb{R}^{r_1+r_2-1} \times (S^1)^{r_2} \tag{3.29}$$

From now on, we fix a subgroup $\mathcal{V} \leq \mathcal{O}_K^\times$ such that $\mathcal{V} \cap \mu_K = \{1\}$ where μ_K is the group of roots of unity in K. In particular, $\mathcal{V}$ embeds in $\mathbf{G}_1$. Let

$$L := L_1 + L_2 \leq \mathbb{R}^{r_1+2r_2-1} \tag{3.30}$$

where $L_1 := \pi_0^{-1}(\gamma_1(\mathcal{V}))$ and $L_2 := \pi_0^{-1}(\{\mathbb{0}_{r_1+r_2-1}\} \times (S^1)^{r_2})$. We clearly have that L_2 is a lattice of rank r_2. Moreover, since $p_1\gamma_1|_{\mathcal{V}} : \mathcal{V} \to \mathbb{R}_+^{r_1+r_2-1}$ is injective it follows at once that $L_1 \leq \mathbb{R}^{r_1+2r_2-1}$ is a lattice of rank $r_1 + r_2 - 1$ such that $L_1 \cap L_2 = \{0\}$. In particular, L must be a lattice of rank $r_1 + 2r_2 - 1$ and therefore

$$\mathbb{R}^{r_1+2r_2-1}/L \simeq (S^1)^{r_1+2r_2-1}. \tag{3.31}$$

Finally since $\mathbf{G}_1/\mathcal{V} \overset{\gamma_1}{\simeq} \mathbb{R}^{r_1+2r_2-1}/L$ we obtain the following:

Proposition 3.11. *There exists a Lie group isomorphism*

$$\mathbf{G}_1/\mathcal{V} \simeq (S^1)^{r_1+2r_2-1}, \tag{3.32}$$

and therefore

$$\mathrm{Hom}^{\mathcal{V}}(\mathbf{G}, \mathbb{C}^\times) \simeq \left(\widehat{\mathbb{S}} \oplus \widehat{(\mathbf{G}_1/\mathcal{V})} \oplus \check{\mathbb{L}}\right) \simeq (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_1+2r_2-1} \times \mathbb{C}. \tag{3.33}$$

In particular, each quasi-character $\chi \in \mathrm{Hom}^{\mathcal{V}}(\mathbf{G}, \mathbb{C}^\times)$ can be uniquely written as

$$\chi = \chi_{\overline{m}} \cdot \chi_1 \cdot |\,\mathbf{N}\,|^{\mathrm{i}\,t} \tag{3.34}$$

for $\chi_{\overline{m}} \in \widehat{\mathbb{S}}$, $\chi_1 \in \mathrm{Hom}^{\mathcal{V}}(\mathbf{G}_1, S^1)$ and $t \in \mathbb{C}$.

Note that for normalization purposes we have chosen to put an i in front of the complex parameter t. In particular, if $t \in \mathbb{R}$ then $g \mapsto |\,\mathbf{N}\,g|^{\mathrm{i}\,t}$ is a unitary character of $\mathbf{G}$.

Definition 3.12. *In order to simplify the notation we let*

$$\mathfrak{X}_{\mathcal{V}} := \mathrm{Hom}^{\mathcal{V}}(\mathbf{G}, \mathbb{C}^\times), \;\; \mathfrak{X}_{\mathcal{V}}^{u} := \mathrm{Hom}^{\mathcal{V}}(\mathbf{G}, S^1), \;\; \mathfrak{X}_{\mathcal{V}}^{u,1} := \mathrm{Hom}^{\mathcal{V}}(\mathbf{G}_1, S^1). \tag{3.35}$$

In this way we get the following internal direct sums

$$\mathfrak{X}_{\mathcal{V}} = \widehat{\mathbb{S}} \oplus \check{\mathbb{L}} \oplus \mathfrak{X}_{\mathcal{V}}^{u,1}, \qquad \textit{and} \qquad \mathfrak{X}_{\mathcal{V}}^{u} = \widehat{\mathbb{S}} \oplus \widehat{\mathbb{L}} \oplus \mathfrak{X}_{\mathcal{V}}^{u,1}. \tag{3.36}$$

Remark 3.13. Note that there exists an abstract group isomorphism

$$\mathfrak{X}_{\mathcal{V}}^{u,1} \simeq \mathbb{Z}^{r_1+2r_2-1}. \tag{3.37}$$

The isomorphism (3.37) is not canonical but once we choose an ordered basis $\mathcal{B}$ of $\mathcal{V}$ we will describe in Section 3.7.1 a standardized isomorphism which depends only on $\mathcal{B}$.

As a direct consequence of Proposition 3.11 we get

Proposition 3.14. *To each quasi-character $\chi \in \mathfrak{X}_{\mathcal{V}}$ we can associate a unique quintuple*

$$\mathfrak{Q}_\chi := (\overline{m}, n, \gamma, \delta; s)_\chi \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \times \mathbb{C}, \tag{3.38}$$

such that

$$\chi = \chi_{\overline{m}} \cdot \chi_n \cdot \chi_\gamma \cdot \chi_\delta \cdot |\,\mathbf{N}\,|^{\mathrm{i}\,t} \;\; \textit{and} \;\; \widetilde{\mathrm{Tr}}(\gamma, \delta) = 0. \tag{3.39}$$

Note that in this case $\chi_1 := \chi_n \cdot \chi_\gamma \cdot \chi_\delta \in \mathfrak{X}_{\mathcal{V}}^{u,1}$. In particular, if $\chi \in \mathfrak{X}_{\mathcal{V}}^{u}$ then it can be uniquely written as

$$\chi = \chi_{\overline{m}} \cdot \chi_1 \cdot |\,\mathbf{N}\,|^{\mathrm{i}\,t} \tag{3.40}$$

where $\chi_{\overline{m}} \in \widehat{\mathbb{S}}$, $\chi_1 \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ and $t \in \mathbb{R}$ so that the quintuple in that case is given by

$$\mathfrak{Q}_\chi = (\overline{m}, n, \gamma, \delta; t)_\chi \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \times \mathbb{R}. \tag{3.41}$$

We call $\mathfrak{Q}_\chi$ the $\mathfrak{Q}$-weight of χ.

Example 3.15. *(i) $\chi \in \mathfrak{X}_{\mathcal{V}}$ is a sign character, i.e. $\chi \in \widehat{\mathbb{S}}$, if and only if $\mathfrak{Q}_\chi = (\overline{m}, \mathbb{0}_{r_2}, \mathbb{0}_{r_1}, \mathbb{0}_{r_2}, 0)$.*

(ii) For each fixed complex number $s \in \mathbb{C}$, the map $x \mapsto |\,\mathbf{N}\,x|^s$, for $x \in \mathbf{G}$, gives a quasi-character $\chi \in \mathfrak{X}_{\mathcal{V}}$ for which $\mathfrak{Q}_\chi = (\overline{0}_{r_1}, \mathbb{0}_{r_2}, \mathbb{0}_{r_1}, \mathbb{0}_{r_2}, -\mathrm{i}\,s)$.

Remark 3.16. In Definition 4.1 we shall give a recipe to associate to $\mathfrak{Q}_\chi$ a unique pair of invariants $w_\chi = (p, q) \in \mathbb{Z}_{\geq 0}^{r_1+2r_2} \times \mathbf{R}_{\pm}$.

Definition 3.17. *Consider a quasi-character $\chi \in \mathfrak{X}_{\mathcal{V}}$ with its associated quintuple $\mathfrak{Q}_\chi = (\overline{m}, n, \gamma, \delta; t)_\chi$. We define*

(1) the discrete weight of χ to be the pair $(\overline{m}, n)$.

(2) the continuous weight of χ to be the triple (γ, δ, t).

(3) the norm parameter of χ to be $\mathrm{i}\,t$.

3.7.1 $\mathcal{B}$-integral parametrization of characters in $\mathfrak{X}_{\mathcal{V}}^{u,1}$ and the invariant vector A_χ

We have explained in the previous section that any character $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ is uniquely determined by a triple

$$\mathfrak{T}_\chi = (n, \gamma, \delta) = (n_\chi, \gamma_\chi, \delta_\chi) \in \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2},$$

such that $\mathrm{Tr}(\chi) = \widetilde{\mathrm{Tr}}(\gamma_\chi, \delta_\chi) = 0$, see Definition 3.7. Note that the way the weight $\mathfrak{T}_\chi$ is computed does not depend on the finite index subgroup $\mathcal{V} \leq \mathcal{O}_K^\times$. We wish in this section to provide an alternative parametrization of a character $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ in terms of a pair $[n, m] \in \mathbb{Z}^{r_2} \times \mathbb{Z}^{r_1+r_2-1}$ where $n = n_\chi$ (the first coordinate of the weight $\mathfrak{T}_\chi$) and $m \in \mathbb{Z}^{r_1+r_2-1}$. The fact that such a parametrization exists follows from the existence of an isomorphism in (3.37); but such a parametrization will depend on $\mathcal{V}$, and in fact it will really depend on a choice of an ordered basis $\mathcal{B}$ of $\mathcal{V}$. So the information contained in m will be equivalent to the information contained in the pair $(\gamma_\chi, \delta_\chi)$. We shall also introduce certain vectors $a_\chi, A_\chi \in \mathbb{R}^r$. The vector a_χ will be entirely determined by the vector n (and the basis $\mathcal{B}$) and the vector A_χ will be entirely determined by the pair $[n, m] \in \mathbb{Z}^{r_2} \times \mathbb{Z}^r$

(and the basis $\mathcal{B}$). When $r_2 = 0$, the pair $[n, m]$ reduces to m and it will be explained that in this case $A_\chi = 2\pi \cdot m$ so that m determines uniquely A_χ. However, when $r_2 \geq 1$, we shall give examples where $A_\chi = A_\psi$ but $\chi \neq \psi$. So at least in the case when $r_2 = 0$, the integral vectors $A_\chi = 2\pi m \in 2\pi \cdot \mathbb{Z}^r$ provide an alternative way of parametrizing the characters $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$.

For the rest of this section, we fix a finite index subgroup $\mathcal{V} \leq \mathcal{O}_K^\times$ such that $\mathcal{V} \cap \mu_K = \{1\}$. Let us choose arbitrarily an ordered $\mathbb{Z}$-basis of $\mathcal{V}$, namely a set of units

$$\mathcal{B} := \{\varepsilon_1, \dots, \varepsilon_r\} \subseteq \mathcal{V} \tag{3.42}$$

such that $\mathcal{V} = \varepsilon_1^{\mathbb{Z}} \cdot \varepsilon_2^{\mathbb{Z}} \dots \varepsilon_r^{\mathbb{Z}}$ (internal direct product) where $r := r_1 + r_2 - 1$.

For $x \in K$ and $1 \leq j \leq 2g$ we let $x^{(j)} := \tau_j(x)$ where $\Sigma = \{\tau_1, \dots, \tau_{2g}\}$ is the ordering associated to an admissible labelling (see Section 2). Let $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ with its associated triple $\mathfrak{T}_\chi = (n, \gamma, \delta) \in \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2}$. For each unit $\varepsilon \in \mathcal{V}$ we have $\chi(\varepsilon) = 1$ and therefore

$$\begin{aligned} &e^{\mathrm{i}\,\gamma_1 \log|\varepsilon^{(1)}|} \dots e^{\mathrm{i}\,\gamma_{r_1} \log|\varepsilon^{(r_1)}|} \cdot e^{\mathrm{i}\,2\delta_1 \log|\varepsilon^{(r_1+1)}|} \dots e^{\mathrm{i}\,2\delta_{r_2} \log|\varepsilon^{(r_1+r_2)}|} \cdot \\ &\qquad \left(\frac{\varepsilon^{(r_1+1)}}{|\varepsilon^{(r_1+1)}|}\right)^{n_1} \dots \left(\frac{\varepsilon^{(r_1+r_2)}}{|\varepsilon^{(r_1+r_2)}|}\right)^{n_{r_2}} = 1. \end{aligned} \tag{3.43}$$

Taking the complex logarithm of (3.43) we find

$$\mathrm{i}\left[\sum_{j=1}^{r_1} \gamma_j \log|\varepsilon^{(j)}| + 2\sum_{j=1}^{r_2} \delta_j \log|\varepsilon^{(r_1+j)}| + \sum_{j=1}^{r_2} n_j \arg \varepsilon^{(r_1+j)}\right] = 2\pi\,\mathrm{i}\,m \;\text{ for some } m \in \mathbb{Z}. \tag{3.44}$$

When $r_2 \geq 1$, the third sum in (3.44) involves the logarithm of complex numbers; for $z \in \mathbb{C}$, we choose the principal value of $\arg(z)$ i.e. $-\pi < \arg(z) \leq \pi$. If we apply (3.44) to each ε_k and recall that $\mathrm{Tr}(\chi) = \widetilde{\mathrm{Tr}}(\gamma, \delta) = 0$ we find that the following system of equations must be verified:

$$\sum_{j=1}^{r_1} \gamma_j + 2\sum_{j=1}^{r_2} \delta_j = 0 \tag{3.45}$$

$$\sum_{j=1}^{r_1} \gamma_j \log|\varepsilon_1^{(j)}| + 2\sum_{j=1}^{r_2} \delta_j \log|\varepsilon_1^{(r_1+j)}| = 2\pi m_1 - a_1$$

$$\sum_{j=1}^{r_1} \gamma_j \log|\varepsilon_2^{(j)}| + 2\sum_{j=1}^{r_2} \delta_j \log|\varepsilon_2^{(r_1+j)}| = 2\pi m_2 - a_2$$

$$\vdots$$

$$\sum_{j=1}^{r_1} \gamma_j \log|\varepsilon_r^{(j)}| + 2\sum_{j=1}^{r_2} \delta_j \log|\varepsilon_r^{(r_1+j)}| = 2\pi m_r - a_r \tag{3.46}$$

where

$$a_j := \sum_{k=1}^{r_2} n_k \cdot \arg\left(\frac{\varepsilon_j^{(r_1+k)}}{|\varepsilon_j^{(r_1+k)}|}\right) \quad \text{for } j = 1, 2, \dots, r \tag{3.47}$$

and where the $m_k \in \mathbb{Z}$. We set

$$m := \begin{pmatrix} m_1 \\ \vdots \\ m_r \end{pmatrix} \qquad \text{and} \qquad a_\chi := a = \begin{pmatrix} a_1 \\ \vdots \\ a_r \end{pmatrix}. \tag{3.48}$$

Notice here that $a_\chi = (a_1, \ldots, a_r)^t$ only depends on $n_\chi = (n_1, \ldots, n_r)$ and not on $(\gamma_\chi, \delta_\chi)$.

Remark 3.18. In the special case where $r = 0$ (so $K = \mathbb{Q}$ or $K = \mathbb{Q}(\sqrt{-d})$) it is understood that the above system of equations is empty. Also when $r_2 = 0$ it is understood that $s = \mathbb{0}_r$.

In matrix notation the above system can be written as

$$M \cdot R_\chi = A_\chi^0 \tag{3.49}$$

where

(3.50)

$$M = \begin{pmatrix} 1 & \ldots & 1 & 1 & \ldots & 1 \\ \log \varepsilon_1^{(1)} & \ldots & \log \varepsilon_1^{(r_1)} & \log |\varepsilon_1^{(r_1+1)}| & \ldots & \log |\varepsilon_1^{(r_1+r_2)}| \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ \log \varepsilon_r^{(1)} & \ldots & \log \varepsilon_r^{(r_1)} & \log |\varepsilon_r^{(r_1+1)}| & \ldots & \log |\varepsilon_r^{(r_1+r_2)}| \end{pmatrix}, \ R_\chi = \begin{pmatrix} \gamma_1 \\ \vdots \\ \gamma_{r_1} \\ 2\delta_1 \\ \vdots \\ 2\delta_{r_2} \end{pmatrix} \ \text{and} \ A_\chi^0 = \begin{pmatrix} 0 \\ 2\pi m_1 - a_1 \\ \vdots \\ 2\pi m_r - a_r \end{pmatrix}.$$

So the matrix M has size $(r_1 + r_2) \times (r_1 + r_2)$ and the column vectors R_χ and A_χ^0 have length r_1+r_2. Let $L_1, \ldots, L_{r+1} = L_{r_1+r_2}$ be lines of M so that $L_1 = (1 \ \ldots, 1), \ldots, L_{r_1+r_2} = (\log |\varepsilon_r^{(1)}|, \ldots, \log |\varepsilon_r^{(r_1+r_2)}|)$. By Dirichlet's unit theorem, we know that $\{L_2, \ldots, L_{r_1+r_2}\}$ are $\mathbb{R}$-linearly independent. Moreover, it follows from Proposition 7.5 on p. 45 of [26] (or it is directly and easily proved) that L_1 cannot be written as a linear combinations of $L_2, \ldots, L_{r_1+r_2}$ so that M is invertible.

Remark 3.19. Note that the first column of M^{-1} is easy to compute. From the identity $MM^{-1} = I_{r_1+r_2}$ we readily find that

$$M^{-1} = \begin{pmatrix} \frac{1}{g} & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ \frac{1}{g} & * & \ldots & * \\ & & & \\ \frac{2}{g} & * & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ \frac{2}{g} & * & \ldots & * \end{pmatrix}.$$

Definition 3.20. *Given a character $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ of weight $\mathfrak{T}_\chi = (n_\chi, \gamma_\chi, \delta_\chi)$ determines a vector*

$$A_\chi^0 := M \cdot R_\chi \in \mathbb{R}^{r_1+r_2}, \tag{3.51}$$

and we let

$$A_\chi := 2\pi \cdot m - a_\chi = \begin{pmatrix} 2\pi m_1 - a_1 \\ \vdots \\ 2\pi m_r - a_r \end{pmatrix} \in \mathbb{R}^r \tag{3.52}$$

So A_χ is the vector A_χ^0 to which one has removed its first entry 0.

Let $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ with its weight $\mathfrak{T}_\chi = (n_\chi, \gamma_\chi, \delta_\chi)$. By definition the vector $A_\chi \in \mathbb{R}^r$ is uniquely determined by the pair $(\gamma_\chi, \delta_\chi)$ (and $\mathcal{B}$); moreover since the matrix M is invertible it also follows that A_χ determines the pair $(\gamma_\chi, \delta_\chi)$. Recall that a_χ is entirely determined by the vector n_χ (and $\mathcal{B}$). In particular, the integral column vector

$$m = \frac{1}{2\pi}\left(A_\chi + a_\chi\right) \in \mathbb{Z}^r \tag{3.53}$$

is entirely determined by $\mathfrak{T}_\chi$ (and $\mathcal{B}$). Conversely, given an integral vector $n \in \mathbb{Z}^{r_2}$ (and $\mathcal{B}$) determines a vector $a_\chi \in \mathbb{R}^r$; moreover given also an integral vector $m \in \mathbb{Z}^r$ determines the vector $A_\chi = 2\pi m - a_\chi$ which in its turn determines a pair (γ, δ), and therefore a triple (n, γ, δ) which determines a character $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$. We thus see that the data $\mathfrak{T}_\chi = (n_\chi, \gamma_\chi, \delta_\chi)$ and $[n_\chi, m_\chi]$ are equivalent.

Definition 3.21. *For a character $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$, we call $[n_\chi, m_\chi]$ the $\mathcal{B}$-integral weight of χ. Conversely, any pair $[n, m] \in \mathbb{Z}^{r_2} \times \mathbb{Z}^r$ determines a unique character in $\mathfrak{X}_{\mathcal{V}}^{u,1}$ which we denote by $\chi_{[n,m];\mathcal{B}}$. When the integral basis $\mathcal{B}$ is clear from the context we shall drop the index $\mathcal{B}$ and simply write $\chi_{[n,m]}$.*

Therefore the notion of $\mathcal{B}$-integral weight provides an alternative way of parametrizing the characters in $\mathfrak{X}_{\mathcal{V}}^{u,1}$. Now the following natural question arises:

Q: Is it possible to find two distinct pairs $[n, m] \neq [n', m']$ such that $A_{\chi_{[n,m]}} = A_{\chi_{[n',m']}}$?

In the case when $r_2 = 0$ the answer is trivially no since $A_\chi = 2\pi m$ determines χ. When K contains a CM field (so that necessarily $r_2 \geq 1$) we shall soon explain a strategy for constructing distinct characters $\chi, \psi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ such that $A_\chi = A_\psi$.

Remark 3.22. The author of this paper did not succeed in constructing such a distinct pairs of characters $\chi, \psi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ when K contains no CM field.

Let us first provide a definition of a CM field which is well suited for what is to come.

Definition 3.23. *We say that K is a CM field if the following* 3 *conditions are satisfied:*

(1) K is totally imaginary so that $[K : \mathbb{Q}] = 2r_2$.

(2) For each $\sigma \in \Sigma$, $\sigma(K)$ is c_∞-stable as a set. Here $c_\infty : \mathbb{C} \to \mathbb{C}, z \mapsto c_\infty(z) = \overline{z}$ is the complex conjugation.

(3) The element $\rho := \sigma^{-1} c_\infty \sigma \in \mathrm{Aut}(K)$ is an involution which is independent of the choice of the embedding σ.

In particular, ρ is an involution in the center of $\mathrm{Aut}(K)$ *which we call the* **canonical involution**. *Note that in general $Z(\mathrm{Aut}(K))$ may contain many distinct involutions.*

Remark 3.24. If we let $K^+ := K^{\langle\rho\rangle}$ then K^+ corresponds to the maximal totally real subfield of K so that K can be viewed as a totally imaginary quadratic extension of the totally real field K^+. Note that many authors define a CM field as a totally imaginary quadratic extension of a totally real field.

So let us assume that K be a CM field and let $\rho \in Z(\mathrm{Aut}(K))$ be its canonical involution. We let $\mathrm{Aut}(K)$ act on the left of K and we let Σ "operate" as well on the left of K. Let us first explain the basic idea for constructing such pairs (χ, ψ). Let $\varepsilon \in \mathcal{O}_K^\times$. It is well-known that for any $\sigma \in \Sigma$ one has that

$$\zeta_\sigma := \sigma\left(\frac{\varepsilon}{\rho(\varepsilon)}\right) \in \mathbb{C}^\times,$$

is a root of unity in K, i.e. $\zeta_\sigma \in \mu_K$; see for example Theorem 4.12 of [31]. In particular, this implies that

$$\arg\left(\frac{\sigma(\varepsilon)}{|\sigma(\varepsilon)|}\right) \in \mathbb{Q}\pi. \tag{3.54}$$

To fix the idea, let assume for simplicity that $m = \mathbb{0}_r$. For $n \in \mathbb{Z}^{r_2}$ let

$$\chi_n := \chi_{[n, \mathbb{0}_r]}. \tag{3.55}$$

Since $m = \mathbb{0}_r$ it follows that $A_{\chi_n} = -a_{\chi_n}$. We thus wish to find two distinct vectors $n, n' \in \mathbb{Z}^{r_2}$ such that

$$a_{\chi_n} = a_{\chi_{n'}}. \tag{3.56}$$

Recall that $a_{\chi_n} = (a_1, \ldots, a_r)$ where the a_j's are defined as in (3.47) and depend only on n. We hope for the system of equations

$$a_j = \sum_{k=1}^{r_2} n'_k \cdot \arg\left(\frac{\varepsilon_j^{(r_1+k)}}{|\varepsilon_j^{(r_1+k)}|}\right) = 0 \quad \text{for } 1 \le j \le r, \tag{3.57}$$

to have infinitely many *integral* solutions in the variables $n' = (n'_1, \ldots, n'_r)$. Since $r = r_2 - 1 < r_2$ this system has infinitely many rational solutions $n' = (n'_1, \ldots, n'_r) \in \mathbb{Q}^{r_2}$. By renormalizing the system (3.57) by multiplying each equation by a suitable large integer N (so the vector $a \in \mathbb{Q}^r$ gets replaced by Na and the coefficients $\arg\left(\frac{\varepsilon_j^{(r_1+k)}}{|\varepsilon_j^{(r_1+k)}|}\right)$ get replaced by $N \arg\left(\frac{\varepsilon_j^{(r_1+k)}}{|\varepsilon_j^{(r_1+k)}|}\right)$) one obtains infinitely many solutions $n' \in \mathbb{Z}^{r_2}$.

To be more specific, let K be a quartic CM field where $\Sigma = \{\tau_1 = \sigma_1, \tau_2 = \sigma_2, \overline{\tau}_1 = \overline{\sigma}_1, \overline{\tau}_2 = \overline{\sigma}_2\}$. In that case $r_1 = 0$, $r_2 = 2$, $r = r_1 + r_2 - 1 = 1$, and the free part of $\mathcal{O}_K^\times$ has rank 1. Let $\varepsilon \in \mathcal{O}_K^\times$ generate a finite index subgroup of $\mathcal{O}_K^\times$ which does not contain any root of unity and assume furthermore that $\mathbb{Q}(\varepsilon) = K$ (in particular ε is totally imaginary). Set $\mathcal{V} := \langle \varepsilon \rangle$ and let

$$\beta_k := \frac{1}{\pi} \cdot \arg\left(\frac{\sigma_k(\varepsilon)}{|\sigma_k(\varepsilon)|}\right) \in \mathbb{Q} \text{ for } k = 1, 2. \tag{3.58}$$

Since ε is totally imaginary it follows that both β_1 and β_2 are non-vanishing. For each $n = (n_1, n_2) \in \mathbb{Z}^2$ such that

$$n_1\beta_1 + n_2\beta_2 = 0, \tag{3.59}$$

one can associate the character

$$\chi_n(x) := \left(\frac{\sigma_1(x)}{|\sigma_1(x)|}\right)^{n_1} \cdot \left(\frac{\sigma_2(x)}{|\sigma_2(x)|}\right)^{n_2}. \tag{3.60}$$

In particular, in this case note that

$$a_{\chi_n} = a_1(\chi_n) = 0 \tag{3.61}$$

is the null vector of length $r = 1$. Now, let $n = (n_1, n_2) \in \mathbb{Z}^2 \backslash \{(0,0)\}$ be a fixed solution to (3.59) so that

$$a_1(\chi_n) = \pi\big(n_1\beta_1 + n_2\beta_2\big) = 0. \tag{3.62}$$

Choose $k \in \mathbb{Z}\backslash\{0\}$ such that $k\beta_1, k\beta_2 \in \mathbb{Z}$ and set

$$n' := (n_1 + k\beta_2, n_2 - k\beta_1) \in \mathbb{Z}^2. \tag{3.63}$$

Then

$$a_1(\chi_{n'}) = \pi\big(n'_1\beta_1 + n'_2\beta_2\big) = 0. \tag{3.64}$$

In this way we have succeeded in producing infinitely many $n' \in \mathbb{Z}^2$ such that $\chi_n \neq \chi_{n'}$ but with $A_{\chi_n} = A_{\chi_{n'}} = 0$.

3.8 An instructive example when K is real quadratic fields

Let us provide an explicit description of a $\mathcal{B}$-integral parametrizations of characters in the special case of real quadratic fields. Consider the embedded quadratic field $K = \mathbb{Q}(\sqrt{d_K}) \subseteq \mathbb{R}$ where $d_K > 0$ is the discriminant of K. Let $\varepsilon_0 \in \mathcal{O}_K^\times$ be the fundamental unit normalized so that $\varepsilon_0 > 1$ and let $\mathcal{V} := \langle \varepsilon_0 \rangle$. Note that the condition $\mathcal{V} \cap \mu_K = \{1\}$ is trivially respected. In that case each character $\chi \in \mathfrak{X}_{\mathcal{V}}^u$ is uniquely parametrized by a pair $(t_1, t_2) \in \mathbb{R}^2$:

$$\begin{aligned} \chi : \mathbf{G}_0 = (K_{\mathbb{R}}^\times)^\circ &\longrightarrow S^1 \\ (x_1, x_2) &\longmapsto x_1^{it_1} \cdot x_2^{it_2}. \end{aligned}$$

Since χ is $\mathcal{V}$-invariant we must have

$$|\varepsilon_0^{(1)}|^{\mathrm{i}\,\gamma_1} \cdot |\varepsilon_0^{(2)}|^{\mathrm{i}\,\gamma_2} = \frac{|\varepsilon_0^{(1)}|^{\mathrm{i}\,\gamma_1}}{|\varepsilon_0^{(1)}|^{\mathrm{i}\,\gamma_2}} = |\varepsilon_0|^{\mathrm{i}(\gamma_1-\gamma_2)} = 1.$$

Taking the complex logarithm we thus find that

$$(\gamma_1 - \gamma_2)\log\varepsilon_0 = 2\pi m \tag{3.65}$$

for some $m \in \mathbb{Z}$. If moreover $\chi \in \mathfrak{X}_{\mathcal{V}}^{u,1}$ we must have $\mathrm{Tr}(\chi) = 0$ i.e. $\gamma_1 + \gamma_2 = 0$. We thus get the following linear system in (γ_1, γ_2)

$$\begin{cases} (\gamma_1 - \gamma_2)\log\varepsilon_0 & = 2\pi m \\ \gamma_1 + \gamma_2 & = 0 \end{cases}$$

which solution is given by

$$\gamma_1 = \frac{\pi m}{\log\varepsilon_0} \ \text{ and } \ \gamma_2 = -\frac{\pi m}{\log\varepsilon_0}. \tag{3.66}$$

It thus follows that the characters in $\mathfrak{X}_{\mathcal{V}}^{u,1}$, when restricted to $K^\times$, admits the following integral parametrization:

$$\begin{aligned} \chi_m : K^\times &\longrightarrow S^1 \\ x &\longmapsto |x|^{\frac{i\pi m}{\log\varepsilon_0}} \cdot |x'|^{-\frac{\mathrm{i}\,\pi m}{\log\varepsilon_0}} = \left|\frac{x}{x'}\right|^{\frac{\mathrm{i}\,\pi m}{\log\varepsilon_0}} \end{aligned} \tag{3.67}$$

where $K \subseteq \mathbb{R}$ via $x \mapsto x^{(1)}$ and $x' := x^{(2)}$.

3.8.1 Hyperbolic Fourier series coefficients for the real analytic lattice Eisenstein series over $\mathbb{Q}$

Consider the classical real analytic "lattice Eisenstein series"

$$G(z,s) := \sum_{(0,0)\neq(m,n)\in\mathbb{Z}^2} \frac{y^s}{|mz+n|^{2s}} \quad \mathrm{Re}(s) > 1. \tag{3.68}$$

Recall that $[s \mapsto G(z,s)]$ admits an analytic continuation to all of $\mathbb{C}\backslash\{1\}$ (with a pole of order one at $s = 1$) and that it satisfies the functional equation

$$\pi^{-s}\Gamma(s)\cdot G(z,s) = \pi^{-(1-s)}\Gamma(1-s)\cdot G(z,1-s). \tag{3.69}$$

To learn more about the motivation behind our choice of the word "lattice" see [7]. Let $\gamma \in \mathrm{SL}_2(\mathbb{Z})$ be a hyperbolic matrix, i.e. $\mathrm{Tr}(\gamma)^2 > 4$. In that case, the two eigenvalues of γ are units $\varepsilon, \varepsilon' \in \mathcal{O}_K^\times$ where $K := \mathbb{Q}(\sqrt{D}) \subseteq \mathbb{R}$, $D = \mathrm{Tr}(\gamma)^2 - 4 = f^2 d_K$ where $d_K = \mathrm{disc}(K)$ and $f \in \mathbb{Z}_{\geq 1}$. We view $K \subseteq \mathbb{R}$ as an embedded number field by taking the positive square root of D. If $x \in K$ we let x' denote its conjugate. Let $\varepsilon_0 \in \mathcal{O}_K^\times$ be the fundamental unit, normalized so that $\varepsilon_0 > 1$. Note that if $\mathbf{N}\,\varepsilon_0 = -1$ then $0 < \varepsilon_0' < -1$ while if $\mathbf{N}\,\varepsilon_0 = 1$ then $0 < \varepsilon_0' < 1$. Let w, w' be the two fixed points of γ on the real axis. Recall that $\varepsilon, \varepsilon' \in \mathbb{R}$ are the two eigenvalues of γ. We shall assume, without loss of generality, that the quantities $\varepsilon, \varepsilon', w$ and w' are normalized so that $\varepsilon > 1 > \varepsilon' > 0$ and that $w > w'$. Let $\mathfrak{n} = \mathbb{Z} + w\mathbb{Z}$ and $\mathcal{O}_f := \mathcal{O}_{\mathfrak{n}} = \mathbb{Z} + f\ caO_K$ be the order of level f. We thus have that $\mathfrak{n}$ is $\mathcal{O}_f$-invertible (the notions of $\mathcal{O}_f$-invertibility and $\mathcal{O}_f$-properness for lattices of real quadratic fields agree, see [6]).

The function $[z \mapsto G(z,s)]$ is invariant under the hyperbolic matrix γ. On p. 84-89 of [27], Siegel provides a *hyperbolic Fourier series expansion along* γ (see also § 4 of [13] for a nice discussion on the hyperbolic Fourier series expansion of a general one variable automorphic form):

$$[x \mapsto G(x + \mathrm{i}\, y, s)] = \sum_{k \in \mathbb{Z}} a_k(s) e^{2\pi \mathrm{i}\, v} \tag{3.70}$$

where

$$v := \frac{\log u}{2 \log \varepsilon}, \qquad u := \frac{z - w'}{w - z} \qquad \text{and} \qquad z = x + \mathrm{i}\, y. \tag{3.71}$$

Let $\mathfrak{g}_{w'w}$ be the oriented geodesic which connects w' to w. It is a half circle with diameter corresponding to the segment $\overline{w'w}$ which has the negative trigonometric orientation. If $z \in \mathfrak{g}_{w'w}$ then it follows from our choices of normalizations that $z^* := \gamma z$ lies on $\mathfrak{g}_{w'w}$ to the right of z; see the first picture on p. 85 of [27]. It also follows that the coefficients $a_k(s)$ do not depend on u in the sense that $\frac{\partial}{\partial u} a_k(s) = 0$ and $\frac{\partial}{\partial \overline{u}} a_k(s) = 0$; this follows from the fact that $\arg u = \frac{\pi}{2}$ as z moves along the geodesic $\mathfrak{g}_{w'w}$ (see the discussion on the top of p. 86 of [27]).

Let $\mathcal{V} := \langle \varepsilon \rangle$, it is a finite index subgroup of $\mathcal{O}_K^\times$ such that $\mathcal{V} \cap \{\pm 1\} = \{1\}$. For each integer $k \in \mathbb{Z}$, consider the character $\chi_k \in \mathfrak{X}_{\mathcal{V}}$ defined by

$$\begin{aligned} \chi_k : K^\times &\to S^1 \\ x &\mapsto \left| \frac{x}{x'} \right|^{\frac{\mathrm{i}\,\pi k}{\log \varepsilon}} \end{aligned} \tag{3.72}$$

On p. 89 of [27], Siegel calculation implies that the k-th Fourier coefficient can be written as

$$a_k(s) = \frac{\pi^s (\Gamma(s))^{-1}}{2 \log \varepsilon} \cdot \widehat{Z}_{\mathfrak{n}}(0, 0, (\chi_k, \mathcal{V}); s). \tag{3.73}$$

Here $\widehat{Z}_{\mathfrak{n}}(0, 0, (\chi_k, \mathcal{V}); s)$ is the completed $\mathcal{LZ}$ for the triple $\mathfrak{T} = (\mathfrak{n}, (0,0); (\chi_k, \mathcal{V}))$. In particular, note that the functional equations for the family of completed zeta functions

$$\{[s \mapsto \widehat{Z}_{\mathfrak{n}}(0, 0, (\chi_k, \mathcal{V}); s)] : \text{for } k \in \mathbb{Z}\},$$

follows at once from the **single** functional equation of $[s \mapsto G(z,s)]$ given in (3.69).

Remark 3.25. In [27], Siegel assumes that $\mathfrak{n}$ is an $\mathcal{O}_K$-ideal but his calculation is still valid for a general lattice $\mathfrak{n} \subseteq K$.

3.9 Relationships between $\mathcal{V}$-invariant characters, Größencharaktere and ideal class group characters

In this section we recall succinctly, for the convenience of the reader, the relationship between the following three types of objects associated to a number field K:

(1) Characters in the spaces $\mathfrak{X}^u_{\mathcal{V}}$,

(2) Größencharaktere of various moduli $\mathfrak{m}$,

(3) Characters of the idele class group $C_K := I_K/K^\times$.

Below we shall use a notation which is mostly consistent with the one found in §6 of [26]. We try in our presentation to be concise but nevertheless recall some the key definitions in order to facilitate the reading. We refer the reader to loc. cit. for a more detailed presentation (including some definitions of objects considered below for which definitions have been omitted) and some of our omitted proofs.

Let K be a fixed number field and $\mathcal{O} := \mathcal{O}_K$ be its ring of algebraic integers (so the maximal order inside K). Let $\mathfrak{m}$ be a finite modulus of K, which is the same as having an integral $\mathcal{O}$-ideal of K. We let

(a) $J_\mathfrak{m}$ be the group of all fractional ideals of K which are relatively prime to $\mathfrak{m}$

(b) $P_{\mathfrak{m},1}$ be the group of principal fractional ideals (a) which can be generated by an element $a \equiv 1 \pmod{\mathfrak{m}}$.

Remark 3.26. The groups $J_\mathfrak{m}$ and $P_{\mathfrak{m},1}$ above are denoted instead in [26] by $J^\mathfrak{m}$, $\overline{P}_\mathfrak{m}$ respectively.

We also let

(i) $R_\mathfrak{m} := \mathcal{O}/\mathfrak{m}$, the residue ring modulo $\mathfrak{m}$,

and

(ii) $\mathcal{C}\ell(\mathfrak{m}) := J_\mathfrak{m}/P_{\mathfrak{m},1}$, the ray class group of modulus $\mathfrak{m}$.

When $\mathfrak{m} = \mathcal{O}_K$ we usually write $\mathcal{C}\ell_K := \mathcal{C}\ell(\mathcal{O}_K)$.

Remark 3.27. The discussion below can be generalized in two ways, we could allow $\mathcal{O}$ to be a general order, and we could also allow $\mathfrak{m}$ to contain a square free factor supported on a subset of the set of real places of K. For simplicity we don't follow this avenue here. We suggest the reader interested in such refinements to consult [21].

Definition 3.28. *A Dirichlet character mod* $\mathfrak{m}$ *(or modulo* $\mathfrak{m}$*) is a character* $\chi \in \widehat{\mathcal{C}\ell(\mathfrak{m})}$.

Definition 3.29. *Consider $J_{\mathfrak{m}}$ as a topological group under the discrete topology. A Größencharakter mod $\mathfrak{m}$ is a character $\chi \in \widehat{J_{\mathfrak{m}}}$ for which there exists a pair of characters*

$$\chi_{\mathrm{f}} : R_{\mathfrak{m}}^{\times} \to S^1, \qquad \chi_{\infty} : \mathbf{G} = K_{\mathbb{R}}^{\times} \to S^1, \tag{3.74}$$

such that for all $a \in \mathcal{O} \backslash \{0\}$, $(a, \mathfrak{m}) = 1$, one has

$$\chi(a\mathcal{O}) = \chi_{\mathrm{f}}(a) \cdot \chi_{\infty}(a). \tag{3.75}$$

We denote the group of Größencharakere mod $\mathfrak{m}$ by $\mathrm{Gr}(\mathfrak{m})$.

We don't put any topology on $\mathrm{Gr}(\mathfrak{m})$ and think of it just as an abstract group. Let I_K be the group of ideles of K and let $C_K := I_K/K^{\times}$ be the idele class group of K. Recall that I_K and C_K are topological locally compact abelian groups.

Definition 3.30. *An idele class character (or a Hecke character in the terminology of [26]) is a continuous character $\chi \in \widehat{C_K}$*

Let

$$\mathcal{O}_{\mathfrak{m}}^{\times} := \{\varepsilon \in \mathcal{O}^{\times} : \varepsilon \equiv 1 \pmod{\mathfrak{m}}\} \tag{3.76}$$

and assume that $\mathcal{O}_{\mathfrak{m}}^{\times} \cap \mu_K = \{1\}$, so that it can be viewed as a discrete group subgroup of $\mathbf{G}$. Let also

$$C(\mathfrak{m}) := I_K/I_{\mathrm{f}}^{\mathfrak{m}} K^{\times} \tag{3.77}$$

be the "idele class group quotient of level $\mathfrak{m}$" considered on p. 481 of [26]. See p. 480 for the definition $I_{\mathrm{f}}^{\mathfrak{m}}$. In particular, by definition $C(\mathfrak{m})$ is a quotient of the idele class group C_K.

There is a well-defined group homomorphism (injective but not surjective)

$$c : J_{\mathfrak{m}} \to C(\mathfrak{m}) \tag{3.78}$$

which takes a prime ideal $\mathfrak{p} \nmid \mathfrak{m}$ and send it to the Frobenius idele $(1, \dots, 1, \pi_{\mathfrak{p}}, 1, \dots, 1)$ where $\pi_{\mathfrak{p}}$ is uniformizer of $K_{\mathfrak{p}}$ placed in position $\mathfrak{p}$; see p. 481 of [26] for more details. Moreover, one has the following short exact sequence (see Proposition 6.12 on p. 481 of [26])

$$1 \to \mathbf{G}/\mathcal{O}_{\mathfrak{m}}^{\times} \xrightarrow{\alpha} C(\mathfrak{m}) \xrightarrow{\beta} \mathcal{C}\ell(\mathfrak{m}) \to 1. \tag{3.79}$$

Remark 3.31. The term $\mathbf{G}/\mathcal{O}_{\mathfrak{m}}^{\times}$ above reads instead as $\mathbf{R}^{\times}/\mathcal{O}_{\mathfrak{m}}^{\times}$ in [26] but recall that $\mathbf{G} \simeq \mathbf{R}^{\times}$ (this follows directly from the ring isomorphism given in (2.10)).

A first key result is the following:

Theorem 3.32. *The induce map*

$$\widehat{C(\mathfrak{m})} \to \mathrm{Gr}(\mathfrak{m}) \qquad (3.80)$$
$$\chi \mapsto \chi \circ c$$

is a (abstract) group isomorphism.

Proof This is Corollary 6.14 on p. 483 of [26]. $\square$

Given a character $\chi_\infty \in \mathfrak{X}^u_{\mathcal{V}}$ we would like now to understand the Größencharakere ψ, of a suitable modulus, for which $\psi_\infty = \chi_\infty$. We shall need the following elementary lemma:

Lemma 3.33. *Let G be a finitely generated abelian group and let $H \leq G$ be a finite index subgroup where $[G : H] = e$. Let $\chi \in \widehat{H}$. Then the character χ can be extended in e different ways to a character $\widetilde{\chi}$ on G, i.e. $\widetilde{\chi} \in \widehat{G}$ and $\widetilde{\chi}|_H = \chi$.*

Proof The Lemma is clear if G/H is cyclic. Using the Smith normal form for a square matrix with integral entries we see that it is possible to find cyclic groups C_i and C_i' such that $G = \bigoplus_{i=1}^r C_i$ (internal direct sum) and $H = \bigoplus_{i=1}^r C_i'$ (internal direct sum), with $C_i' \leq C_i$ for $1 \leq i \leq r$. Since $[G : H] < \infty$ it follows that for each i, $[C_i : C_i'] < \infty$. Finally, extending χ to G is equivalent to extending separately $\chi|_{C_i'}$ to C_i and therefore the result follows. $\square$

If $\mathcal{V}, \mathcal{V}' \leq \mathcal{O}^\times$ are finite index subgroups such that $\mathcal{V}' \leq \mathcal{V}$, then we have a natural injection map $\mathfrak{X}^u_{\mathcal{V}} \to \mathfrak{X}^u_{\mathcal{V}'}$. Given a character $\chi_\infty \in \mathfrak{X}^u_{\mathcal{V}}$ we can always choose a suitable finite modulus $\mathfrak{m}$ such that $\mathcal{O}^\times_{\mathfrak{m}} \leq \mathcal{V}$ so that $\chi_\infty \in \mathfrak{X}^u_{\mathcal{O}^\times_{\mathfrak{m}}}$. Now applying Lemma 3.33 and using (3.79) we see that χ_∞, when viewed as an element of $\chi_\infty \in \mathfrak{X}^u_{\mathcal{O}^\times_{\mathfrak{m}}}$, can be extended in exactly $\#\mathcal{C}\ell(\mathfrak{m})$ ways to a character of $C(\mathfrak{m})$. As a consequence we get

Corollary 3.34. *Let $\chi_\infty \in \mathfrak{X}^u_{\mathcal{V}}$. Then there are infinitely many distinct Größencharaktere $\psi \in \mathrm{Gr}(\mathfrak{m})$, for different moduli $\mathfrak{m}$, such that $\psi_\infty = \chi_\infty$.*

It follows from the definition of a Größencharakter mod $\mathfrak{m}$ that if $\chi \in \mathrm{Gr}(\mathfrak{m})$ then

(i) $\chi_\infty \in \mathfrak{X}^u_{\mathcal{O}^\times_{\mathfrak{m}}}$, $\chi_{\mathrm{f}} \in \widehat{R^\times_{\mathfrak{m}}}$

and

(ii) for all $\varepsilon \in \mathcal{O}^\times$, $\chi_{\mathrm{f}}(\varepsilon) \cdot \chi_\infty(\varepsilon) = 1$.

Conversely, to each pair of characters as in (i) for which (ii) is satisfied we would like to better understand the Größencharactere $\psi \in \mathrm{Gr}(\mathfrak{m})$ such that $(\psi_{\mathrm{f}}, \psi_\infty) = (\chi_{\mathrm{f}}, \chi_\infty)$. As an intermediate step let us introduce the following group

$$\mathrm{Gr}_0(\mathfrak{m}) := \left\{(\psi_1, \psi_2) \in \widehat{R^\times_{\mathfrak{m}}} \times \mathfrak{X}^u_{\mathcal{O}^\times_{\mathfrak{m}}} : \text{for all } \varepsilon \in \mathcal{O}^\times,\ \psi_1(\varepsilon) \cdot \psi_2(\varepsilon) = 1.\right\} \qquad (3.81)$$

Consider the sequence

$$\mathrm{Gr}(\mathfrak{m}) \xrightarrow{\pi_{\mathfrak{m}}} \mathrm{Gr}_0(\mathfrak{m}) \xrightarrow{p_2} \mathfrak{X}^u_{\mathcal{O}^\times_{\mathfrak{m}}} \qquad (3.82)$$

where $\pi_{\mathfrak{m}}(\chi) = (\chi_{\mathrm{f}}, \chi_\infty)$ and $p_2(\chi_{\mathrm{f}}, \chi_\infty) = \chi_\infty$.

Proposition 3.35. *The maps $\pi_{\mathfrak{m}}$ and p_2 are onto. Moreover $\#(\ker \pi_{\mathfrak{m}}) = \#(J_{\mathfrak{m}}/P_{\mathfrak{m}})$ where*

(3.83) $P_{\mathfrak{m}} \leq J_{\mathfrak{m}}$ *is the subgroup of principal fractional ideals (a) which are coprime to $\mathfrak{m}$.*

Proof Consider the exact sequence

$$1 \to \mathcal{O}_{\mathfrak{m}}^{\times} \xrightarrow{t} \mathcal{O}^{\times} \xrightarrow{u} R_{\mathfrak{m}}^{\times} \xrightarrow{v} \mathcal{C}\ell(\mathfrak{m}) \tag{3.84}$$

where t, u and v are the natural maps. Let $U = u(\mathcal{O}^{\times})$ and set $e := [R_{\mathfrak{m}}^{\times} : U]$. Let us first show the surjectivity of p_2. Let $\psi_2 \in \mathfrak{X}^u_{\mathcal{O}_{\mathfrak{m}}^{\times}}$ be given. We wish to find a character $\psi_1 \in \widehat{R_{\mathfrak{m}}^{\times}}$ such that $(\psi_1, \psi_2) \in \mathrm{Gr}_0(\mathfrak{m})$. If we let $[\mathcal{O}^{\times} : \mathcal{V}] = n$ and $\varepsilon \in \mathcal{O}^{\times}$, it follows from the $\mathcal{V}$-invariance of ψ_2 that $\psi_2(\varepsilon) \in \mu_n$; and therefore looking at condition (ii) we set $\widetilde{\psi}_1(\varepsilon) := (\psi_2(\varepsilon))^{-1} = \overline{\chi_{\infty}(\varepsilon)}$. So $\widetilde{\psi}_1$ is a well-defined character on U which is completely determined by ψ_2. Using Lemma 3.33 we see that $\widetilde{\psi}_1$ can be lifted in e different ways to a character $\psi_1 \in \widehat{R_{\mathfrak{m}}^{\times}}$ which proves that p_2 is onto. Now let $(\psi_1, \psi_2) \in \mathrm{Gr}_0(\mathfrak{m})$. We wish to construct a character $\Psi \in \mathrm{Gr}(\mathfrak{m})$ such that $\pi_{\mathfrak{m}}(\Psi) = (\psi_1, \psi_2)$. First note that the pair $(\psi_1, \psi_2) \in \mathrm{Gr}_0(\mathfrak{m})$ determines a character $\widetilde{\Psi}$ on the set of principal ideals $P_{\mathfrak{m}}$ by the rule $\widetilde{\Psi}((a)) = \psi_1(a) \cdot \psi_2(a)$. Reciprocally, given a character $\chi \in \widehat{P_{\mathfrak{m}}}$ determines a unique pair $(\chi_{\mathrm{f}}, \chi_{\infty}) \in \mathrm{Gr}_0(\mathfrak{m})$. Now the character $\widetilde{\Psi}$ does not necessarily determine a character on $J_{\mathfrak{m}}$ since not all ideals in $J_{\mathfrak{m}}$ are principal. Since the class number of K is finite it follows that $h := [J_{\mathfrak{m}} : P_{\mathfrak{m}}] < \infty$. Again, using Lemma 3.33 we see that $\widetilde{\Psi}$ can be extended in h different ways to a character Ψ on $J_{\mathfrak{m}}$. Finally, by construction we have $\Psi \in \mathrm{Gr}(\mathfrak{m})$ and $\pi_{\mathfrak{m}}(\Psi) = \widetilde{\Psi}$ which proves simultaneously that $\pi_{\mathfrak{m}}$ is surjective and that $\#(\ker \pi_{\mathfrak{m}}) = h$. □

4 The w-weight, the Euler factor at ∞ and a key change of variables

We shall use systematically in this section the notation introduced in Section 2. Here $(K, {}^{\ell}\Sigma)$ is a framed number field of signature (r_1, r_2) so that $g = [K : \mathbb{Q}] = r_1 + 2r_2$; and $\Sigma = \Sigma_r \bigsqcup \Sigma_c' \bigsqcup \Sigma_c''$ is the set of embdeddings of K into $\mathbb{C}$ with a choice of an admissible labelling as described in (2.2), (2.3) and (2.4).

4.1 The w-weight of a character of G

Recall that $\mathbf{G} = K_{\mathbb{R}}^{\times}$.

Definition 4.1. *An element $p = (p_1, \ldots, p_{r_1+2r_2}) \in \mathbb{Z}_{\geq 0}^{r_1+2r_2} \subseteq \mathbf{R}$ is called* **admissible** *if for $1 \leq i \leq r_1$, $p_i \in \{0, 1\}$, and for $r_1 + 1 \leq i \leq r_1 + r_2$, $p_i \cdot p_{i+r_2} = 0$. Let $\chi \in \widehat{\mathbf{G}}$ and let*

$$\mathfrak{Q}_{\chi} = \mathfrak{Q} = (\overline{m}, n, \gamma, \delta, t) \in (\mathbb{Z}/2\mathbb{Z})^{r_1} \times \mathbb{Z}^{r_2} \times \mathbb{R}^{r_1} \times \mathbb{R}^{r_2} \times \mathbb{R} \quad \textit{and} \quad \widetilde{\mathrm{Tr}}(\gamma, \delta) = 0, \tag{4.1}$$

be its $\mathfrak{Q}$-weight, see Proposition 3.14). Following Neukrich's notation introduced on p. 476-477 of [26] we let

$$p_\chi = p := ([\overline{m}_1], \dots, [\overline{m}_{r_1}]; \widetilde{n}_1, \dots, \widetilde{n}_{r_2}; \widehat{n}_1, \dots, \widehat{n}_{r_2}) \in \mathbf{R} \tag{4.2}$$

where $\widetilde{n}_j = n_j$ and $\widehat{n}_j = 0$ if $n_j \geq 0$, while $\widetilde{n}_j = 0$ and $\widehat{n}_j = |n_j|$ if $n_j < 0$; and

$$q_\chi = q := t \cdot \mathbb{1}_{r_1+2r_2} + (\gamma_1, \dots, \gamma_{r_1}; \delta_1, \dots, \delta_{r_2}; \delta_1, \dots, \delta_{r_2}) \in \mathbf{R}_\pm. \tag{4.3}$$

In particular note that by definition p is admissible and that $p - \mathrm{i}\, q$ lies in the weight space $\mathbf{W}$ (see Definition 2.16). Moreover, by definition of (p, q), for $x \in \mathbf{G} = K_{\mathbb{R}}^\times$, we have

$$\chi(x) = \left(\frac{\alpha_1(x)}{|\alpha_1(x)|}\right)^p \cdot |\alpha_1(x)|^{\mathrm{i}\, q} = \frac{\alpha_1(x)^p}{|\alpha_1(x)|^{p - \mathrm{i}\, q}} \tag{4.4}$$

where $\alpha_1|_{\mathbf{G}} : \mathbf{G} \to \mathbf{R}^\times$ is the isomorphism induced by α_1 (see (2.10)). Recall here that for $y \in \mathbf{R}^\times$, $|y| \in \mathbf{R}_+$ is the absolute value vector and that $|y|^{p - \mathrm{i}\, q} \in \mathbf{C}$ is the power vector, see Section 2.1. We let $w_\chi = (p, q)$ and call w_χ the w-weight of χ. We call

(1) $\mathfrak{Q}_\chi$ or w_χ the weight of χ,

(2) $\overline{m}$ the (real) signature of χ,

(3) n the complex signature of χ,

(4) $(\overline{m}, n)$ or p the discrete weight of χ,

(5) (γ, δ) or q the continuous weight of χ.

Note that the continuous weight q is always an element of $\mathbf{R}_\pm$ while the discrete weight p ($\in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$) does not necessarily lie in $\mathbf{R}_\pm$ (in fact it lies outside $\mathbf{R}_\pm$ precisely when $n \neq \mathbb{0}_{r_2}$).

Remark 4.2. Let $\chi \in \mathbf{G}$. Depending on the context, it is sometimes preferable to work with the weight $\mathfrak{Q}_\chi$ while at other times (p_χ, q_χ) is more handy. Even though strictly speaking the weight $\mathfrak{Q}_\chi$ (a quintuple) and (p, q) (an ordered pair of vectors) are different they contain the same information so this slight ambiguity in the terminology should not cause any difficulty.

Example 4.3. *Let $\chi \in \mathfrak{X}_{\mathcal{V}}^u$ and suppose that $\mathfrak{Q}_\chi = (\overline{m}, n, \gamma, \delta, t)$ and $w_\chi = (p_\chi, q_\chi)$ are the normalized weights of χ. Let $\overline{\chi}$ be the complex conjugate character of χ. Then $\mathfrak{Q}_{\overline{\chi}} = (\overline{m}, -n, -\gamma, -\delta, -t)$ and $(p_{\overline{\chi}}, q_{\overline{\chi}}) = (c_1(p), -q)$ are the weights of $\overline{\chi}$. Here c_1 is the involution introduced in Section 2.*

When χ is fixed and clear from the context we shall usually (p, q) instead of (p_χ, q_χ).

4.2 The Euler factor at infinity

Let $(K, {}^{\ell}\Sigma)$ be a framed number field of signature (r_1, r_2) and let $\mathbf{C} = \mathbf{C}_\Sigma$ (see Section 2). Recall that $\mathbf{W} = \mathbf{W}_\Sigma$ is the weight space (see Definition 2.16) and that for $\mathfrak{s} \in \mathbf{C}$, such that $\mathfrak{s}_i \notin \mathbb{Z}_{\leq 0}$ for all i, $\Gamma_\Sigma(\mathfrak{s})$ was defined as the product given in (2.45).

Definition 4.4. *For $s \in \mathbb{C}$, $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$ admissible and $q \in \mathbf{R}_\pm$, we set*

$$\mathfrak{s}[s] = s \cdot \mathbb{1} + (p - \mathrm{i}\, q) \in \mathbf{W} \subseteq \mathbf{C}. \tag{4.5}$$

Let $s \in \mathbb{C}$ be fixed and set $\mathfrak{s} := \mathfrak{s}[s]$ to lighten the notation. The Euler factor of parameters (p, q) evaluated at $s \in \mathbb{C}$ is defined by

$$\mathrm{F}_{(p,q)}(s) := |d_K|^{\frac{s}{2}} \cdot \left(\prod_{i=1}^{r_1} \left(\pi^{-\frac{\mathfrak{s}_i}{2}} \cdot \Gamma\left(\frac{\mathfrak{s}_i}{2}\right) \right) \right) \cdot \left(\prod_{i=r_1+1}^{r_1+r_2} \left(2(2\pi)^{-\frac{\mathfrak{s}_i + \mathfrak{s}_{i'}}{2}} \cdot \Gamma\left(\frac{\mathfrak{s}_i + \mathfrak{s}_{i'}}{2}\right) \right) \right) \tag{4.6}$$

$$= |d_K|^{\frac{s}{2}} \cdot \pi^{-\frac{1}{2}\operatorname{Tr}(\mathfrak{s})} \cdot \Gamma_\Sigma\left(\frac{\mathfrak{s}}{2}\right), \tag{4.7}$$

where for $r_1 + 1 \leq i \leq r_1 + r_2$, $i' = i + r_2$. It is also convenient to define the "p-reduced Euler factor of parameters (p, q) evaluated at s" by

$$\widetilde{\mathrm{F}}_{(p,q)}(s) := |d_K|^{\frac{s}{2}} \cdot \pi^{-\frac{sg}{2}} \cdot \left(\cdot 2^{\mathrm{i} \sum_{j=r_1+1}^{r_1+r_2} q_j} \cdot \pi^{\frac{\mathrm{i}}{2}\left(\sum_{j=1}^{r_1} q_j\right) + \mathrm{i}\left(\sum_{j=r_1+1}^{r_1+r_2} q_j\right)} \right) \cdot \Gamma_\Sigma(\mathfrak{s})$$

$$= \left(|d_K|^{\frac{s}{2}} \cdot \pi^{-\frac{sg}{2}} \cdot 2^{r_2(1-s) + \mathrm{i} \sum_{j=r_1+1}^{r_1+r_2} q_j} \cdot \pi^{\frac{\mathrm{i}}{2}\left(\sum_{j=1}^{r_1} q_j\right) + \mathrm{i}\left(\sum_{j=r_1+1}^{r_1+r_2} q_j\right)} \right) \cdot \prod_{j=1}^{r_1} \Gamma\left(\frac{\mathfrak{s}_j}{2}\right) \cdot \prod_{j=r_1+1}^{r_1+r_2} \Gamma\left(\frac{\mathfrak{s}_j + \mathfrak{s}_{j'}}{2}\right). \tag{4.8}$$

In particular, note that the first factor within the large parentheses in the last equality of (4.8) does not depend on the parameter p which explains our choice of terminology. Note that by definition the argument of the Euler factors $\mathrm{F}_{(p,q)}(\_)$ and $\widetilde{\mathrm{F}}_{(p,q)}(\_)$ is a one dimensional complex variable.

Recall that $c_1 : \mathbf{C} \to \mathbf{C}$ is the involution which swaps the two coordinates for each conjugate pair of complex embeddings (see Section 2). Using the observations that $\sum_{j=r_1+1}^{r_1+2r_2} p_j = \sum_{j=r_1+1}^{r_1+2r_2} (c_1(p))_j$ and that $q_j = q_{j'}$ for $r_1 + 1 \leq j \leq r_1 + r_2$ one readily sees from the definitions of $\mathrm{F}_{(p,q)}(s)$ and $\widetilde{\mathrm{F}}_{(p,q)}(s)$ that

$$\frac{\mathrm{F}_{(p,q)}(s)}{\mathrm{F}_{(c_1(p),-q)}(1-s)} = \frac{\widetilde{\mathrm{F}}_{(p,q)}(s)}{\widetilde{\mathrm{F}}_{(c_1(p),-q)}(1-s)}. \tag{4.9}$$

Proposition 4.5. *Let $\chi \in \mathfrak{X}_{\mathcal{V}}^u$ with weight equal to $(p, q) \in \mathbf{W}$. Then*

$$\mathrm{F}_\chi(s) = \mathrm{F}_{(p,q)}(s), \tag{4.10}$$

where $\mathrm{F}_\chi(s)$ was the Euler factor defined in (1.8).

Proof This is clear when one compares the product expansion given in (4.6) with the one given in (1.8). □

4.3 The key change of variable for the multi-dimensional Gamma function

Lemma 4.6. *For $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$ admissible, $q \in \mathbf{R}_\pm$ and $s \in \Pi_1$ we let*

$$\widehat{\mathfrak{s}}[s] := s \cdot \mathbb{1} + \frac{1}{2} \cdot (p - \mathrm{i}\, q) \in \mathbf{W}. \tag{4.11}$$

Notice the factor $\frac{1}{2}$ in front of $p - \mathrm{i}\, q$. Let also

$$\mathfrak{s}[s] := s \cdot \mathbb{1} + (p - \mathrm{i}\, q) \tag{4.12}$$

be as in (4.5)*. For* $\mathrm{Re}(\widehat{\mathfrak{s}}[s]) > 0$ *it follows from Proposition 2.19 that*

$$\Gamma_\Sigma(\widehat{\mathfrak{s}}[s]) = \int_{\mathbf{R}_+} \mathbf{N}(e^{-y} y^{\widehat{\mathfrak{s}}[s]}) d^* y. \tag{4.13}$$

Here the variable $y \in \mathbf{R}_+$ and d^*y is the Haar measure that was defined in Section 2.3. Let $\mathfrak{n} \subseteq K$ be a lattice. Recall that $\mathbf{N}\, \mathfrak{n}$ is the absolute norm of $\mathfrak{n}$ and $d_\mathfrak{n}$ is the absolute value discriminant of $\mathfrak{n}$, see (2.26) *and* (2.25) *respectively. Then for $x \in \mathbf{R}^\times$ we have*

$$\int_{\mathbf{R}_+} e^{-\langle\, \pi |x|^2 / d_\mathfrak{n}^{1/g} y, \mathbb{1}\, \rangle_c} \mathbf{N}(y^{\widehat{\mathfrak{s}}[s]}) d^* y = |d_K|^{\frac{\lambda_0}{2}} \cdot \mathrm{F}_{(p,q)}(2s) \cdot (\mathbf{N}\, \mathfrak{n})^{\lambda_0} \cdot \frac{(\mathbf{N}\, \mathfrak{n})^{2s}}{\mathbf{N}\left(|x|^{p - \mathrm{i}\, q}\right) \cdot |\,\mathbf{N}(x)|^{2s}}, \tag{4.14}$$

where

$$\lambda_0 := \lambda_0(p,q) = \frac{\mathrm{Tr}(p - \mathrm{i}\, q)}{g}. \tag{4.15}$$

For the defining formula for $\mathrm{F}_{(p,q)}(2s)$ see Definition 4.4.

Proof For $x \in \mathbf{R}^\times$ we perform the substitution $y \mapsto \pi |x|^2 y / d_\mathfrak{n}^{1/g}$ into (4.13). From the multiplicative invariance of d^*y we find that

$$\int_{\mathbf{R}_+} e^{-\pi \langle\, |x|^2 y / d_\mathfrak{n}^{1/g}, 1\, \rangle_c} \cdot \mathbf{N}(y^{\widehat{\mathfrak{s}}[s]}) d^* y = \pi^{-\mathrm{Tr}(\widehat{\mathfrak{s}}[s])} \cdot \Gamma_\Sigma(\widehat{\mathfrak{s}}[s]) \cdot (d_\mathfrak{n})^{\frac{\mathrm{Tr}(\widehat{\mathfrak{s}}[s])}{g}} \cdot \frac{1}{\mathbf{N}\left(|x|^{2\widehat{\mathfrak{s}}[s]}\right)}, \quad \mathrm{Re}(\widehat{\mathfrak{s}}[s]) > 0. \tag{4.16}$$

By definition we have

$$\frac{\mathfrak{s}[2s]}{2} = \widehat{\mathfrak{s}}[s], \tag{4.17}$$

and therefore it follows that

$$\Gamma_\Sigma(\widehat{\mathfrak{s}}[s]) = \Gamma_\Sigma\left(\frac{\mathfrak{s}[2s]}{2}\right) \quad \text{and} \quad \pi^{-\frac{1}{2}\operatorname{Tr}(\mathfrak{s}[2s])} = \pi^{-\operatorname{Tr}(\widehat{\mathfrak{s}}[s])}. \tag{4.18}$$

Using (4.18) and unfolding the definitions of $\mathrm{F}_{(p,q)}(2s)$ reveals that

$$\pi^{-\operatorname{Tr}(\widehat{\mathfrak{s}}[s])} \cdot \Gamma_\Sigma(\widehat{\mathfrak{s}}[s]) = \frac{1}{|d_K|^s} \cdot \mathrm{F}_{(p,q)}(2s). \tag{4.19}$$

Since $d_\mathfrak{n} = |d_K| \cdot (\mathbf{N}\,\mathfrak{n})^2$ it follows that

$$d_\mathfrak{n}^{\frac{\operatorname{Tr}\widehat{\mathfrak{s}}[s]}{g}} = |d_K|^s \cdot (\mathbf{N}\,\mathfrak{n})^{2s} \cdot (\mathbf{N}(\mathfrak{n}))^{\lambda_0} \cdot |d_K|^{\frac{\lambda_0}{2}}. \tag{4.20}$$

Moreover, unfolding the definition of the norm, the power map (see Section 2.1) and $\widehat{\mathfrak{s}}[s]$, we see that

$$\frac{1}{\mathbf{N}\left(|x|^{2\widehat{\mathfrak{s}}[s]}\right)} = \frac{1}{\mathbf{N}\left(|x|^{p-\mathrm{i}\,q}\right) \cdot |\,\mathbf{N}(x)|^{2s}}. \tag{4.21}$$

Finally, if we combine (4.19), (4.20) and (4.21) in order to rewrite the right hand side of (4.16), we deduce that

(4.22)
$$\int_{\mathbf{R}_+} e^{-\pi\langle\, |x|^2 y/d_\mathfrak{n}^{1/g}, 1\,\rangle_c} \cdot \mathbf{N}(y^{\widehat{\mathfrak{s}}[s]}) d^*y = |d_K|^{\frac{\lambda_0}{2}} \cdot \mathrm{F}_{(p,q)}(2s) \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \frac{(\mathbf{N}\,\mathfrak{n})^{2s}}{\mathbf{N}\left(|x|^{p-\mathrm{i}\,q}\right) \cdot |\,\mathbf{N}(x)|^{2s}}.$$

This concludes the proof. □

5 Spherical polynomials and theta functions

Recall that $(K, {}^\ell\Sigma)$ is a framed number field of signature (r_1, r_2) so that $g = [K : \mathbb{Q}] = r_1 + 2r_2$; and $\Sigma = \Sigma_r \bigsqcup \Sigma_c' \bigsqcup \Sigma_c''$ is the set of embdeddings of K into $\mathbb{C}$ with a choice of an admissible labelling as described in (2.2), (2.3) and (2.4).

5.1 The quadratic spaces $\mathbb{R}_\rho$ and $\mathbb{C}_\tau$ and their spherical polynomials

For this section we shall sometimes appeal to some basic notions on spherical polynomials for which we refer the reader to Appendix B for a short overview.

Definition 5.1. *(1) For $\rho \in \Sigma_r$ we let $\mathbb{R}_\rho := (\mathbb{R}, Q)_\rho$ where $Q(x) = x^2$ where $x \in \mathbb{R}$ is the coordinate. The Laplacian on $\mathbb{R}_\rho$ is defined as $\Delta_\rho := -\partial_x^2$.*

(2) For $\sigma \in \Sigma_c'$ we let $\mathbb{C}_\sigma := (\mathbb{C}, Q)_\sigma$ where $Q(z) = 2(x^2 + y^2)$ where $z = x + \mathrm{i}\,y \in \mathbb{C}$ is the coordinate. The Laplacian on $\mathbb{C}_\sigma$ is defined by $\Delta_\sigma := -\left(\partial_x^2 + \partial_y^2\right)$.

Note that we put a subscript ρ on $(\mathbb{R}, \mathbb{Q})_\rho$ since we wish to view the quadratic spaces $\mathbb{R}_\rho$'s as being distinct as the ρ's vary in Σ_r. Similarly, we keep the subscript σ on $(\mathbb{C}, Q_c)_\sigma$ since we wish to view those as well as distinct quadratic spaces when the σ's vary in Σ'_c.

For $\rho \in \Sigma_r$ and $\sigma \in \Sigma'_c$ we have canonical injections of real quadratic spaces

$$\iota_\rho : \mathbb{R}_\rho \hookrightarrow (\mathbf{R}, \langle , \rangle_c) \qquad \text{and} \qquad \iota_\sigma : \mathbb{C}_\sigma \hookrightarrow (\mathbf{R}, \langle , \rangle_c). \tag{5.1}$$

which provide a canonical internal direct sum decomposition, as quadratic spaces, of $(\mathbf{R}, \langle , \rangle_c)$:

$$(\mathbf{R}, \langle , \rangle_c) = \left(\bigoplus_{j=1}^{r_1} \mathbb{R}_{\rho_j} \right) \oplus \left(\bigoplus_{j=1}^{r_2} \mathbb{C}_{\sigma_j} \right). \tag{5.2}$$

If we denote by Δ_c the Laplacian on $(\mathbf{R}, \langle , \rangle_c)$ we find the following canonical decomposition into partial Laplacians:

$$\Delta_c = \left(\bigoplus_{i=1}^{r_1} \Delta_{\rho_i} \right) \oplus \left(\bigoplus_{i=1}^{r_2} \Delta_{\sigma_i} \right). \tag{5.3}$$

Now, let $(V := \mathbb{R}^n, Q)$ be a positive definite real quadratic space of rank n. We let $\mathcal{H}(V, Q)$ be the space of spherical polynomials with **complex** coefficients associated to (V, Q). $\mathcal{H}(V, Q)$ is a graded vector subspace of $\mathbb{C}[x_1, \ldots, x_n]$ where $(x_1, \ldots, x_n) \in \mathbb{R}^n$ is the usual coordinate chart, so that $\mathcal{H}(V, Q) = \bigoplus_{k \geq 1} \mathcal{H}_k(V, Q)$. For example, if $\rho \in \Sigma_r$ then

$$\mathcal{H}(\mathbb{R}_\rho) = \mathbb{C} + \mathbb{C}x, \tag{5.4}$$

and if $\sigma \in \Sigma'_c$ then

$$\mathcal{H}(\mathbb{C}_\sigma) = \mathbb{C} \oplus z\mathbb{C}[z] \oplus \overline{z}\mathbb{C}[\overline{z}] \subseteq \mathbb{C}[x, y] \tag{5.5}$$

where $z = x + \mathrm{i}\, y$. In particular, $\mathcal{H}(\mathbb{R}_\rho)$ has only two non-zero graded components in degree 0 and 1 respectively. As for $\mathcal{H}(\mathbb{R}_\sigma)$ its k-th graded component, for $k \geq 1$, is given by

$$\mathcal{H}_k(\mathbb{C}_\sigma) = \mathbb{C}z^k \oplus \mathbb{C}\overline{z}^k = \mathbb{C}\,\mathrm{Re}(z^k) \oplus \mathbb{C}\,\mathrm{Im}(z^k), \tag{5.6}$$

so that $\dim_{\mathbb{C}} \mathcal{H}_k(\mathbb{C}_\sigma) = 2$ (for $k \geq 1$).

Let us write a coordinate $z \in \mathbf{R} \subseteq \mathbf{C}$ as

$$z = (z_1, \ldots, z_{r_1}; z_{r_1+1}, \ldots, z_{r_1+r_2}; \overline{z}_{r_1+r_2+1}, \ldots, \overline{z}_{r_1+2r_2}) \tag{5.7}$$

so that $\mathcal{H}(\mathbf{R}, \langle , \rangle_c)$ can be viewed as a subspace of $\mathbb{C}[z_1, \ldots, z_{r_1+r_2}, \overline{z}_{r_1+r_2+1}, \ldots, \overline{z}_{r_1+2r_2}]$.

Definition 5.2. *Let $f(z) \in \mathcal{H}_k(\mathbf{R}, \langle , \rangle_c)$ be a homogeneous spherical polynomial of degree k. We say that $f(z)$ is* **diagonal** *with respect to the decomposition* (5.2) *if*

$$\text{for all } \tau \in \Sigma_r \textstyle\bigsqcup \Sigma'_c, \;\; \Delta_\tau f = 0. \tag{5.8}$$

It follows directly from the definition that a homogeneous polynomial $f(z) \in \mathcal{H}_k(\mathbf{R}, \langle , \rangle_c)$ is spherical diagonal if and only if there exists an **admissible** tuple $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$ (see Definition 2.16) and a scalar $\lambda \in \mathbb{C}$ such that

$$f(z) = \lambda\, \mathbf{N}(z^p). \tag{5.9}$$

Recall here that if $p \in \mathbb{Z}_{\geq 0}^g$ and $z \in \mathbf{C}$ then z^p means $(z_1^{p_1}, \ldots, z_g^{p_g}) \in \mathbf{C}$ (see Section 2.1). Note that the homogeneous degree of f is such that $k = \sum_i p_i$.

Definition 5.3. *A homogeneous spherical diagonal polynomial which satisfies* (5.9) *is said to have (admissible) weight* $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$.

5.2 The number field theta function

For the definitions of the sets $(\mathbf{C}, \langle , \rangle_c)$, $\mathbf{R}_\pm$, $\mathbf{R}$ and $\mathbf{H}$ see Section 2.

Definition 5.4. *We define the decorated gaussian function as the map*

$$\begin{aligned} \mathcal{G}_{\_}(\_, \_) : \mathbf{R} \times \mathbf{C} \times \mathbf{H} &\longrightarrow \mathbb{C} \\ (a, w, z) &\longmapsto \mathcal{G}_a(w, z) := e^{\pi\,\mathrm{i}\langle az, a\rangle_c + 2\pi\,\mathrm{i}\langle w, a\rangle_c}. \end{aligned} \tag{5.10}$$

Usually, we think of a as a fixed parameter which corresponds to the function

$$\begin{aligned} &\mathbf{C} \times \mathbf{H} \to \mathbb{C} \\ &(w, z) \mapsto \mathcal{G}_a(w, z) = e^{\pi\,\mathrm{i}\langle az, a\rangle_c + 2\pi\,\mathrm{i}\langle w, a\rangle_c}. \end{aligned} \tag{5.11}$$

Since $\langle , \rangle_c$ is $(\mathbb{C}, \overline{\mathbb{C}})$-bilinear, it follows that $(w, z) \mapsto \mathcal{G}_a(w, z)$ is holomorphic.

Definition 5.5. *Let* $(a, w, z) \in \mathbf{R} \times \mathbf{C} \times \mathbf{H}$. *Let* $L \subseteq \mathbf{R}$ *be a lattice of maximal rank and let* $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$ *be admissible (see Definition 2.16). We define the number field (spherical) theta function of weight p as*

$$\theta_L^p(a, w; z) = \sum_{\ell \in L} \mathbf{N}((a+\ell)^p) \cdot \mathcal{G}_{a+\ell}(w, z) = \sum_{\ell \in L} \mathbf{N}((a+\ell)^p) \cdot e^{\pi i\langle (a+\ell)z, (a+\ell)\rangle_c + 2\pi i\langle w, a+\ell\rangle_c}. \tag{5.12}$$

The sum in (5.12) converges absolutely, this is Proposition 3.5 of [26]. Note that Neukirch in his statement assumes that $w \in \mathbf{R}$ but this is not needed in his proof. It is also clear that $[(w, z) \mapsto \theta_L^p(a, w; z)]$ is holomorphic.

Remark 5.6. If we ignore the choice of an admissible labelling on Σ, we see that the dependence of $\theta_L^p(a, w; z)$ on K is only a dependence on the signature (r_1, r_2) of K.

Remark 5.7. Our definition of $\theta_L^p(a, w; z)$ is slightly different from the one appearing on the bottom of page 450 of [26]. Namely, if we denote by $\theta_L^p(a, w; z)_{\text{Neuk}}$ the theta function introduced by Neukirch on p.450 we find the simple relation

$$e^{2\pi\,\mathrm{i}\langle a, w\rangle_c} \cdot \theta_L^p(a, w; z)_{\text{Neuk}} = \theta_L^p(a, w; z). \tag{5.13}$$

Remark 5.8. The reader may wonder about the relationship, if any, between $\theta^p(a, w; z)$ and the classical Riemann theta functions which are use to embed polarized complex tori into projective space. In Appendix C we explain such a relationship.

Recall that if $\mathfrak{n} \subseteq K$ is a lattice then the rational index of $\mathfrak{n}$ in $\mathcal{O}_K$ is denoted by $[\mathcal{O}_K : \mathfrak{n}] \in \mathbb{Q}_{>0}$. Moreover the covolume of $\mathfrak{n}$ inside $\mathcal{O}_K$ is defined as $\operatorname{cov}(\mathfrak{n}) := \sqrt{|d_K|}[\mathcal{O}_K : \mathfrak{n}]$, see Section 2.2. Recall that the quantity $\sqrt{d_L}$ can be interpreted as the covolume of the lattice $L \subset \mathbf{R}$ with respect to the measure vol_c on $(\mathbf{R}, \langle\ ,\ \rangle_c)$ (see Section 2.2) which ascribes the volume 1 to the cube spanned by an orthonormal basis.

The key ingredient to prove the functional equation appearing in Theorem 1.4 is the following transformation formula for the theta function.

Theorem 5.9. *(theta transformation formula) Let $a, b \in \mathbf{R}_\pm$ and $p \in \mathbb{Z}_{\geq 0}^{r_1+2r_2}$ be an admissible discrete weight. Then one has*

(5.14)
$$\theta^p_{\mathfrak{n}}(a, b; -1/z) = (-\mathrm{i})^{\operatorname{Tr}(p)} \cdot e^{2\pi \mathrm{i}\langle a,b\rangle_c} \cdot (d_{\mathfrak{n}})^{-1/2} \cdot \mathbf{N}((z/\mathrm{i})^{p+\frac{1}{2}\cdot\mathbb{1}}) \cdot \theta^p_{\mathfrak{n}^*}(-b, a; z) \ \textit{for all } z \in \mathbf{H},$$

where $\mathbb{1} = \mathbb{1}_g$ is the unit element of $\mathbf{R}$ and $\mathfrak{n}^*$ is the dual lattice of $\mathfrak{n}$ (see Definition 1.8). Moreover, one has that

$$\theta^p_{\mathfrak{n}}(-a, -b; z) = (-1)^{\operatorname{Tr}(p)} \cdot \theta^p_{\mathfrak{n}}(a, b; z) \tag{5.15}$$

Proof For the proof of (5.14) see equation (19) on p. 264 of [16] or (3.6) on the top of p. 452 of [26]. The proof of (5.15) follows directly from the definition $\theta^p_{\mathfrak{n}}(-a, -b; z)$ given in (5.12). □

Remark 5.10. Note that the analogue of the factor $(-\mathrm{i})^{\operatorname{Tr}(p)} \cdot e^{2\pi \mathrm{i}\langle a,b\rangle_c} \cdot (d_{\mathfrak{n}})^{-1/2}$ in (5.14) reads instead as in [26] (top of p. 452)

$$(\mathrm{i}^{\operatorname{Tr}(p)}\, e^{2\pi \mathrm{i}\langle a,b\rangle} \operatorname{vol}(\mathfrak{n}))^{-1} = (-\mathrm{i})^{\operatorname{Tr}(p)} \cdot e^{-2\pi \mathrm{i}\langle a,b\rangle} \cdot \operatorname{vol}(\mathfrak{n})^{-1} \tag{5.16}$$

Here $\operatorname{vol}(\mathfrak{n})^{-1}$ in Neukrich's notation which corresponds to our $d_{\mathfrak{n}}^{-1/2}$. There is thus a sign discrepancy when one compares the factor $e^{2\pi \mathrm{i}\langle a,b\rangle_c}$ which appears in (5.14) with the analogous factor $e^{-2\pi \mathrm{i}\langle a,b\rangle}$ which appears in (5.16). The reason for this discrepancy is that we chose a slightly different normalization of our theta function than Neukirch's one. Using (5.13) one can see that the two functional equations are indeed equivalent.

6 Proof of the functional equation

6.1 Fixing the notation

We have $K \subseteq K_{\mathbb{R}}$ and we view K as a subset of $\mathbf{R}$ via the map $\alpha_1 : K_{\mathbb{R}} \to \mathbf{R}$ (see (2.10)) We let $\mathfrak{n} \subseteq K$ be a lattice and $a, b \in K$. A typical element of $\mathfrak{n}$ will be denoted by n. Recall that $\mathcal{V}_{\mathfrak{n};a,b}$ is the group of unit which appears in Definition 1.10. Let $\mathcal{V}$ be a finite index subgroup of $\mathcal{V}_{\mathfrak{n};a,b}$ such that $\mathcal{V}\cap\mu_K=\{1\}$. We fix once and for all a character $\chi\in\mathfrak{X}^u_{\mathcal{V}}$ (see Definition 3.12). Various objects that will appear in the proof will depend on the data $(\mathfrak{n},(a,b),(\chi,\mathcal{V}))$; so in order to make the notation more compact we let

$$\mathfrak{T}=(\mathfrak{n},(a,b),\chi)) \text{ and } \mathfrak{T}^*=(\mathfrak{n}^*,(-b,a),\overline{\chi}), \tag{6.1}$$

where χ is viewed as a character on $K_{\mathbb{R}}^\times/\mathcal{V}$. Note that the definition of $\mathfrak{T}^*$ makes sense since $\mathcal{V}_{\mathfrak{n};a,b}=\mathcal{V}_{\mathfrak{n}^*;-b,a}$ ((a) of Proposition 1.12) and therefore $\mathcal{V}\le\mathcal{V}_{\mathfrak{n}^*;-b,a}$. We let $w_\chi:=(p,q))$ be the w-weight vector of χ (see Definitions 4.1); so that $w_{\overline{\chi}}=(p_{\overline{\chi}},q_{\overline{\chi}})=(c_1(p),-q)$. In particular $p\in\mathbb{Z}^{r_1+2r_2}_{\ge0}$ is admissible and $q\in\mathbf{R}_\pm$. It is also convenient to let

$$\lambda_0:=\frac{\operatorname{Tr}(p-\mathrm{i}\,q)}{g} \tag{6.2}$$

6.2 Preliminaries: Lemmas and Propositions

For $s\in\Pi_1$ we let

$$\widehat{\mathfrak{s}}:=s\cdot\mathbb{1}+\frac{1}{2}\cdot(p-\mathrm{i}\,q). \tag{6.3}$$

For $x\in\mathbf{R}^\times$, it follows from Lemma 4.6 that

(6.4)
$$\int_{\mathbf{R}_+}e^{-\pi\langle\,|x|^2y/d_{\mathfrak{n}}^{1/g},1\,\rangle_c}\,\mathbf{N}(y^{\widehat{\mathfrak{s}}})d^*y=|d_K|^{\frac{\lambda_0}{2}}\cdot\mathrm{F}_{(p,q)}(2s)\cdot(\mathbf{N}\,\mathfrak{n})^{\lambda_0}\cdot\frac{\mathbf{N}(\mathfrak{n})^{2s}}{\left|\mathbf{N}\left(|x|^{p-\mathrm{i}\,q}\right)\right|\cdot|\,\mathbf{N}(x)|^{2s}}.$$

Let us recall the notation in (6.4). Here $\langle,\rangle_c$ is the canonical metric on $\mathbf{R}$ (see Definition (2.7)), d^*y is the normalized multiplicative Haar measure on $\mathbf{R}_+$ defined in Section 2.3, $d_{\mathfrak{n}}>0$ is the absolute value discriminant of $\mathfrak{n}$ (see (2.25)), $\mathbf{N}(\mathfrak{n})>0$ is the absolute norm of $\mathfrak{n}$ (see (2.26)) and $\mathrm{F}_{(p,q)}(s)$ was defined in (4.6).

Now let $\mathfrak{R}=\{n_i\in\mathfrak{n}\}_{i\in I}$ be a complete set of representatives of $\{a+\mathfrak{n}\}/\mathcal{V}$ in the sense that every element $0\ne a+n\in a+\mathfrak{n}$ can be written uniquely as $\epsilon(a+n_i)$ for some $n_i\in\mathfrak{R}$ and $\epsilon\in\mathcal{V}$. We associate to $\mathfrak{R}$ the following "coset theta function"

$$\theta^p_{\mathfrak{R}}(a,b;z):=\sum_{\substack{n\in\mathfrak{R}\\a+n\ne0}}\mathbf{N}((a+n)^p)\cdot e^{\pi\,\mathrm{i}\langle|a+n|^2\,z,1\rangle_c+2\pi i\langle b,a+n\rangle_c}, \tag{6.5}$$

where $z\in\mathbf{H}$ and $|a+n|\in\mathbf{R}_\pm$ is the absolute value vector.

Combining (6.4), (6.5) and (4.4) we find that

$$\begin{aligned}\int_{\mathbf{R}_+}\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,y/d_{\mathfrak{n}}^{1/g})\cdot\mathbf{N}(y^{\widehat{\mathfrak{s}}})\,d^*y&=|d_K|^{\frac{\lambda_0}{2}}\cdot\mathrm{F}_{(p,q)}(2s)\cdot(\mathbf{N}\,\mathfrak{n})^{\lambda_0}\cdot(\mathbf{N}\,\mathfrak{n})^{2s}\cdot\sum_{\substack{n\in\mathfrak{R}\\a+n\ne0}}\chi(a+n)\cdot\frac{e^{2\pi\,\mathrm{i}\langle b,a+n\rangle_c}}{|\,\mathbf{N}_{K/\mathbb{Q}}(a+n)|^{2s}}\\&=|d_K|^{\frac{\lambda_0}{2}}\cdot(\mathbf{N}\,\mathfrak{n})^{\lambda_0}\cdot\mathrm{F}_{(p,q)}(2s)\cdot Z_{\mathfrak{n}}(a,b,\chi;2s)\\&=|d_K|^{\frac{\lambda_0}{2}}\cdot(\mathbf{N}\,\mathfrak{n})^{\lambda_0}\cdot\widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s),\end{aligned} \tag{6.6}$$

where $Z_{\mathfrak{n}}(a,b,\chi;2s)$ is the uncomplete lattice zeta function given in (1.6) and where

$$\widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s) := \mathrm{F}_{(p,q)}(2s)\cdot Z(a,b,\chi;2s), \tag{6.7}$$

is the completed lattice zeta function. Note that (6.7) agrees with the definition given in equation (1.9) of the introduction since $\mathrm{F}_{\chi}(s)=\mathrm{F}_{(p,q)}(2s)$ (see Proposition 4.5).

For $z\in\mathbf{H}$, recall that the number field theta function of weight p (see Definition 5.5) is defined as

$$\theta^p_{\mathfrak{n}}(a,b;z)=\sum_{n\in\mathfrak{n}}\mathbf{N}((a+n)^p)\cdot e^{\pi\,\mathrm{i}\langle|a+n|^2\,z,1\rangle_c+2\pi i\langle b,a+n\rangle_c} \tag{6.8}$$

where z, a and b are as before. We let

$$c^p_{\mathfrak{n}}(a,b) := \lim_{\substack{z\to\mathrm{i}\,\infty\cdot\mathbb{1}_g\\ z\in\mathbf{H}}}\theta^p_{\mathfrak{n}}(a,b;z) \tag{6.9}$$

be the "constant term c_0" (i.e. independent of z) of $[z\mapsto\theta^p_{\mathfrak{n}}(a,b;z)]$.

Lemma 6.1. *We have $c^p_{\mathfrak{n}}(a,b)=0$ unless the following two conditions are verified:*

(i) for all $\tau\in\Sigma$, $p_\tau=0$,

and

(ii) $a\in\mathfrak{n}$.

When (i) and (ii) hold true then $c^p_{\mathfrak{n}}(a,b)=1$.

Proof The constant term c_0 is potentially non-trivial only if $-a\in\mathfrak{n}$. If this happens then it is given by $c_0=\mathbf{N}(\mathbb{0}^p)$ which vanishes unless $p=\mathbb{0}_g=\mathbb{0}_{\mathbf{R}_\pm}$. □

Remark 6.2. In particular, note that the indicator function $c_{\mathfrak{n}}(a,b)$ does not depend on b but we find it more convenient to leave the argument b within the notation.

The next lemma gives an exact relation between the coset theta function $\theta^p_{\mathfrak{R}}$ and the number field theta function $\theta^p_{\mathfrak{n}}$.

Lemma 6.3. *For $x\in\mathbf{S}$ and $t\in\mathbb{R}_+$ we have*

(6.10)
$$\mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)})\cdot\left(\theta^p_{\mathfrak{n}}(a,b;\mathrm{i}\,xt^{1/g}/d^{1/g}_{\mathfrak{n}})-c^p_{\mathfrak{n}}(a,b)\right)=\sum_{\epsilon\in\mathcal{V}}\mathbf{N}\left((|\epsilon|^2x)^{\frac{1}{2}(p-\mathrm{i}\,q)}\right)\cdot\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,|\epsilon^2|\cdot xt^{1/g}/d^{1/g}_{\mathfrak{n}}).$$

Proof For $\chi(\epsilon)=1$, since $\epsilon\in\mathcal{V}$, it follows from (4.4) that

$$\mathbf{N}(|\epsilon|^{p-\mathrm{i}\,q})=\mathbf{N}(\epsilon^p). \tag{6.11}$$

To lighten the notation we put $\xi = xt^{1/g}/d_{\mathfrak{n}}^{1/g}$. Using (6.11) and (1.15) we find that

$$\begin{aligned}&\sum_{\epsilon\in\mathcal{V}} \mathbf{N}\left((|\epsilon|^2 x)^{\frac{1}{2}(p-\mathrm{i}\,q)}\right)\cdot \theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,|\epsilon^2|\xi)\\ &= \mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)}) \sum_{\epsilon\in\mathcal{V}_{\mathfrak{n};a,b}} \sum_{\substack{n\in\mathfrak{R}\\ a+n\neq 0}} \mathbf{N}((\epsilon(a+n))^p)\cdot e^{\pi\,\mathrm{i}\langle|\epsilon(a+n)|^2\,\xi,1\rangle_c+2\pi i\langle b,\epsilon(a+n)\rangle_c}\\ &= \mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)})\cdot(\theta^p_{\mathfrak{n}}(a,b;\mathrm{i}\,\xi)-c^p_{\mathfrak{n}}(a,b))\,.\end{aligned}$$

This proves the lemma. □

Proposition 6.4. *The completed Zeta function $\widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s)$ can be obtained as the following one dimensional Mellin transform:*

$$\widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s) = |d_K|^{-\frac{\lambda_0}{2}}\cdot(\mathbf{N}\,\mathfrak{n})^{-\lambda_0}\cdot\int_0^\infty (f_{\mathfrak{T}}(t)-f_{\mathfrak{T}}(\infty))\cdot t^{s+\frac{\lambda_0}{2}}\cdot\frac{dt}{t},\quad \mathrm{Re}(s)>1, \tag{6.12}$$

where

$$f_{\mathfrak{T}}(t) := \int_{\mathcal{F}} \theta^p_{\mathfrak{n}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g})\cdot\mathbf{N}\left(x^{\frac{1}{2}(p-\mathrm{i}\,q)}\right)d^*x \tag{6.13}$$

and

$$f_{\mathfrak{T}}(\infty) = \lim_{t\to\infty} f_{\mathfrak{T}}(t) = c^p_{\mathfrak{n}}(a,b)\cdot\int_{\mathcal{F}}\mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)})\,d^*x. \tag{6.14}$$

Here $\mathbf{S}$ *is the unit sphere in* $\mathbf{R}_+$ *and* d^*x *is the normalized multiplicative Haar measure on* $\mathbf{S}$*, see Section 2.3. Note that the map* $|\_| : \mathcal{V}\to|\mathcal{V}|$ *given by* $\epsilon\mapsto|\epsilon|$ *is an injection since* $\mathcal{V}\cap\mu_K=\{1\}$*. Moreover* $|\mathcal{V}|\le\mathbf{S}$ *acts freely on* $\mathbf{S}$ *and therefore, a fortiori,* $|\mathcal{V}^2|$ *acts freely on* $\mathbf{S}$*. Here* $\mathcal{V}^2$ *is the group of the* **square** *of the units of* $\mathcal{V}$*. Finally, we let* $\mathcal{F}\subseteq\mathbf{S}$ *be a fundamental domain for the action of* $|\mathcal{V}^2|$ *on* $\mathbf{S}$ *so that*

$$\mathbf{S} = \bigcup_{\epsilon\in\mathcal{V}} |\epsilon^2|\cdot\mathcal{F}. \tag{6.15}$$

Remark 6.5. It follows from Lemma 6.1 that $f_{\mathfrak{T}}(\infty)=0$ unless $p=\mathbb{0}$ and $a\in\mathfrak{n}$. Moreover, when $p=\mathbb{0}_g$ and $a\in\mathfrak{n}$ it follows that

$$f_{\mathfrak{T}}(\infty) = f_{\mathfrak{T}^*}(\infty) = \overline{f_{\mathfrak{T}}(\infty)}. \tag{6.16}$$

The first equality in (6.16) follows from the observation that $\mathcal{F}^{-1}$ is again a fundamental domain for the action of $|\mathcal{V}^2|$ on $\mathbf{S}$ and that d^*x is invariant under the involution $x\mapsto x^{-1}$. The second equality follows from the continuity of the complex conjugation. Finally, note that if $(p,q)=(\mathbb{0}_g,\mathbb{0}_g)$ and $a\in\mathfrak{n}$ then using Proposition 2.15 we find that

$$f_{\mathfrak{T}}(\infty) = e\cdot 2^{r_1+r_2-1}\cdot R_K \tag{6.17}$$

where $e := [\langle(\mathcal{O}_K^\times)^2,\mu_K\rangle : \langle\mathcal{V}^2,\mu_K\rangle]$. In particular, in this special case we always have $f_{\mathfrak{T}}(\infty)\neq 0$.

Proof of Proposition 6.4 Recall the internal direct sum decomposition of $\mathbf{R}_+$ (see Section 2.3)

$$\mathbf{R}_+ = \mathbf{S} \oplus \widetilde{\mathbb{L}}, \tag{6.18}$$

where $\mathbb{R}_+ \simeq \widetilde{\mathbb{L}}$ via the map $t \mapsto \mathbb{1}_{r_1+r_2} \cdot t^{1/g} = (t^{1/g}, \dots, t^{1/g})$. It follows from (6.18) that each $y \in \mathbf{R}_+$ can be uniquely written in the form

$$y = x \cdot t^{1/g}, \quad \text{where } x = \frac{y}{\mathbf{N}(y)^{1/g}} \in \mathbf{S} \quad \text{and} \quad t = \mathbf{N}(y) \in \mathbb{R}_+. \tag{6.19}$$

Recall that d^*x is the unique Haar measure on the multiplicative group $\mathbf{S}$ such that the canonical Haar measure d^*y on $\mathbf{R}_+$ becomes the product measure $d^*y = d^*x \times \frac{dt}{t}$ on $\mathbf{S} \times \mathbb{R}_+$. It follows from (6.6) that

$$\begin{aligned} |d_K|^{\frac{\lambda_0}{2}} \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s) &= \int_{\mathbf{R}_+} \left(\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,y/d_{\mathfrak{n}}^{1/g})\right) \mathbf{N}(y^{\widehat{\mathfrak{s}}}) d^*y \\ &= \int_0^\infty \int_{\mathbf{S}} \left(\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g})\right) \mathbf{N}((xt^{1/g})^{\widehat{\mathfrak{s}}}) d^*x \frac{dt}{t} \\ &= \int_0^\infty \int_{\mathbf{S}} \left(\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g})\right) \mathbf{N}(x^{\widehat{\mathfrak{s}}}) d^*x \; t^{s+\frac{\lambda_0}{2}} \frac{dt}{t} \end{aligned} \tag{6.20}$$

where for the last equality we have used the fact that $\mathbf{N}((xt^{1/g})^{\widehat{\mathfrak{s}}}) = \mathbf{N}(x^{\widehat{\mathfrak{s}}}) \cdot t^{s+\frac{\lambda_0}{2}}$. Now using the fact that d^*x is invariant under $x \mapsto |\epsilon^2|x$, since $\mathbf{N}(|\epsilon|) = 1$ for $\epsilon \in \mathcal{V}$, and that $\mathbf{N}(x) = 1$ for $x \in \mathbf{S}$, we find that

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{S}} \left(\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g})\right) \mathbf{N}(x^{\widehat{\mathfrak{s}}}) d^*x \; t^{s+\frac{\lambda_0}{2}} \frac{dt}{t} \\ &= \int_0^\infty \int_{\mathcal{F}} \sum_{\epsilon \in \mathcal{V}} \theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,|\epsilon^2|xt^{1/g}/d_{\mathfrak{n}}^{1/g})\, \mathbf{N}\left((|\epsilon|^2 x)^{\frac{1}{2}(p-\mathrm{i}\,q)}\right) d^*x \; t^{s+\frac{\lambda_0}{2}} \frac{dt}{t}. \end{aligned} \tag{6.21}$$

where $\mathcal{F}$ is the fundamental domain for the action of $|\mathcal{V}^2|$ on $\mathbf{S}$ as in (6.15). Now, applying Lemma 6.3 to the right hand side of (6.21) we find

$$\begin{aligned} &\int_0^\infty \int_{\mathbf{S}} \left(\theta^p_{\mathfrak{R}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g})\right) \mathbf{N}(x^{\widehat{\mathfrak{s}}}) d^*x \; t^{s+\frac{\lambda_0}{2}} \frac{dt}{t} \\ &= \int_0^\infty \int_{\mathcal{F}} \mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)}) \cdot \left(\theta^p_{\mathfrak{n}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g}) - c^p_{\mathfrak{n}}(a,b)\right) d^*x \, t^{s+\frac{\lambda_0}{2}} \frac{dt}{t}. \end{aligned} \tag{6.22}$$

Stacking together (6.20) and (6.22) gives

$$\begin{aligned} |d_K|^{\frac{\lambda_0}{2}} \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s) &= \int_0^\infty \int_{\mathcal{F}} \mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\,q)}) \cdot \left(\theta^p_{\mathfrak{n}}(a,b;\mathrm{i}\,xt^{1/g}/d_{\mathfrak{n}}^{1/g}) - c^p_{\mathfrak{n}}(a,b)\right) d^*x \, t^{s+\frac{\lambda_0}{2}} \frac{dt}{t} \\ &= \int_0^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) t^{s+\frac{\lambda_0}{2}} \frac{dt}{t}, \end{aligned} \tag{6.23}$$

where the last equality follows from the definition of $f_{\mathfrak{T}}(t)$ and of $f_{\mathfrak{T}}(\infty)$. This proves (6.12) □

Proposition 6.6. *The function* $f_{\mathfrak{T}}(t)$ *(for* $t \in \mathbb{R}_+$*) satisfies the following functional equation*

$$f_{\mathfrak{T}}\left(\frac{1}{t}\right) = (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i \langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot t^{\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}} \cdot f_{\mathfrak{T}^*}(t), \tag{6.24}$$

and moreover

$$f_{\mathfrak{T}^{**}}(t) = (-1)^{\mathrm{Tr}(p)} \cdot f_{\mathfrak{T}}(t). \tag{6.25}$$

Recall that the index $\mathfrak{T}^*$ *on the right hand side of the above equality is equal to* $\mathfrak{T}^* = (\mathfrak{n}^*, (-b, a), (\overline{\chi}, \mathcal{V}))$*. Moreover,*

$$f_{\mathfrak{T}}(t) = f_{\mathfrak{T}}(\infty) + O(e^{-\alpha t^{1/g}}) \ \ \text{and} \ \ f_{\mathfrak{T}^*}(t) = f_{\mathfrak{T}^*}(\infty) + O(e^{-\beta t^{1/g}}) \text{ for } t \to \infty, \tag{6.26}$$

where $\alpha, \beta > 0$ *are suitable constants*

Proof The proof of (6.25) follows directly from (5.15). The proof of (6.26) is exactly the same as the one of Proposition 5.8 of [26] so there is no need to reproduce it here. So it remains to prove (1.11). Using the invariance of d^*x under $x \mapsto x^{-1}$ we find

$$\begin{aligned} f_{\mathfrak{T}}(1/t) &= \int_{\mathcal{F}} \theta^p_{\mathfrak{n}}(a, b, \chi; \mathrm{i}\, x t^{-1/g}/d_{\mathfrak{n}}^{1/g}) \cdot \mathbf{N}(x^{\frac{1}{2}(p-\mathrm{i}\, q)})d^*x \\ &= \int_{\mathcal{F}^{-1}} \theta^p_{\mathfrak{n}}\left(a, b, \chi; \frac{-1}{(\mathrm{i}\, x(d_{\mathfrak{n}}t)^{1/g})}\right) \cdot \mathbf{N}(x^{-\frac{1}{2}(p-\mathrm{i}\, q)})d^*x. \end{aligned} \tag{6.27}$$

Now if we use the functional equation (5.14) won $z = \mathrm{i}\, x(d_{\mathfrak{n}}t)^{1/g} \in \mathbf{H}$ in the right hand side of (6.27) we get

$$\begin{aligned} &\int_{\mathcal{F}^{-1}} \theta^p_{\mathfrak{n}}\left(a, b, \chi; \frac{-1}{(\mathrm{i}\, x(d_{\mathfrak{n}}t)^{1/g})}\right) \cdot \mathbf{N}(x^{-\frac{1}{2}(p-\mathrm{i}\, q)})d^*x \\ &= (-\mathrm{i})^{\mathrm{Tr}(p)} e^{2\pi\,\mathrm{i}\langle a,b\rangle_c} \cdot (d_{\mathfrak{n}})^{-1/2} \int_{\mathcal{F}^{-1}} \mathbf{N}(x^{-\frac{1}{2}(p-\mathrm{i}\, q)}) \cdot \mathbf{N}\left((x(d_{\mathfrak{n}}t)^{1/g})^{p+\frac{1}{2}\mathbb{1}}\right) \cdot \theta^p_{\mathfrak{n}^*}(-b, a; \mathrm{i}\, x(d_{\mathfrak{n}}t)^{1/g}))d^*x \\ &= (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi\,\mathrm{i}\langle a,b\rangle_c}(d_{\mathfrak{n}})^{-1/2} \cdot d_{\mathfrak{n}}^{\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}} \int_{\mathcal{F}^{-1}} \mathbf{N}(x^{\frac{1}{2}(c_1(p)+\mathrm{i}\, q)}) \cdot t^{\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}}\Big) \cdot \theta^p_{\mathfrak{n}^*}(-b, a; \mathrm{i}\, x t^{1/g}/d_{\mathfrak{n}^*}^{1/g}))d^*x \\ &= (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi\,\mathrm{i}\langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot t^{\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}} \cdot f_{\mathfrak{T}^*}(t). \end{aligned} \tag{6.28}$$

For the second equality we have used the fact that for $x \in \mathbf{S}$, $\mathbf{N}(x) = 1$ and $\mathbf{N}(x^p) = \mathbf{N}(x^{c_1(p)})$, and that $d_{\mathfrak{n}}^{1/g} = (d_{\mathfrak{n}^*})^{-1/g}$ (for the last equality see Lemma 2.12). Stacking together (6.27) and (6.28) proves the result. □

6.3 Proof of Theorem 1.4

We are now ready to prove Theorem 1.4. Assume that $s \in \Pi_{\frac{3}{2}+\frac{3}{2}\frac{\mathrm{Tr}(p)}{g}}$. Starting with Proposition 6.4 we find

$$|d_K|^{\frac{\lambda_0}{2}} \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s) = \int_0^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) \cdot t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t}$$
$$= \int_0^1 (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) \cdot t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t} + \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) t^{s+\frac{\lambda_0}{2}} \frac{dt}{t}$$
$$= \int_1^\infty f_{\mathfrak{T}}(1/t) - f_{\mathfrak{T}}(\infty)) \cdot t^{-s-\frac{\lambda_0}{2}} \cdot \frac{dt}{t} + \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) \cdot t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t}$$
$$(6.29) \qquad = \int_1^\infty f_{\mathfrak{T}}(1/t) \cdot t^{-s-\frac{\lambda_0}{2}} \cdot \frac{dt}{t} + \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) \cdot t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t} - \frac{f_{\mathfrak{T}}(\infty)}{s - \frac{\lambda_0}{2}}$$

Substituting (6.24) in the first integral of (6.29) we get

$$|d_K|^{\frac{\lambda_0}{2}} \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s)$$
$$= (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i \langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \int_1^\infty f_{\mathfrak{T}^*}(t) \cdot t^{-s-\frac{\lambda_0}{2}+\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{dt}{t}$$
$$+ \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t} - \frac{f_{\mathfrak{T}}(\infty)}{s - \frac{\lambda_0}{2}}$$
(6.30)
$$= (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i \langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \int_1^\infty (f_{\mathfrak{T}^*}(t) - f_{\mathfrak{T}^*}(\infty)) \cdot t^{-s-\frac{\lambda_0}{2}+\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{dt}{t}$$
$$+ \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) t^{s+\frac{\lambda_0}{2}} \cdot \frac{dt}{t} - \frac{f_{\mathfrak{T}}(\infty)}{s + \frac{\lambda_0}{2}} - (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i \langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{f_{\mathfrak{T}^*}(\infty)}{-s - \frac{\lambda_0}{2} + \frac{1}{2} + \frac{\mathrm{Tr}(p)}{g}}.$$

The two integrals of the last equality converge for all complex number $s \in \mathbb{C}$. It follows from this that $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s)]$ admits an analytic continuation to all $s \in \mathbb{C}\backslash\{\frac{-\lambda_0}{2}, -\frac{\lambda_0}{2}+\frac{1}{2}+\frac{\mathrm{Tr}(p)}{g}\}$. Recall that λ_0 was defined in (6.2). In particualr, if $p \neq \mathbb{0}$ then $f_{\mathfrak{T}}(\infty) = f_{\mathfrak{T}^*}(\infty) = 0$ and therefore $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s)]$ is analytic on $\mathbb{C}$. When $p = \mathbb{0}$ note that $-\lambda_0 + 1 + 2\frac{\mathrm{Tr}(p)}{g} = -\lambda_0 + 1$ and therefore the set of possible poles for $[s \mapsto \widehat{Z}_{\mathfrak{n}}(a,b,\chi;s)]$ agrees with Theorem 1.4. It remains to prove the functional equation (1.11).

Recall that $(p_{\overline{\chi}}, q_{\overline{\chi}}) = (c_1(p), -q)$. Set

$$\lambda_0^* := \frac{\mathrm{Tr}(c_1(p) + \mathrm{i}\,q)}{g} = \frac{\mathrm{Tr}(p + \mathrm{i}\,q)}{g} = \overline{\lambda_0}.$$

Replacing s by $\frac{1}{2} - s$ and $\mathfrak{T}$ by $\mathfrak{T}^*$ in (6.30) and using the facts that $\mathrm{Tr}(p) = \mathrm{Tr}(c_1(p))$ and that

$$(6.31) \qquad f_{\mathfrak{T}^{**}}(t) = (-1)^{\mathrm{Tr}(p)} \cdot f(t)$$

we find

$$|d_K|^{\frac{\lambda_0^*}{2}} \cdot (\mathbf{N}\,\mathfrak{n}^*)^{\lambda_0^*} \cdot \widehat{Z}_{\mathfrak{n}^*}(-b,a,\overline{\chi};1-2s)$$

(6.32)

$$= (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i\langle -b,a\rangle_c} \cdot d_{\mathfrak{n}^*}^{\frac{\mathrm{Tr}(p)}{g}} \cdot (-1)^{\mathrm{Tr}(p)} \cdot \int_1^\infty (f_{\mathfrak{T}}(t) - f_{\mathfrak{T}}(\infty)) \cdot t^{\frac{1}{2}-s+\frac{\lambda_0^*}{2}+\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{dt}{t}$$

$$+ \int_1^\infty (f_{\mathfrak{T}^*}(t) - f_{\mathfrak{T}^*}(\infty))t^{\frac{1}{2}-s-\frac{\lambda_0^*}{2}} \cdot \frac{dt}{t} - \frac{f_{\mathfrak{T}^*}(\infty)}{\frac{1}{2}-s+\frac{\lambda_0^*}{2}} - (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i\langle -b,a\rangle_c} \cdot (d_{\mathfrak{n}^*})^{\frac{\mathrm{Tr}(p)}{g}} (-1)^{\mathrm{Tr}(p)} \frac{f_{\mathfrak{T}}(\infty)}{s-\frac{\lambda_0^*}{2}+\frac{\mathrm{Tr}(p)}{g}}.$$

If $p \neq 0$ then $f_{\mathfrak{T}}(\infty) = f_{\mathfrak{T}^*}(\infty) = 0$ and when $p = 0$, $-\lambda_0 = \lambda_0^*$. Taking the last observation into and comparing (6.30) with (6.32) with the fact that $d_{\mathfrak{n}} \cdot d_{\mathfrak{n}^*} = 1$ gives

(6.33)

$$(-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi\,\mathrm{i}\langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot |d_K|^{\frac{\lambda_0^*}{2}} \cdot (\mathbf{N}\,\mathfrak{n}^*)^{\lambda_0^*} \cdot \widehat{Z}_{\mathfrak{n}^*}(-b,a,\overline{\chi},1-2s) = |d_K|^{\frac{\lambda_0}{2}} \cdot (\mathbf{N}\,\mathfrak{n})^{\lambda_0} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s).$$

Now using the fact that $d_{\mathfrak{n}} = |d_K|(\mathbf{N}\,\mathfrak{n})^2$ (and similarly for $d_{\mathfrak{n}^*}$) we can rewrite (6.33) as

$$(-1)^{\mathrm{Tr}(p)} \cdot e^{2\pi\,\mathrm{i}\langle a,b\rangle_c} \cdot d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{d_{\mathfrak{n}^*}^{\frac{\lambda_0^*}{2}}}{d_{\mathfrak{n}}^{\frac{\lambda_0}{2}}} \cdot \widehat{Z}_{\mathfrak{n}^*}(-b,a,\overline{\chi},1-2s) = \widehat{Z}_{\mathfrak{n}}(a,b,\chi;2s). \tag{6.34}$$

Finally, note that

$$d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g}} \cdot \frac{d_{\mathfrak{n}^*}^{\frac{\lambda_0^*}{2}}}{d_{\mathfrak{n}}^{\frac{\lambda_0}{2}}} = d_{\mathfrak{n}}^{\frac{\mathrm{Tr}(p)}{g} - \frac{\mathrm{Tr}(p-\mathrm{i}\,q)}{2g} - \frac{\mathrm{Tr}(p+\mathrm{i}\,q)}{2g}} = 1, \tag{6.35}$$

which, when used in (6.34), implies that

$$(-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi i\langle a,b\rangle_c} \cdot \widehat{Z}_{\mathfrak{n}^*}(-b,a,\chi;2s) = \widehat{Z}_{\mathfrak{n}}(a,b,\chi;1-2s),$$

which is in turn equivalent to (1.11) and therefore concludes the proof. □

7 Basic properties of lattice zeta functions

7.1 Symmetries of $\mathcal{LZ}$

We record in this section some basic symmetries which are satisfied by lattice zeta functions ($\mathcal{LZ}$). Consider an admissible triple $\mathfrak{T} = (\mathfrak{n},(a,b),(\chi,\mathcal{V}))$ as in the introduction so that $\mathcal{V} \leq \mathcal{V}_{\mathfrak{n};a,b}$. For the definition of $\mathcal{V}_{\mathfrak{n};a,b}$ see Definition 1.10. In particular $\chi \in \mathfrak{X}_{\mathcal{V}}^u$ (see Definition 3.12). We shall view χ as a character of $K_{\mathbb{R}}^\times/\mathcal{V}$ and therefore drop the argument $\mathcal{V}$ in $\mathfrak{T}$. Recall that the dual triple to $\mathfrak{T}$ is defined as $\mathfrak{T}^* := (\mathfrak{n}^*,(-b,a),\overline{\chi})$.

The uncompleted $\mathcal{LZ}$ function associated to $\mathfrak{T}$ is given by

$$Z_{\mathfrak{T}}(s) = Z_{\mathfrak{n}}(a,b,\chi;s). \tag{7.1}$$

its dual $\mathcal{LZ}$ is given by

$$Z_{\mathfrak{T}^*}(s) = Z_{\mathfrak{n}^*}(-b, a, \overline{\chi}; s). \tag{7.2}$$

Note that the map $\mathfrak{T} \mapsto \mathfrak{T}^*$, on the set of admissible triples, is an automorphism of order four which reflects the fact that the Fourier transform is an automorphism of order 4 on the space of Schwartz functions.

It follows directly from Proposition 1.12 that

Proposition 7.1. *(i) If $a \equiv a' \pmod{\mathfrak{n}}$ and $b \equiv b' \pmod{\mathfrak{n}^*}$, then*

$$Z_{\mathfrak{n}}(a, b; \chi; s) = e^{2\pi \mathrm{i} \operatorname{Tr}((b-b')a)} \cdot Z_{\mathfrak{n}}(a', b'; \chi; s). \tag{7.3}$$

(ii) For all $\lambda \in K\backslash\{0\}$, one has that

$$Z_{\lambda\mathfrak{n}}\left(\lambda a, \frac{b}{\lambda}; \chi; s\right) = \chi(\lambda) \cdot Z_{\mathfrak{n}}(a, b; \chi; s). \tag{7.4}$$

In particular, if we apply the change of parameters $\mathfrak{T} \mapsto \mathfrak{T}^*$ in the functional equation (1.11) and use (7.4) with $\lambda = -1$ we find the relation

Corollary 7.2.

$$Z_{\mathfrak{n}}(-a, -b; \chi; s) = Z_{\mathfrak{T}^{**}}(s) = \chi(-1) \cdot Z_{\mathfrak{T}}(s) = \chi(-1) \cdot Z_{\mathfrak{n}}(a, b; \chi; s). \tag{7.5}$$

If we let $\mathfrak{Q}_\chi = (\overline{m}, n, \gamma, \delta, t)$ and $w_\chi = (p, q)$ note that

$$\chi(-1) = (-1)^{\operatorname{Tr}(p)} = (-1)^{\operatorname{Tr}(([\overline{m}], |n|))}.$$

Consider the group of units

$$\mathcal{V}^{\pm}_{\mathfrak{n};a,b} := \langle \mathcal{V}_{\mathfrak{n};a,b}, \pm 1 \rangle. \tag{7.6}$$

Let $\eta = \mu\epsilon \in \mathcal{V}^{\pm}_{\mathfrak{n};a,b}$ where $\epsilon \in \mathcal{V}_{\mathfrak{n};a,b}$ and $\mu \in \{\pm 1\}$. If we set $\lambda = \eta$ in (7.4) and use (7.5) and (7.3) we find that

$$Z_{\mathfrak{n}}(a, b, \chi, s) = e^{2\pi \mathrm{i} \operatorname{Tr}\left((b-\epsilon^{-1}b)a\right)} \cdot \chi(\mu) \cdot \chi(\epsilon) \cdot Z_{\mathfrak{n}}(a, b, \chi, s). \tag{7.7}$$

In particular, if there exists $\eta \in \mathcal{V}^{\pm}_{\mathfrak{n};a,b}$ such that

$$e^{2\pi \mathrm{i} \operatorname{Tr}\left((b-\epsilon^{-1}b)a\right)} \cdot \chi(\eta) \neq 1 \tag{7.8}$$

then $[s \mapsto Z_{\mathfrak{n}}(a, b, \chi, s)]$ must be identically equal to 0. We raise the following natural question:

Question 7.3. *If χ is an admissible character for the triple $(\mathfrak{n}, a, b)$ and if for all $\eta = \mu\epsilon \in \mathcal{V}^{\pm}_{\mathfrak{n},a,b}$ (7.8) is satisfied, do we necessarily have that $[s \mapsto Z_{\mathfrak{n}}(a, b, \chi, s)]$ is not identically equal to 0 ?*

Example 7.4. *Let us check this conjecture in a very special case in order to see what is at play. Let us assume that K is a Galois totally real with degree $[K:\mathbb{Q}]=g$ is even, $\mathfrak{n}$ is an integral $\mathcal{O}_K$-ideal such that for all $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$, $\mathfrak{n}^\sigma=\mathfrak{n}$. $a=b=0$ and $\overline{m}=\overline{\mathbb{1}_g}$. Note that if $[K:\mathbb{Q}]$ were odd then automatically $[s\mapsto Z_\mathfrak{n}(0,0;\omega_{\overline{\mathbb{1}}};s)]\equiv 0$ since $-1\in\mathcal{V}_{0,0,\mathfrak{n}}$. Then, under these very restrictive assumptions, we claim that if*

$$[s\mapsto Z_\mathfrak{n}(a,b;\omega_{\overline{m}};s)\equiv 0]\Longrightarrow \exists\epsilon\in\mathcal{O}_K^\times=\mathcal{V}_{0,0,\mathfrak{n}},\ \text{such that } \omega_{\overline{\mathbb{1}}}(\epsilon)=-1.$$

Let us prove it. From the Chebotarev's density theorem, there exists $\lambda\in\mathfrak{n}$, such that $|\mathbf{N}(\lambda\mathfrak{n}^{-1})|=p$ is a rational prime (in fact there are infinitely many such pairs (λ,p)). From the uniqueness of the writing of Dirichlet series, the vanishing of $[s\mapsto Z_\mathfrak{n}(0,0;\omega_{\overline{m}};s)]\equiv 0$ implies that there exists $\lambda'\in\mathfrak{n}$, such that $\omega_{\overline{\mathbb{1}}}(\lambda)|\mathbf{N}_{K/\mathbb{Q}}(\lambda)|=-\omega_{\overline{\mathbb{1}}}(\lambda')|\mathbf{N}_{K/\mathbb{Q}}(\lambda')|$. Note that $\mathfrak{p}=\lambda\mathfrak{n}^{-1}$ and $\mathfrak{p}'=\lambda'\mathfrak{n}^{-1}$ are prime ideals of K above p. Since $K/\mathbb{Q}$ is Galois, there must exists $\sigma\in\mathrm{Gal}(K/\mathbb{Q})$ such that $\lambda\mathfrak{n}^{-1}=\mathfrak{p}=\mathfrak{p}'^\sigma=\lambda'^\sigma(\mathfrak{n}^\sigma)^{-1}=\lambda'^\sigma\mathfrak{n}^{-1}$. Let $\epsilon=\frac{\lambda}{\lambda'^\sigma}$. Then $\epsilon\in\mathcal{V}_{0,0,\mathcal{O}_K}$ and $\omega_{\overline{\mathbb{1}}}(\epsilon)=\frac{\omega_{\overline{\mathbb{1}}}(\lambda)}{\omega_{\overline{\mathbb{1}}}(\lambda'^\sigma)}=\frac{\omega_{\overline{\mathbb{1}}}(\lambda)}{\omega_{\overline{\mathbb{1}}}(\lambda')}=-1$.

7.2 Regions of holomorphicity and order of vanishing at non-positive integers of the uncompleted $\mathcal{LZ}$

As in the previous section we consider an admissible triple $\mathfrak{T}=(\mathfrak{n},(a,b),(\chi,\mathcal{V}))$ and its associated (uncompleted) lattice zeta function $Z_\mathfrak{T}(s)$.

Recall that $\mathrm{F}_\chi(s)$ was the Euler factor defined in (1.8) of the introduction and that if $w_\chi=(p,q)$ then $\mathrm{F}_\chi=\mathrm{F}_{(p,q)}(s)$ (Proposition 4.5) where $\mathrm{F}_{(p,q)}(s)$ was given in (4.6).

Definition 7.5. *For $\chi\in\mathfrak{X}_\mathcal{V}^u$ with $w_\chi=(p,q)$ and $s\in\mathbb{C}$ we define the quotient Euler factor*

$$\lambda_\chi(s)=\lambda_{(p,q)}(s):=\frac{\mathrm{F}_\chi(s)}{\mathrm{F}_{\overline{\chi}}(1-s)}=\frac{\mathrm{F}_{(p,q)}(s)}{\mathrm{F}_{(c_1(p),-q)}(1-s)}=\frac{\widetilde{\mathrm{F}}_{(p,q)}(s)}{\widetilde{\mathrm{F}}_{(c_1(p),-q)}(1-s)}. \tag{7.9}$$

For the definition of $\widetilde{\mathrm{F}}_{(p,q)}(s)$ see (4.8).

It follows readily from the definition of $\lambda_\chi(s)$ that

$$\lambda_{\overline{\chi}}(1-s)=(\lambda_\chi(s))^{-1}. \tag{7.10}$$

Corollary 7.6. *Assume that $[s\mapsto Z_\mathfrak{n}(a,b,\chi;s)]$ is not identically equal to 0. It follows at once from Theorem 1.4 that*

$$\left[s\mapsto\frac{Z_\mathfrak{T}(s)}{Z_{\mathfrak{T}^*}(1-s)}=\frac{Z_\mathfrak{n}(a,b;\chi;s)}{Z_{\mathfrak{n}^*}(-b,a;\overline{\chi};1-s)}\right] \tag{7.11}$$

is a non-zero meromorphic function such that

$$\frac{Z_\mathfrak{T}(s)}{Z_{\mathfrak{T}^*}(1-s)}=(-\mathrm{i})^{\mathrm{Tr}(p)}\cdot e^{2\pi\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)}\cdot\lambda_{\overline{\chi}}(1-s). \tag{7.12}$$

In particular, it follows from (7.12) that the quotient on the left hand side of (7.12) is independent of the choice of $\mathfrak{n}$. Moreover, its dependence on the pair (a,b), is only a dependence given by the root of unity $e^{2\pi\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)}$.

The next proposition gives some analytic properties of the meromorphic functions which appear in the functional equation (7.12).

Proposition 7.7. *(i) The meromorphic function $s \mapsto \lambda_\chi(s)$ does not vanish on the left half-plane $-\Pi_0$.*

(ii) $[s \mapsto Z_{\mathfrak{T}}(s)]$ is holomorphic on the right half-plane Π_1.

(iii) $[s \mapsto \lambda_{\overline{\chi}}(s) \cdot Z_{\mathfrak{T}^*}(s)]$ is holomorphic on the left half-plane $-\Pi_0$.

(iv) $[s \mapsto Z_{\mathfrak{T}^*}(s)]$ is holomorphic on the left half-plane $-\Pi_0$.

Proof (i) follows the facts that $[s \mapsto \Gamma(s)]$ does not vanish on $\mathbb{C}$ and that its only poles (all of order one) are located at $s \in \mathbb{Z}_{\leq 0}$. (ii) follows from Proposition 1.16. (iii) follows from (ii) and the functional equation (7.12). Finally (iv) follows from (iii) and (i). □

The next Theorem gives some lower bound for the order of vanishing of the uncompleted zeta function $[s \mapsto Z_{\mathfrak{T}}(s)]$ at *non-positive integers* and also an exact relationship between a higher order derivative $Z_{\mathfrak{T}}^{(e_\ell)}(\ell)$ and the value $Z_{\mathfrak{T}^*}(1-\ell)$ for $\ell \in \mathbb{Z}_{\leq 0}$. Here e_ℓ, soon to be defined, is the generic order of vanishing of $[s \mapsto Z_{\mathfrak{T}}(s)]$ at $s = \ell$.

Theorem 7.8. *Recall that $w_\chi = (p, q)$. For $\ell \in \mathbb{Z}$, let*

(i) $r_\chi(\ell) := \#\{1 \leq k \leq r_1 : \ell + p_k - \mathrm{i}\, q_k \in \mathbb{Z}_{\leq 0}\}$

(ii) $s_\chi(\ell) = \#\{r_1 + 1 \leq k \leq r_1 + r_2 : 2\ell + p_k + p_{k'} - \mathrm{i}\, q_k - \mathrm{i}\, q_{k'} \in \mathbb{Z}_{\leq 0}\}$,

where $k' := k + r_2$ for $r_1 + 1 \leq k \leq r_1 + r_2$. For $\ell \in \mathbb{Z}_{\leq 0}$ let

$$e_\ell = e_{\ell,\chi} = r_\chi(\ell) + s_\chi(\ell). \tag{7.13}$$

We call e_ℓ the generic order of vanishing of $[s \mapsto Z_{\mathfrak{T}}(s)]$ at $s = \ell$. Then for $\ell \in \mathbb{Z}_{\leq -1}$, we have

$$\operatorname{ord}_{s=\ell} Z_{\mathfrak{T}}(s) \geq e_\ell \tag{7.14}$$

Furthermore, if we assume that

$$\operatorname{ord}_{s=1} Z_{\mathfrak{T}^*}(s) \geq 0, \tag{7.15}$$

(i.e. condition (a) of Theorem 1.4 is not fulfilled), then

$$\operatorname{ord}_{s=0} Z_{\mathfrak{T}}(s) \geq e_0. \tag{7.16}$$

Note that the integral value $\ell = 0$ lies precisely on the boundary of the open left half-plane $-\Pi_0$. Moreover for $\ell \in \mathbb{Z}_{\leq 0}$ one has that

$$\frac{C_\ell}{e_\ell!} \cdot Z_{\mathfrak{T}}^{(e_\ell)}(\ell) = (-\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi \mathrm{i}\, \mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot Z_{\mathfrak{T}^*}(1-\ell) \tag{7.17}$$

where $Z_{\mathfrak{T}}^{(e_\ell)}(\ell)$ corresponds to the e-th derivative of $Z_{\mathfrak{T}}(s)$ at $s = \ell$ and $C_\ell := \lim_{s \to \ell} s^{e_\ell} \lambda_\chi(s)$.

Proof Let us first prove (7.14). From (iii) of Proposition 7.7 we know that

$$[s \mapsto \lambda_\chi(s) \cdot Z_{\mathfrak{T}}(s)] \tag{7.18}$$

is holomorphic on $-\Pi_0$. Therefore, a pole of order e of $[s \mapsto \lambda_\chi(s)]$ at $\ell \in \mathbb{Z}_{\leq 1}$ forces $[s \mapsto Z_{\mathfrak{T}}(s)]$ to have at least a zero of order e at ℓ. Now recall that $\Gamma(s)$ does not vanish on Π_1 (in fact in does not vanish on all of $\mathbb{C}$) and that its only poles are of order one and located at the values $s \in \mathbb{Z}_{\leq 0}$. Applying this to the definition of $\lambda_\chi(s)$ with the fact that that $p \geq \mathbb{0}_g$, a direct calculation shows that for $\ell \in \mathbb{Z}_{\leq -1}$

$$\operatorname{ord}_{s=\ell} \lambda_\chi(s) = -e_\ell, \tag{7.19}$$

from which the result follows. Let us now prove (7.16). Using the assumption 7.15 it follows from (7.12) that

$$[s \mapsto \lambda_\chi(s) \cdot Z_{\mathfrak{T}}(s)] \tag{7.20}$$

is holomorphic at $s = 0$ and thus the previous reasoning applies as well which proves (7.16). Finally, the proof of (7.17) follows from the identity

$$(s^{e_\ell}\lambda_\chi(s)) \cdot (\frac{Z_{\mathfrak{T}}(s)}{s^{e_\ell}}) = (-\,\mathrm{i})^{\mathrm{Tr}(p)} \cdot e^{2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot Z_{\mathfrak{T}^*}(1-s) \tag{7.21}$$

for which one lets $s \to \ell$ combined with the relation $\frac{1}{e_\ell!} \cdot Z_{\mathfrak{T}}^{(e_\ell)}(\ell) = \lim_{s\to\ell} \frac{Z_{\mathfrak{T}}(s)}{s^{e_\ell}}$. □

Remark 7.9. When $[s \mapsto Z_{\mathfrak{T}}(s)]$ is not identically equal to 0 we expect generically the inequality (7.14) and (7.16) to be equalities. However, it is possible to give simple examples where $\operatorname{ord}_{s=\ell} Z_{\mathfrak{T}}(s) > e_\ell$. Let us give one such example for $K = \mathbb{Q}$ and $\ell = 0$. Consider the triple

$$\mathfrak{T} = \left(\mathbb{Z}, \left(\frac{a}{f}, 0\right), (\chi_0, \{1\})\right) \tag{7.22}$$

where χ_0 is the trivial character and $a, f \in \mathbb{Z}_{>0}$ are to be specified. We have $r_{\chi_0}(0) = 1$ and $s_{\chi_0}(0) = 0$ so that $e_0 = 1$ and therefore

$$\operatorname{ord}_{s=0} \zeta_{\mathfrak{T}}(s) \geq 1. \tag{7.23}$$

Using (7.17), it follows that

$$\zeta'_{\mathfrak{T}}(0) = \frac{1}{2}\zeta_{\mathfrak{T}^*}(1). \tag{7.24}$$

In particular it follows from (7.24) that $\operatorname{ord}_{s=0} \zeta_{\mathfrak{T}}(s) = 1 \iff \zeta_{\mathfrak{T}^*}(1) \neq 0$. We have

$$\zeta_{\mathfrak{T}^*}(s) = \sum_{n\in\mathbb{Z}} \frac{e^{-\frac{2\pi\,\mathrm{i}\,a}{f}}}{|n|^s}. \tag{7.25}$$

It is well known that for any $\theta \in \mathbb{R}\backslash\mathbb{Z}$ (see for example Section 6.5 of [10]) that

$$\sum_{k\geq 1} \frac{e^{2\pi\,\mathrm{i}\,k\theta}}{k} = -\log(1 - e^{2\pi\,\mathrm{i}\,\theta}) \tag{7.26}$$

where the branch of log is specified by $-\pi < \mathrm{Im}(\log z) \leq \pi$. Combining (7.24), (7.25) and (7.26) we find that

$$\zeta'_{\mathfrak{T}}(0) = -\frac{1}{2}\log 4\left(\sin(\frac{\pi a}{f})\right)^2. \tag{7.27}$$

In particular, if we choose $f = 6$ and $a = 1$ so that $1 = 4\left(\sin(\frac{\pi}{6})\right)^2$ we obtain from (7.27) that

$$\mathrm{ord}_{s=0}\,\zeta_{\mathfrak{T}}(s) \geq 2. \tag{7.28}$$

A Signature lattice zeta functions

Let $(K, {}^{\ell}\Sigma)$ be a framed number field of signature (r_1, r_2) with $g = r_1 + 2r_2$ where

$$\Sigma_K = \Big\{\tau_1 := \rho_1,\, \tau_2 := \rho_2,\, \ldots,\, \tau_{r_1} = \rho_{r_1},\\ \tau_{r_1+1} := \sigma_1,\, \tau_{r_1+2} := \sigma_2,\, \ldots, \tau_{r_1+r_2} := \sigma_{r_2}, \tau_{r_1+r_2+1} = \overline{\sigma}_1,\, \ldots,\, \tau_{r_1+2r_2} := \overline{\sigma}_{r_2}\Big\},$$

corresponds to a choice of an admissible labelling (see Section 2). In particular,

$$\Sigma_r = \{\tau_1 = \rho_1,\, \tau_2 = \rho_2,\, \ldots,\, \tau_{r_1} = \rho_{r_2}\} \tag{A.1}$$

is the set of real embeddings of K and

(A.2)
$$\Sigma'_c = \{\tau_{r_1+1} := \sigma_1,\, \tau_{r_1+2} := \sigma_2,\, \ldots, \tau_{r_1+r_2} := \sigma_{r_2}\} \text{ and } \Sigma'_c = \tau_{r_1+r_2+1} = \overline{\sigma}_1,\, \ldots,\, \tau_{r_1+2r_2} := \overline{\sigma}_{r_2}\}$$

are half sets of complex embeddings of K, so that $\Sigma = \Sigma_r \bigsqcup \Sigma'_c \bigsqcup \Sigma''_c$.

For $x \in K$ and $1 \leq i \leq g$ we let $x^{(i)} := \tau_i(x)$. The set of embeddings $\Sigma_r \cup \Sigma'_c$ give rise to an injection

$$\iota : K_{\mathbb{R}} = K \otimes_{\mathbb{Q}} \mathbb{R} \to V := \mathbb{R}^{r_1} \times \mathbb{C}^{r_2}\\ x \otimes \lambda \mapsto (\rho_1(x)\lambda, \ldots, \rho_{r_1}(x)\lambda; \sigma_1(x)\lambda, \ldots, \sigma_{r_2}(x)\lambda) \tag{A.3}$$

for which $\iota(\mathcal{O}_K)$ is a lattice of V. We let

$$\overline{\mathfrak{S}}_K = \overline{\mathfrak{S}} := (\mathbb{Z}/2\mathbb{Z})^{r_1} \tag{A.4}$$

and call it the *signature group*. We also let

$${}^*V := \{v \in V : \text{for all } 1 \leq i \leq r_1, v_i \neq 0\} \tag{A.5}$$

and consider the *signature map*

$$\mathfrak{sg} : {}^*V \to \overline{\mathfrak{S}} \tag{A.6}$$

where $\mathfrak{sg}(x) = \overline{m} = (\overline{m}_1, \ldots, \overline{m}_{r_1}) \in \overline{\mathfrak{S}}$ where for $1 \le i \le r_1$, $\overline{m_i} = \overline{0}$ if $x^{(i)} > 0$ and $\overline{m_i} = \overline{1}$ if $x^{(i)} < 0$. Note that $K^\times$ is a subset of ${}^*V$ via ι. Using the $\mathfrak{sg}$-map, for each $\overline{m} \in \overline{\mathfrak{S}}$, it makes sense to introduce the "orthant" of signature $\overline{m} = (\overline{m}_1, \ldots, \overline{m}_{r_1}) \in \overline{\mathfrak{S}}$ which we define as

$$V_{\overline{m}} := \{x \in {}^*V : \mathfrak{sg}(x) = \overline{m}\}. \tag{A.7}$$

In this manner the space ${}^*V$ is decomposed into 2^{r_1} disjoint orthants.

Remark A.1. The group $\overline{\mathfrak{S}}$ is the additive version of the signature group $\mathbb{S}$ that was introduced in (3.8). In the setting of "GL_2-lattice Eisenstein series", it is convenient to introduce what we call the "complex signature group" $\mathfrak{S}_K = \mathfrak{S} := \mathbb{Z}^{r_1}$ for which one has a projection map $\pi : \mathfrak{S} \to \overline{\mathfrak{S}}$. In fact for each $m \in \mathfrak{S}$ which lifts a signature $\overline{m} \in \overline{\mathfrak{S}}$ one can construct a "signature lattice Eisenstein series" for which its 0-Fourier coefficients interpolates the signature lattice zeta function of weight $\omega_{\overline{m}}$. For more on this circle of ideas see [4].

Let

$$\widehat{\overline{\mathfrak{S}}} := \mathrm{Hom}(\overline{\mathfrak{S}}, \mathbb{C}^\times) = \mathrm{Hom}(\overline{\mathfrak{S}}, \{\pm 1\}) \tag{A.8}$$

be the group of characters of $\overline{\mathfrak{S}}$ and label them as $\omega_{\overline{m}}$, for $\overline{m} \in \overline{\mathfrak{S}}$, where

$$\begin{aligned} \omega_{\overline{m}} : K_{\mathbb{R}}^\times &\to \{\pm 1\} \\ x \otimes \lambda &\mapsto \prod_{i=1}^{r_1} (\mathrm{sign}(x^{(i)}\lambda))^{[\overline{m}_i]} \end{aligned} \tag{A.9}$$

where $[\overline{m}_i] = 0$ if $\overline{m}_i = \overline{0}$ and $[\overline{m}_i] = 1$ if $\overline{m}_i = \overline{1}$. We call an element of $\widehat{\overline{\mathfrak{S}}}$ a *sign character of K*.

We wish now to state Theorem 1.4 and Theorem 7.8 in the special case when $\chi = \omega_{\overline{m}}$ is a sign character of K. If we let $w_{\omega_{\overline{m}}} = (p, q)$ be the w-weight (see Definition 4.1) then note that $q = \mathbb{0}_g$ and $p = ([\overline{m}]; \mathbb{0}_{2r_2})$. Let

$$\mathfrak{T} = (\mathfrak{n}, (a, b), (\omega_{\overline{m}}, \mathcal{V})) \tag{A.10}$$

where $\mathfrak{n} \subseteq K$ is a lattice, $a, b \in K$ and $\mathcal{V} \le \mathcal{V}_{\mathfrak{n};a,b}$ with $\mathcal{V} \cap \mu_K = \{1\}$. See Definition 1.10 for the meaning of $\mathcal{V}_{\mathfrak{n};a,b}$. We define the "Euler factor at ∞" associated to the uncompleted $Z_{\mathfrak{n}}(a, b; \omega_{\overline{m}}; s)$ as

$$F_{\overline{m}}(s) := |d_K|^{s/2} \cdot \pi^{-\frac{gs}{2}} \cdot 2^{r_2} \cdot \Gamma(s)^{r_2} \cdot \prod_{i=1}^{r_1} \Gamma\left(\frac{s + [\overline{m}_i]}{2}\right),$$

where $\Gamma(x)$ stands for the usual gamma function evaluated at x. Recall here that $[\overline{m}_i] = 0 \in \mathbb{Z}$ if $\overline{m}_i = \overline{0}$, and $[\overline{m}_i] = 1 \in \mathbb{Z}$ if $\overline{m}_i = 1$. Note that the Euler factor at infinity does not depend on the lattice $\mathfrak{n}$.

Remark A.2. Note that $F_{\overline{m}}(s) = \widetilde{\mathrm{F}}_{(p,q)}(s) = \pi^{\frac{\mathrm{Tr}([\overline{m}])}{2}} \cdot \mathrm{F}_{(p,q)}(s)$. For the definitions of $\mathrm{F}_{(p,q)}(s)$ and $\mathrm{F}_{\omega_{\overline{m}}}(s)$ see Section 4.2.

We may now state the main theorem that was proved in [2] and which can be viewed as a special case of Theorem 1.4.

Theorem A.3. *Let*

$$\widehat{Z}_{\mathfrak{n}}(a,b,\omega_{\overline{m}};s) := F_{\overline{m}}(s) \cdot Z_{\mathfrak{n}}(a,b;\omega_{\overline{m}};s) \tag{A.11}$$

be the completed zeta function of $Z_{\mathfrak{n}}(a,b;\omega_{\overline{m}};s)$. Firstly, $\widehat{Z}_{\mathfrak{n}}(a,b,\omega_{\overline{m}};s)$ admits an analytic continuation to $\mathbb{C}\backslash\{0,1\}$ and has at most a pole of order one at $s \in \{0,1\}$. Secondly, $\widehat{Z}_{\mathfrak{n}}(a,b,\omega_{\overline{m}};s)$ satisfies the following functional equation:

$$(\mathrm{i})^{\mathrm{Tr}([\overline{m}])} \cdot e^{-2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot \widehat{Z}_{\mathfrak{n}}(a,b,\omega_{\overline{m}};s) = \widehat{Z}_{\mathfrak{n}^*}(-b,a,\omega_{\overline{m}},1-s). \tag{A.12}$$

Thirdly, the function $s \mapsto \widehat{Z}_{\mathfrak{n}}(a,b,\omega_{\overline{m}};s)$ admits

(a) a pole of order one at $s=1$, if and only if, $\overline{m}_i = \overline{0}$ for all i and $-b \in \mathfrak{n}^*$.

(b) a pole of order one at $s=0$, if and only if, $\overline{m}_i = \overline{0}$ for all i and $a \in \mathfrak{n}$,

The next Theorem gives some lower bound for the order of vanishing of the uncompleted zeta function $[s \mapsto Z_{\mathfrak{n}}(a,b;\omega_{\overline{m}};s)]$ at non-positive integers and also an exact relationship between a higher order derivative $Z_{\mathfrak{n}}^{(e_\ell)}(a,b;\omega_{\overline{m}};\ell)$ and the value $Z_{\mathfrak{n}^*}(-b,a;\omega_{\overline{m}};1-\ell)$ at $\ell \in \mathbb{Z}_{\leq 0}$.

Theorem A.4. *Define*

(a) $r_{\overline{m}}^0 := \#\{1 \leq j \leq r_1 : \overline{m}_j = \overline{0}\}$.

(b) $r_{\overline{m}}^1 := \#\{1 \leq j \leq r_1 : \overline{m}_j = \overline{1}\}$.

Note that $r_{\overline{m}}^0 + r_{\overline{m}}^1 = r_1$. For $\ell \in \mathbb{Z}_{\leq 0}$ let

$$e_\ell := \begin{cases} r_{\overline{m}}^0 & \text{if} \quad \ell \equiv 0 \pmod 2 \\ r_{\overline{m}}^1 & \text{if} \quad \ell \equiv 1 \pmod 2 \end{cases} \tag{A.13}$$

and note that for any $\ell \in \mathbb{Z}_{\leq 0}$

$$\mathrm{ord}_{s=\ell} \frac{F_{\overline{m}}(s)}{F_{\overline{m}}(1-s)} = -e_\ell. \tag{A.14}$$

We call e_ℓ the generic order of vanishing of $[s \mapsto Z_{\mathfrak{n}}(a,b;\omega_{\overline{m}};s)]$. Then for $\ell \in \mathbb{Z}_{\leq -1}$, we have

$$\mathrm{ord}_{s=\ell} Z_{\mathfrak{n}}(a,b;\omega_{\overline{m}};s) \geq e_\ell. \tag{A.15}$$

Furthermore, if we assume that

$$\mathrm{ord}_{s=1} Z_{\mathfrak{n}^*}(-b, a; \omega_{\overline{m}}; s) \geq 0,$$

(i.e. condition (a) of Theorem A.3 is nots fulfilled), then

$$\mathrm{ord}_{s=0} Z_{\mathfrak{n}}(a, b; \omega_{\overline{m}}; s) \geq e_0.$$

Note that the integral value $\ell = 0$ lies precisely on the boundary of the left half-plane $-\Pi_0$. Finally one has that

$$\text{(A.16)} \qquad \frac{C_\ell}{e_\ell!} \cdot Z_{\mathfrak{n}}^{(e_\ell)}(a, b; \omega_{\overline{m}}; \ell) = (-\,\mathrm{i})^{\mathrm{Tr}(\overline{m})} \cdot e^{2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot Z_{\mathfrak{n}^*}(-b, a; \omega_{\overline{m}}; 1-\ell),$$

where $Z_{\mathfrak{n}}^{(e_\ell)}(a, b; \omega_{\overline{m}}; \ell)$ corresponds to the e_ℓ-th derivative of $[s \mapsto Z_{\mathfrak{n}}(a, b; \omega_{\overline{m}}; s)]$ at $s = \ell$ and $C_\ell := \lim_{s \to \ell} s^{e_\ell} \frac{F_{\overline{m}}(s)}{F_{\overline{m}}(1-s)}$.

Proof This is a special case of Theorem 7.8 which follows rather directly from the functional equation

$$\text{(A.17)} \qquad \frac{F_{\overline{m}}(s)}{F_{\overline{m}}(1-s)} \cdot Z_{\mathfrak{n}}(a, b; \omega_{\overline{m}}; s) = (-\,\mathrm{i})^{\mathrm{Tr}([\overline{m}])} \cdot e^{2\pi\,\mathrm{i}\,\mathrm{Tr}_{K/\mathbb{Q}}(ab)} \cdot Z_{\mathfrak{n}^*}(-b, a; \omega_{\overline{m}}; 1-s).$$

B Spherical polynomials

Let $Q = (q_{ij}) \in \mathrm{Sym}_n(\mathbb{R})$ be a symmetric matrix with $\det(Q) \neq 0$ (i.e. Q is non-degenerate). In particular, Q is not necessarily positive definite. We let $x = (x_1, \dots, x_n) \in \mathbb{R}^n$ be the usual coordinate chart of $\mathbb{R}^n$. Let $V := T_0\mathbb{R}^n = \sum_{i=1}^n \mathbb{R}\partial_{x_i}$ be the tangent space of $\mathbb{R}^n$ at the origin $0 \in V$. We endow V with the pseudo-metric $\langle , \rangle_Q$ defined by

$$\text{(B.1)} \qquad \langle \partial_{x_i}, \partial_{x_j} \rangle_Q = q_{ij}.$$

Let $Q^{-1} = (q^{ij})$ and set

$$\text{(B.2)} \qquad \Delta_Q = \sum_{ij} q^{ij} \partial_{x_i} \partial_{x_j}.$$

When $Q \gg 0$ (positive definite) then Δ_Q is the usual Laplace-Beltrami operator, acting on functions of $\mathbb{R}^n$, for the metric $\langle , \rangle_Q$.

Definition B.1. *A complex coefficient polynomial $f \in \mathbb{C}[x_1, \dots, x_n]$ is said to be Q-spherical if $\Delta_Q f = 0$. We denote the vector space of Q-spherical polynomials with complex coefficients by $\mathcal{H}(Q)$.*

Let $f \in \mathbb{C}[x_1, \dots, x_n]$ and $f = \sum_i f_i$ be its decomposition into homogeneous polynomials. Since Δ_Q is a homogeneous differential operator of degree 2 it follows that f is Q-spherical if and only if each f_i is Q-spherical so that one has the direct sum decomposition

$$\text{(B.3)} \qquad \mathcal{H}(Q) = \bigoplus_n \mathcal{H}_n(Q)$$

where $\mathcal{H}_n(Q)$ is the vector space of degree n homogeneous polynomials which are Q-spherical.

Let $\mathrm{SO}(Q) := \{g \in \mathrm{GL}_n(\mathbb{R}) : g^t Q g = I_n\}$. Since Δ_Q is $\mathrm{SO}(Q)$-invariant it follows that $\mathrm{SO}(Q)$ acts on the homogeneous factors $\mathcal{H}_k(Q)$.

Remark B.2. Since Q is symmetric it can be diagonalized and since the square root of a diagonal matrix exists over $\mathbb{C}$ this means that we can always find a matrix $L \in M_n(\mathbb{C})$ such that $Q = (L^t)L$. It therefore follows that, up to the invertible complex linear transformation L, one can identify the space $\mathcal{H}(Q)$ with $\mathcal{H}(I_n)$. When $Q = I_n$, $\Delta_Q = -(\sum_i \partial_{x_i}^2)$ is the usual (positive definite) Laplacian one has (see for example Corollary 1.5 of [14])

$$\dim_{\mathbb{C}} \mathcal{H}_m(I_n) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}, \tag{B.4}$$

and therefore, for any non-degenerate $Q \in \mathrm{Sym}_n(Q)$ we also have that $\dim_{\mathbb{C}} \mathcal{H}_m(Q) = \binom{n+m-1}{n-1} - \binom{n+m-3}{n-1}$.

Example B.3.

(1) Let $V = \mathbb{R}$ and $x \in \mathbb{R}$ be the standard coordinate. For $Q = 1$ one has $\Delta_Q = -\partial_x^2$ and $\mathcal{H}(Q) = \mathbb{R} + \mathbb{R}x$ where $\partial_x = \frac{\partial}{\partial x}$.

(2) Let $V = \mathbb{C}$ where $z = x + \mathrm{i}\, y \in \mathbb{C}$ is the usual complex chart. For $Q = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$ one has $\Delta_Q = -(\partial_x^2 + \partial_y^2) = -4\partial_z\partial_{\overline{z}}$ where $\partial_z = \frac{1}{2}(\partial_x - \mathrm{i}\,\partial_y)$. In that case one has $\mathcal{H}_n(Q) = \mathbb{C}z^n + \mathbb{C}\overline{z}^n$.

Definition B.4. *For $x \in \mathbb{C}^n$ (viewed as a column vector) we let*

$$Q[x] := x^t Q x$$

We say that $\xi \in \mathbb{C}^n$ is Q-isotropic if $Q[\xi] = 0$.

Proposition B.5. *Let $\xi \in \mathbb{R}^n$ be Q-isotropic and set*

$$P_{\xi,k}(x) := \left(\sum_{i=1}^n \xi_i x_i\right)^k \tag{B.5}$$

Then $P_{\xi,k}(x) \in \mathcal{H}_k(Q)$. Moreover, the vector $\mathbb{C}$-vector space W_k generated by the set $\{P_{\xi,k}(x) : \xi \text{ is } Q\text{-isotropic}\}$ is all of $\mathcal{H}_k(Q)$.

Proof The fact that $\Delta_Q(P_{\xi,k}(x)) = 0$ is a straight forward calculation. For the second part, in light of Remark B.2, we can without loss of generality assume that $Q = I_n$. The fact that $W_k = \mathcal{H}_k(I_n)$ follows from the well-known irreducibility of $\mathcal{H}_k(I_n)$ as an $\mathrm{SO}(n)$-module. This irreducibility result can be viewed as a consequence of the Peter-Weyl theorem applied to the Gelfand pair $(\mathrm{SO}(n+1), \mathrm{SO}(n))$, see for example p. 35-38 of [14]. The irreducibility can also be proved more directly using character theory, see for example Proposition 5.10 in Chapter II of [1] where it is proved for the special case $n = 3$. $\square$

It is possible to generalize the above discussion to the class of pluriharmonic polynomials.

Definition B.6. *(Definition 7.2 of [25]) Let $X = (X_{ij})_{ij} \in \mathbb{C}^{(n,m)}$ be a matrix variable. A polynomial $P(X)$ is called* **pluri-harmonic** *if*

$$\sum_{k=1}^{m} \frac{\partial^2 p(X)}{\partial X_{ik} \partial X_{jk}} = 0, \qquad 1 \leq i, j \leq n. \tag{B.6}$$

In particular if we set $n = 1$ in Definition B.6 and let $x = (X_1, \ldots, X_m)$ we recover as a special case the notion of Q-spherical polynomials of Definition B.1 with- $Q = I_m$; so that $Q[x] = \sum_{i=1}^{m} x_i^2$ corresponds to the standard positive definite quadratic form of rank m. Finally, let us point out that one can construct theta functions weighted by pluriharmonic polynomials. For a concise and very explicit account on this topic we refer to p. 229-234 of [25].

C Relationship between $\theta^p(a, w; z)$ and the classical Riemann theta function

In this appendix we wish to provide some explicit relation between the number field theta function $\theta_L^p(a, w; z)$ considered in Section 5.2 (for $p = \mathbb{0}$ and $L \subseteq \mathbf{R}$ a lattice) and the classical Riemann theta function which definition we recall below.

Let us fix some positive integer $g \geq 1$. Let $W := \mathbb{C}^g$ and $c_\infty : W \to W$ be the standard $\overline{\mathbb{C}}$-linear involution given by $w = (w_k)_k \mapsto (\overline{w_k})_k$. We endow W with its standard real structure (see Definition 2.2) which comes from c_∞ so that $W^+ = \mathbb{R}^g$ and $W^- = \mathrm{i}\, W^+ = \mathrm{i}\, \mathbb{R}^g$.

Recall that the Siegel upper half-space of degree g is defined as

$$\mathfrak{H}_g = \{Z = X + \mathrm{i}\, Y \in M_n(\mathbb{C}) : Z^t = Z, Y \gg 0\}. \tag{C.1}$$

Following Mumford's convention in [25] we consider elements on $\mathbb{C}^g = W$ as column vectors and we put arrows on these vectors. For $\vec{z} \in \mathbb{C}^g$ and $\Omega \in \mathfrak{H}_g$ recall that the *classical Riemann theta function of genus* g is defined as

$$\mathit{\Theta}(\vec{z}, \Omega) := \sum_{\vec{n} \in \mathbb{Z}^g} e^{\pi \mathrm{i}\, \vec{n}^t \cdot \Omega \cdot \vec{n} + 2\pi \mathrm{i}\, \vec{n}^t \cdot \vec{z}}. \tag{C.2}$$

In particular, note that $[(\vec{z}, \Omega) \mapsto \mathit{\Theta}(\vec{z}, \Omega)]$ is holomorphic. It satisfies the transformation formulae

(i) periodicity in $\mathbb{Z}^g$: $\mathit{\Theta}(\vec{z} + \vec{n}, \Omega) = \mathit{\Theta}(\vec{z}, \Omega)$ for all $\vec{n} \in \mathbb{Z}^g$

(ii) quasi-periodicity in $\Omega\mathbb{Z}^g$: $\mathit{\Theta}(\vec{z} + \Omega\vec{n}, \Omega) = e^{-\pi \mathrm{i}\, \vec{n}^t \cdot \Omega \cdot \vec{n} - 2\pi \mathrm{i}\, \vec{n}^t \cdot \vec{z}} \cdot \mathit{\Theta}(\vec{z}, \Omega)$ for all $\vec{n} \in \mathbb{Z}^g$.

Moreover, for a fixed $\Omega \in \mathfrak{H}_g$ and up to a non-zero scalar, one can show that there exists a unique entire function on $\mathbb{C}^g$ which satisfies (i) and (ii) (see p. 121-122 of [25]). For

$\vec{a}, \vec{b} \in \mathbb{R}^g$ recall that the *Riemann theta function of genus g and real characteristics* $\begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix}$ is defined as

$$\text{(C.3)} \quad \Theta \begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix} (\vec{z}, \Omega) := e^{\pi \mathrm{i}\, \vec{a}^t \cdot \Omega \cdot \vec{a} + 2\pi \mathrm{i}\, \vec{a}^t \cdot (\vec{z} + \vec{b})} \cdot \Theta(\vec{z}, \Omega) = \sum_{\vec{n} \in \mathbb{Z}^g} e^{\pi \mathrm{i} (\vec{n} + \vec{a})^t \cdot \Omega \cdot (\vec{n} + \vec{a}) + 2\pi \mathrm{i} (\vec{n} + \vec{a})^t \cdot (\vec{z} + \vec{b})}.$$

The functions $\left[\vec{z} \mapsto \begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix} (\vec{z}, \Omega)\right]$ are also quasi-periodic with respect to the lattice

$$\text{(C.4)} \qquad \mathcal{L}_\Omega := \mathbb{Z}^g + \Omega \mathbb{Z}^g \leq W.$$

See the bottom of p. 123 for explicit formulas. In particular, note that the defining summation of $\Theta \begin{bmatrix}\vec{a}\\ \vec{b}\end{bmatrix} (\vec{z}, \Omega)$ is over the **real part** of the lattice $\mathcal{L}_\Omega \leq W$ namely over $\mathcal{L}_\Omega^+ := \mathcal{L}_\Omega \cap W^+ = \mathbb{Z}^g$.

Consider now a framed number field K of signature (r_1, r_2) with $r_1 + 2r_2 = g$ (see Section 2). In particular $\Sigma = \mathrm{Hom}(K, \mathbb{C})$ comes with an admissible labelling. We view $\mathbf{C} = \mathbf{C}_\Sigma$ as a $\mathbb{C}$-vector space with the real structure coming from the $\overline{\mathbb{C}}$-linear involution c_3. In particular $\mathbf{R} := \mathbf{C}^+$ and $\mathrm{i}\,\mathbf{R} = \mathbf{C}^-$. Let $L \subsetneq \mathbf{R}$ be a lattice.

For the sequel we wish to explain how the theta functions with real characteristics are related to the theta functions θ_L^p introduced in Section 5.2 when $p = \mathbb{0}_g$. Recall that

$$\text{(C.5)} \qquad \mathrm{f}_1 : \mathbf{R} \to \mathbb{R}^g = W^+$$

was the $\mathbb{R}$-vector isomorphism defined in (2.22) where the target $\mathbb{R}^g$ is identified with W^+, the real part of W. We let

$$\text{(C.6)} \qquad \mathrm{g}_1 := \mathrm{f}_1^{-1} : \mathbb{R}^g \to \mathbf{R}$$

be the reciprocal map to f_1. We choose to extend the map f_1 to a map f on $\mathbf{C}$ in the following way:

$$\text{(C.7)} \qquad \begin{aligned} \mathrm{f} : \mathbf{C} &\to \mathbb{C}^g = W \\ x + \mathrm{i}\, y &\mapsto \mathrm{f}_1(x) + \mathrm{i}\, \mathrm{f}_1(y) \end{aligned}$$

In this way f is a $\mathbb{C}$-linear isomorphism of $\mathbb{C}$-vector spaces with real structures. In particular, we have $\mathrm{f}(\mathbf{C}^+) = W^+$ and $\mathrm{f}(\mathbf{C}^-) = W^-$. Let us denote by

$$\text{(C.8)} \qquad \mathrm{g} := \mathrm{f}^{-1} : \mathbb{C}^g = W \to \mathbf{C}$$

the reciprocal map of f.

Each of the spaces $\mathbf{C}$ and $\mathbb{C}^g$ admit a natural structure of $\mathbb{C}$-algebra. In fact, recall that we have a $\mathbb{C}$-algebra isomorphism

$$\text{(C.9)} \qquad \psi : \mathbf{C} \to W = \mathbb{C}^g$$

given by (2.6). However note that $\psi \neq \mathrm{f}$. In fact the map f fails to be an isomorphism of $\mathbb{C}$-algebras in two ways

(i) $\mathrm{f}(\mathbb{1}_{\mathbf{C}}) = (\underbrace{1,\ldots,1}_{r_1+r_2 \text{ times}};\underbrace{0,\ldots,0}_{r_2 \text{ times}}) \neq 1_W$ (if $r_2 \geq 1$),

(ii) and in general f fails to preserve the multiplication.

However, the following remarkable identity is satisfied (a kind of "partial multiplication preservation"): for all $x \in \mathbf{R}$ and all $z \in \mathbf{C}$ such that $c_1(z) = z$ (see Section 2 for the definition of c_1) one may check that

$$\mathrm{f}(x \cdot z) = \mathrm{f}(x) \cdot \psi(z). \tag{C.10}$$

Let $L \subseteq \mathbf{R}$ be a lattice of rank g (so of maximal rank) and let $\mathcal{B} = (\ell_1, \ldots, \ell_g)$ be a $\mathbb{Z}$-ordered basis of L. To the ordered basis $\mathcal{B}$ we associate the matrix

$$P = P_{\mathcal{B}} := \begin{pmatrix} | & \vdots & | \\ \mathrm{f}_1(\ell_1) & \vdots & \mathrm{f}_1(\ell_g) \\ | & \vdots & | \end{pmatrix} \tag{C.11}$$

where $\mathrm{f}_1(\ell_k)$ is viewed as a column vector in $\mathbb{R}^g = W^+$. Since L has rank g it follows that $P \in \mathrm{GL}_g(\mathbb{R})$. In particular,

$$P\mathbb{Z}^g = \mathrm{f}_1(L). \tag{C.12}$$

We think here of $\mathbb{Z}^g$ as the standard normalized lattice of $W^+ = \mathbb{R}^g$. Consider the block matrix

$$J = J_{r_1,r_2} := \left(\begin{array}{c|c} I_{r_1} & \mathbf{0}_{r_1 \times 2r_2} \\ \hline \mathbf{0}_{2r_2 \times r_1} & 2I_{2r_2} \end{array}\right) \in \mathrm{GL}_g(\mathbb{R}). \tag{C.13}$$

For all $x, y \in \mathbf{R}$ a direct calculation shows that

$$\langle x, y\rangle_c = (\mathrm{f}_1(y))^t \cdot J \cdot \mathrm{f}_1(x) = \langle \mathrm{f}_1(x), \mathrm{f}_1(y)\rangle_c, \tag{C.14}$$

where the right-most inner product $\langle \_, \_\rangle_c$ is the canonical metric of type (r_1, r_2) on $\mathbb{R}^g$ (see Definition 2.6). We now extend the canonical euclidean inner product $\langle , \rangle_c$ on $W^+ = \mathbb{R}^g$ to all of W in the natural way: for $z, w \in \mathbf{C}$, we declare

$$\langle \mathrm{f}(z), \mathrm{f}(w)\rangle_c := \langle z, w\rangle_c, \tag{C.15}$$

which is equivalent to

$$\langle \mathrm{f}(z), \mathrm{f}(w)\rangle_c = \overline{(\mathrm{f}(w))}^t \cdot J \cdot \mathrm{f}(z). \tag{C.16}$$

In this way, the map $\mathrm{f} : \mathbf{C} \to W$ becomes an isomorphism of hermitian vector spaces with real structures which extends the euclidean vector space isomorphism $\mathrm{f}_1 : \mathbf{R} \to W^+ = \mathbb{R}^g$.

Recall that $\mathbf{H} = \mathbf{R}_{\pm} + \mathrm{i}\,\mathbf{R}_{+} \subseteq \mathfrak{h}^g$ and therefore

$$\mathbf{H} \subseteq \mathbf{C}^{\langle c_1 \rangle} := \{z \in \mathbf{C} : c_1(z) = z\}. \tag{C.17}$$

Let $\Delta : \mathfrak{h}^g \to \mathfrak{H}_g$ be the diagonal embedding and set

$$\delta := \Delta|_{\mathbf{H}} : \mathbf{H} \to \mathfrak{H}_g. \tag{C.18}$$

Now for $\tau \in \mathbf{H}$, $\vec{z} \in \mathbb{C}^g = W$, $\vec{a}, \vec{b} \in \mathbb{R}^g = W^+$ we make the following change of variables:

$$\vec{Z} := P^t \vec{z}, \;\; \vec{B} := P^t \vec{b}, \;\; \vec{A} := P\vec{a}, \;\; \Omega_\tau := P^t (J\tau^\delta) P, \tag{C.19}$$

$$\ell := \mathrm{g}_1(P\vec{n}) \text{ so that } \ell \in L, \;\; a = \mathrm{g}_1(P\vec{a}) \text{ so that } a \in \mathbf{R}. \tag{C.20}$$

and

$$w := \mathrm{g}(J^{-1}\vec{z}) \in \mathbf{C}, \quad b := \mathrm{g}_1(J^{-1}\vec{b}) \in \mathbf{R}. \tag{C.21}$$

Putting everything together we find that

$$\begin{aligned}
\Theta \begin{bmatrix} \vec{A} \\ \vec{B} \end{bmatrix} (\vec{Z}, \tau^\delta) &= \sum_{\vec{n} \in \mathbb{Z}^g} e^{\pi\,\mathrm{i}(\vec{n}+\vec{a})^t \cdot P^t (J\tau^\delta) P \cdot (\vec{n}+\vec{a}) + 2\pi\,\mathrm{i}(\vec{n}+\vec{a})^t \cdot P^t \cdot (\vec{z}+\vec{b})} \\
&= \sum_{\ell \in L} e^{\pi\,\mathrm{i}\langle \mathrm{f}_1(\ell+a)) \cdot \psi(\tau), \mathrm{f}_1(\ell+a) \rangle_c + 2\pi\,\mathrm{i}(\mathrm{f}_1(\ell+a))^t \cdot (\vec{z}+\vec{b})} && (\text{ since } J\tau^\delta = \tau^\delta J,\ (\mathrm{f}_1(\ell+a)\cdot\psi(\tau))^t = (\mathrm{f}_1(\ell+a))^t \tau^\delta \text{ and (C.14) }) \\
&= \sum_{\ell \in L} e^{\pi\,\mathrm{i}\langle \mathrm{f}(\ell+a)\cdot\tau), \mathrm{f}_1(\ell+a) \rangle_c + 2\pi\,\mathrm{i}\,\overline{(\mathrm{f}(\ell+a))}^t \cdot J \cdot \mathrm{f}(w+b)} && (\text{ by (C.10) and } \mathrm{f}_1(\ell+a) = \overline{\mathrm{f}(\ell+a)} \text{ }) \\
&= \sum_{\ell \in L} e^{\pi\,\mathrm{i}\langle \mathrm{f}((\ell+a)\cdot\tau), \mathrm{f}(\ell+a) \rangle_c + 2\pi\,\mathrm{i}\langle \mathrm{f}(w+b), \mathrm{f}(\ell+a) \rangle_c} && (\text{ since } \mathrm{f}(\ell+a) = \mathrm{f}_1(\ell+a) \text{ and (C.16) }) \\
&= \sum_{\ell \in L} e^{\pi\,\mathrm{i}\langle (\ell+a)\cdot\tau), \ell+a \rangle_c + 2\pi\,\mathrm{i}\langle w+b, \ell+a \rangle_c} && (\text{ by (C.15) }) \\
&= \theta_L^p(a, w+b; \tau), && (\text{ by definition of } \theta_L^p, \text{ see (5.12) })
\end{aligned}$$

where $p = \mathbb{0}_g$. In this way we obtain an explicit relationship between the classical Riemann's theta function and the number field theta function $\theta_L^p(a, w+b; \tau)$ when $p = \mathbb{0}_g$.

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