Functional equation for partial zeta functions twisted by additive characters

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Abstract

In this paper we prove a functional equation for a certain class of zeta functions attached to an arbitrary number field K. The proof of the functional equation relies on the transformation formula of a multivariables theta function. The techniques which are used are classical and are due essentially to Riemann and Hecke. As a special case, we obtain a functional equation for partial zeta functions of K twisted by sign characters.

Contents

1 Introduction

Let K be a number field of degree n over $\mathbb{Q}$ and let $\{\sigma_1, \ldots, \sigma_{r_1}\}$ be a complete set of real embeddings of K where $r_1 + 2r_2 = n$. Let ω be a sign character of K, i.e., a product over elements of a subset of the characters

$$sign \circ \sigma_i : K^\times \to \mathbb{R}^\times \to \{\pm 1\}.$$

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Let V be a lattice of maximal rank of K and let

$$\mathcal{O}_V = \{\alpha \in K : \alpha V \subseteq V\}.$$

Note that $\mathcal{O}_V$ is an order of K and V is an invertible $\mathcal{O}_V$-ideal with inverse given explicitely by $V^{-1} = \{x \in K : xV \subset \mathcal{O}_V\}$. Let

$$V^* = \{x \in K : Tr_{K/\mathbb{Q}}(xv) \in \mathbb{Z} \text{ for all } v \in V\},$$

be the *dual lattice* of V. Note that V^* is an invertible $\mathcal{O}_V$-module and that $V^{**} = V$. For elements $a, b \in K$ we define

(1.1)
$$\Gamma_{a,b,V} = \{\epsilon \in \mathcal{O}_V : \sigma_i(\epsilon) > 0 \ \forall i, (\epsilon - 1)a \in V, (\epsilon - 1)b \in V^*, (\epsilon - 1)ab \in \mathfrak{d}_K^{-1}\},$$

where $\mathfrak{d}_K^{-1} = (\mathcal{O}_K)^* = \{x \in K : Tr_{K/\mathbb{Q}}(xy) \in \mathbb{Z}, \text{ for all } y \in \mathcal{O}_K\}$ is the inverse of the different ideal of K. One can verify that $\Gamma_{a,b,V} = \Gamma_{-b,a,V^*}$ is a subgroup of finite index in $\mathcal{O}_K^\times$. For the set of data (a, b, ω, V) we define a *partial zeta function twisted by an additive character* as

$$\Psi_V(a, b, \omega, s) := [\mathcal{O}_K : V]^s \sum_{\substack{v \in \mathfrak{R} \\ a+v \neq 0}} \omega(a + v) \frac{e^{2\pi i Tr_{K/\mathbb{Q}}(b(a+v))}}{|\mathbf{N}_{K/\mathbb{Q}}(a + v)|^s}, \tag{1.2}$$

where $[\mathcal{O}_K : V]$ is a positive rational number which plays the role of an index (see Definition 3.3) and $\mathfrak{R} = \{v_i \in V\}_{i \in I}$ is a complete set of representatives of $\{a + V\}/\Gamma_{a,b,V}$ in the sense that every element $0 \neq (a + v) \in a + V$ can be written uniquely as $\epsilon(a + v_i)$ for some $v_i \in \mathfrak{R}$ and $\epsilon \in \Gamma_{a,b,V}$. It is easy to see that (1.2) doesn't depend on the set of representatives $\mathfrak{R}$ and that it converges absolutely for any complex number s such that $\Re(s) > 1$.

Let $p = \{p_i\}_{i=1}^{r_1}$ be the signature of ω, i.e., $\omega = \prod_{i=1}^{r_1} (sign \circ \sigma_i)^{p_i}$ where $p_i \in \{0, 1\}$. Then we define

$$F_V^p(s/2) := |d_K|^{s/2} \pi^{-ns/2} \prod_{i=1}^{r_1} \Gamma\left(\frac{s + p_i}{2}\right) \prod_{i=1}^{r_2} \left(2^{1-s}\Gamma(s)\right),$$

where $\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{d}_K) = d_K$ is the discriminant of K and $\Gamma(x)$ stands for the usual gamma function evaluated at x.

We can now state the main theorem which is proved in this paper.

Theorem 1.1 *Let*

$$Z_V(a, b, \omega, s) = F_V^p(s/2)\Psi_V(a, b, \omega, s)$$

be the completed zeta function of $\Psi_V(a, b, \omega_p, s)$. *Then* $Z_V(a, b, \omega, s)$ *admits an analytic continuation to* $\mathbb{C}\backslash\{0, 1\}$ *and has at most a pole of order one at*

$s \in \{0,1\}$. *A pole of order one at* $s=0$ *occurs exactly when* $p_i = 0$ *for all* i *and* $a \in V$. *Similarly, a pole of order one at* $s = 1$ *occurs exactly when* $p_i = 0$ *for all* i *and* $-b \in V^*$. *Moreover,* $Z_V(a,b,\omega,s)$ *satisfies the following functional equation*

$$(1.3)\qquad (-i)^{Tr(p)} e^{-2\pi i Tr_{K/\mathbb{Q}}(ab)} Z_V(a,b,\omega,s) = Z_{V^*}(-b,a,\omega,1-s).$$

The ideas which are used in the proof of Theorem 1.1 are due for the large part to Riemann and Hecke. Let $\zeta_{\mathbb{Q}}(s) = \sum_{n\geq 1}\frac{1}{n^s} = \prod_p \frac{1}{1-\frac{1}{p^s}}$ be the Riemann zeta function. The idea of using the transformation formula of the one variable theta function $\theta(z) = \sum_{n\in\mathbb{Z}} e^{\pi i n^2 z}$ to prove the functional equation

$$(1.4)\qquad \pi^{-s/2}\Gamma(s/2)\zeta_{\mathbb{Q}}(s) = \pi^{-(1-s)/2}\Gamma((1-s)/2)\zeta_{\mathbb{Q}}(1-s),$$

is due to Riemann, see [Rie90]. Its generalization to an arbitrary number field K of degree n over $\mathbb{Q}$, namely

$$(1.5)\qquad A^s\Gamma(s)^{r_2}\Gamma(s/2)^{r_1}\zeta_K(s) = A^{1-s}\Gamma(1-s)^{r_2}\Gamma((1-s)/2)^{r_1}\zeta_K(1-s),$$

where $\zeta_K(s) = \sum_{\mathfrak{a}\trianglelefteq\mathcal{O}_K}\frac{1}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\mathfrak{p}}\frac{1}{1-\frac{1}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})^s}}$ and $A = 2^{-r_2}\pi^{-n/2}\sqrt{|d_K|}$ is due to Hecke, see [Hec59b].

The functional equation (1.3) can be viewed, in some sense, as a natural generalisation of (1.4) and (1.5). However, there is one important aspect in which the zeta function $\Psi_V(a,b,\omega,s)$ differs from $\zeta_K(s)$, namely that in general, $\Psi_V(a,b,\omega,s)$ doesn't have any Euler product. The latter observation might be a reason why Hecke never published the functional equation (1.3).

It is clear that Hecke had at its disposal all the necessary tools to prove Theorem 1.1. In fact, in [Hec59a], Hecke proves a functional equation for the most general class of zeta functions which admit a degree one Euler product, namely for

$$(1.6)\qquad \zeta_K(\lambda,s) = \sum_{\mathfrak{a}\trianglelefteq\mathcal{O}_K}\frac{\lambda(\mathfrak{a})}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})^s} = \prod_{\mathfrak{p}}\frac{1}{1-\frac{\lambda(\mathfrak{p})}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{p})^s}}$$

where λ is a so-called *Größencharakter*. It would be fair to say that the proof of the functional equation of $\zeta_K(\lambda,s)$ requires more ideas than the proof of Theorem 1.1. For example, Hecke introduced the notion of "idealer Zahlen [1] "(see p.255 of [Hec59a]) in order to work with complex numbers

[1]The english translation of *idealer Zahlen* is *ideal numbers*. These ideal numbers, which have the drawback of not being defined in a canonical way, can be viewed in some sense as a precursor notion to the notion of *idèles* introduced by Chevalley in the mid-1930s

rather than ideals. These "idealer Zahlen" allowed him in particular to define certain Gauss sums (depending on λ) which play a crucial role in the proof of the functional equation of $\zeta_K(\lambda, s)$.

The zeta function $\Psi_V(a, b, \omega_p, s)$ arose naturally in some of the previous work of the author, see for example section 2 of [Cha07]. Let us explain in more details the context in which the zeta function $\Psi_V(a, b, \omega_p, s)$ arose. Let K be a totally real number field and let $V = \frac{\mathfrak{a}}{\mathfrak{f}\mathfrak{d}_K}$ where $\mathfrak{a}, \mathfrak{f}$ are integral ideals which are coprime. Assume that the sign character ω_p is chosen so that $\omega_p = 1$ or $\omega_p = sign \circ \mathbf{N}_{K/\mathbb{Q}}$. In section 2.3 of [Cha07], it is explained how the special values at negative integers of

$$\Psi_{V^*}(1, 0, \omega_p, s) = \mathbf{N}_{K/\mathbb{Q}}(\mathfrak{f})^s \left(\sum_{\{0 \neq \mu \in 1 + \mathfrak{f}\mathfrak{a}^{-1}\}/\Gamma_{\mathfrak{a}}} \frac{\omega_p(\mu)}{|\mathbf{N}_{K/\mathbb{Q}}(\mu\mathfrak{a})|^s} \right) \tag{1.7}$$

(see equation (5.1) for more details) can be related to special values at negative integers of classical partial zeta functions, namely

$$\zeta(\mathfrak{a}, \mathfrak{f}\infty, s) := \mathbf{N}_{K/\mathbb{Q}}(\mathfrak{a})^{-s} \sum_{\Gamma_{\mathfrak{a}} \backslash \{\lambda \in \mathfrak{a}^{-1}, \lambda \equiv 1 (mod\ \mathfrak{f}\mathfrak{a}^{-1}), \lambda \gg 0\}} \frac{1}{|\mathbf{N}_{K/\mathbb{Q}}(\lambda)|^s} \tag{1.8}$$

$$= \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O}_K \\ \mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{a}}} \frac{1}{\mathbf{N}_{K/\mathbb{Q}}(\mathfrak{b})^s}, \tag{1.9}$$

where $\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{c}$ means that $\mathfrak{b}$ and $\mathfrak{c}$ lie in the same narrow ray ideal class modulo $\mathfrak{f}$ and $\Gamma_{\mathfrak{a}} = \mathcal{O}_K(\infty)^\times \cap (1 + \mathfrak{f}\mathfrak{a}^{-1})$ where $\mathcal{O}_K(\infty)^\times$ denotes the group of totally positive units of $\mathcal{O}_K$. Note that the summation in (1.8) is taken over totally positive elements of $\mathfrak{a}^{-1}$. Special values of $\zeta(\mathfrak{a}, \mathfrak{f}\infty, s)$ at negative integers turns out to be rational numbers, see [Kli62], [Sie69] and [Shi76]. These rational numbers satisfy many remarkable congruence relations which have been exploited by many number theorists to construct various p-adic objects. In particular, one can construct p-adic L-functions which interpolates these special values, see [DR80] and [CN79]. In the last section of this paper we show essentially that unless K is a totally real number field and $\omega_p = 1$ or $sign \circ \mathbf{N}_{K/\mathbb{Q}}$ then no such p-adic L-function exists since the special values at negative integers of (1.7) are all equal to zero.

When the author wrote the paper, he was unaware that a special case of the functional equation (1.3) had already appeared in a paper of Siegel (see equation (10) of [Sie70]). In the paragraph below equation (10) of [Sie70], Siegel writes

> Es ist sonderbar, daß (10) bisher in der Literatur nicht erwähnt worden ist. Auch wenn man die Funktionalgleichung der L-Reihen im Auge hat, erscheint es übrigens durchsichtiger, zunächst

die einfachere Formel (10) zu beweisen und erst nachher die notwendigen algebraischen arithmetischen Sätze über Charaktere herzuleiten.

Thirty eight years later, the author of this paper shares the exact same view.

Finally, let us mention some connection between the Lerch zeta function and the zeta function $\Psi_V(a,b,\omega,s)$ in the case where $K = \mathbb{Q}$. For real numbers $0 < u, v \leq 1$ consider the Lech zeta function

$$\varphi(u,v,s) = \sum_{n=0}^{\infty} \frac{e^{2\pi inv}}{|n+u|^s}, \quad \Re(s) > 1.$$

Let $\omega = 1$, $V = \mathbb{Z}$ and $a, b \in \mathbb{Q} \cap (0,1)$. Then a direct computation shows that

$$\begin{aligned}\Psi_V(a,b,\omega,s) &= e^{2\pi iab} \sum_{n\in\mathbb{Z}} \frac{e^{2\pi ibn}}{|n+a|^s} \\ &= e^{2\pi iab}(\varphi(a,b,s) + \varphi(1-a,b,s)).\end{aligned}$$

The contribution of this paper consists essentially in filling a gap in the literature by providing a detailed proof of the functional equation (1.3). In writting the paper, we have decided to follow a more modern account of the work of Riemann and Hecke on zeta functions: namely we borrow most of our notation from chapter 7 of Neukirch's book on algebraic number theory, see [Neu99]. The interested reader may consult as well this chapter in order to find a proof of the functional equation for $\zeta_K(\lambda, s)$.

2 Notation

Let K be a number field of degree n over $\mathbb{Q}$ and let $X = Hom(K,\mathbb{C})$ be a complete set of embeddings of K into $\mathbb{C}$.The set X can be written in the following way

(2.1) $$X = \{\sigma_1, \dots, \sigma_{r_1}, \rho_1, \overline{\rho}_1, \dots, \rho_{r_2}, \overline{\rho}_{r_2}\},$$

where $r_1 + 2r_2 = n$, the σ_i's are the real embeddings, the ρ_i's are the complex embeddings such that $c \circ \rho_i = \overline{\rho}_i$ where $c : \mathbb{C} \to \mathbb{C}$ corresponds to complex conjugation. Note that up to permutation there is still a choice in our way of writting the set X which corresponds to a choice of a privileged representative ρ_i for every pair of complex embeddings of K. Usually for an element $a \in \mathbb{C}$ we will denote its complex conjugate $c(a)$ by $\overline{a}$. Note that X is naturally a left $Gal(\mathbb{C}/\mathbb{R})$-set. For $\tau \in X$ we will also denote $c \circ \tau$ by $\overline{\tau}$.

We consider the n-dimensional $\mathbb{C}$-algbera attached to X

$$\mathbf{C}_X := \prod_{\tau \in X} \mathbb{C}$$

of all tuples $z = (z_\tau)_{\tau \in X}, z_\tau \in \mathbb{C}$ with componentwise addition and multiplication. Since the subset X is fixed from the beginning we will denote $\mathbf{C}_X$ simply by $\mathbf{C}$.

For the sequel we will use the set of notations attached to $\mathbf{C}$ which is introduced on pages 444 and 445 of [Neu99]. The $\mathbb{C}$-algebra $\mathbf{C}$ is endowed with three involutions. For every element $z = (z_\tau)_{\tau \in X} \in \mathbf{C}$ we define the elements $z^*, {}^*z, \overline{z} \in \mathbf{C}$ as

$$(z^*)_\tau = z_{\overline{\tau}}, \qquad ({}^*z)_\tau = \overline{(z_\tau)} \qquad \text{and} \quad (\overline{z})_\tau = ({}^*z^*)_\tau = \overline{(z_{\overline{\tau}})}.$$

The $\mathbb{C}$-algebra $\mathbf{C}$ is equiped with certain distinguished subsets namely

(1) $\mathbf{R} = \{z \in \mathbf{C} : \overline{z} = z\}$,

(2) $\mathbf{R}_\pm = \{x \in \mathbf{R} : x = x^*\}$,

(3) $\mathbf{R}_+^\times = \{x \in \mathbf{R}_\pm : x > 0\}$,

(4) $\mathbf{H} = \mathbf{R}_\pm + i\mathbf{R}_+^\times$.

If $\delta \in \mathbb{R}$ the notation $x > \delta$ means that $x_\tau > \delta$ for all $\tau \in X$. By definition we have the following inclusions

$$\mathbf{H} \subseteq \mathbf{C} \supseteq \mathbf{R} \supseteq \mathbf{R}_\pm \supseteq \mathbf{R}_+^\times.$$

Note that subset $\mathbf{R}$ is naturally an $\mathbb{R}$-subalgebra of the $\mathbb{C}$-algebra $\mathbf{C}$.

For every infinite place ν of K there exists a *unique* field inclusion $\iota_\nu : \mathbb{R} \to K_\nu$. Because of the uniqueness of ι_ν we can view the set $\mathbb{R}$ as being *naturally* included in K_ν. We thus have a natural isomorphism $K \otimes_{\mathbb{Q}} \mathbb{R} \to \prod_{\nu|\infty} K_\nu$ given by $\alpha \otimes \beta \mapsto (\alpha\beta)_{\nu|\infty}$. Our choice of a complete set of pairwise non conjugate complex embeddings $\{\rho_i\}_{i=1}^{r_2}$ gives rise to a *natural* ismomorphism $\prod_{\nu|\infty} K_\nu \to \mathbf{R}$ given by $(x_\nu)_{\nu|\infty} \mapsto (y_\tau)_{\tau \in X}$ where $y_\tau = x_\nu$ if τ is the real embedding corresponding to the real place ν, $y_\tau = x_\nu$ (resp. $y_\tau = \overline{x}_\nu$) if $\tau = \rho_i$ (resp. $\tau = \overline{\rho}_i$) is a complex embedding corresponding to the complex place ν. In this way we obtain a natural isomorphism of $\mathbb{R}$-algebras

$$\iota : K \otimes_{\mathbb{Q}} \mathbb{R} \xrightarrow{\sim} \mathbf{R}.$$

From now on we will think of the number field K as being naturally included in the $\mathbb{R}$-algebra $K \otimes_{\mathbb{Q}} \mathbb{R}$ via the natural map $\alpha \mapsto \alpha \otimes 1$.

The $\mathbb{C}$-algebra $\mathbf{C}$ and certain of its subsets are equipped with various maps. For the additive group $\mathbf{C}$ (resp. multiplicative group $\mathbf{C}^\times$) we have the homomorphisms

$$Tr : \mathbf{C} \to \mathbb{C}, \quad Tr(z) = \sum_\tau z_\tau,$$
$$N : \mathbf{C}^\times \to \mathbb{C}^\times, \quad N(z) = \prod_\tau z_\tau.$$

We have on $\mathbf{C}$ an hermitian scalar product

$$\langle x, y\rangle = \sum_\tau x_\tau \overline{y}_\tau = Tr(x({}^*y)) \quad \text{and} \quad ||z|| = \sqrt{\langle z, z\rangle}.$$

It is invariant under conjugation, i.e., $\overline{\langle x, y\rangle} = \langle \overline{x}, \overline{y}\rangle$ and restricting it, yields a euclidian metric on the $\mathbb{R}$-vector space $\mathbf{R}$. If $z \in \mathbf{C}$, then ${}^*z$ is the adjoint element i.e., $\langle xz, y\rangle = \langle x, {}^*zy\rangle$. For two tuples $z = (z_\tau)_\tau, (p_\tau)_\tau \in \mathbf{C}$ the power

$$z^p = (z_\tau^{p_\tau}) \in \mathbf{C} \quad \text{where} \quad z_\tau^{p_\tau} = e^{p_\tau \log z_\tau},$$

is well defined if we agree to take the principal branch of logarithm and assume that the z_τ's move only in the plane cut along the negative real axis. Finally we define

$$|| \; || : \mathbf{R}^\times \to \mathbf{R}_+^\times, \quad x = (x_\tau)_\tau \mapsto ||x|| = (|x_\tau|)_\tau,$$
$$\log : \mathbf{R}_+^\times \xrightarrow{\sim} \mathbf{R}_\pm, \quad x = (x_\tau)_\tau \mapsto \log x = (\log x_\tau)_\tau.$$

3 Multivariables θ-function and Γ-function

Definition 3.1 *We say that a tuple $p = (p_\tau)_{\tau \in X}$ of non-negative integers is admissible (resp. strictly admissible) if $p_\tau \in \{0, 1\}$ when $\overline{\tau} = \tau$ and $p_\tau p_{\overline{\tau}} = 0$ (resp. $p_\tau = p_{\overline{\tau}} = 0$) if $\tau \neq \overline{\tau}$.*

Definition 3.2 *Let $V \subseteq \mathbf{R}$ be a lattice of maximal rank, $a, b \in \mathbf{R}$ and let $p \in \prod_{\tau \in X} \mathbb{Z}$ be admissible. We define the theta series*

$$\theta_V^p(a, b, z) = \sum_{v \in V} N((a+v)^p) e^{\pi i \langle (a+v)z, (a+v)\rangle} e^{2\pi i \langle b, a+v\rangle}$$

which converges absolutely for every $z \in \mathbf{H}$.

Remark 3.1 Note that our definition of $\theta_V^p(a, b, z)$ is slightly different from the one appearing on the bottom of page 450 of [Neu99].

Definition 3.3 *Let V be a lattice of maximal rank in K. Let $\{e_1, \ldots, e_n\}$ be a $\mathbb{Z}$-basis of $\mathcal{O}_K$ and let $\{e'_1, \ldots, e'_n\}$ be a $\mathbb{Z}$-basis of V. Let $M \in M_n(\mathbb{Q})$ be the matrix which sends the ordered basis $(e_i)_{i=1}^n$ to the ordered basis $(e'_i)_{i=1}^n$. Then we define the index $[\mathcal{O}_K : V]$ to be the rational number $|det(M)|$.*

It is easy to see that $[\mathcal{O}_K : V]$ is well defined positive rational number independent of the choice of the bases. It is also convenient to define the positive rational number

$$d_V = [\mathcal{O}_K : V]^2 |d_K|,$$

where d_K is the discriminant of K. The quantity $\sqrt{d_V}$ can be interpreted as the covolume of the lattice $\iota(V) \subset \mathbf{R}$ with respect to the Haar measure dx on $(\mathbf{R}, \langle\ ,\ \rangle)$ which ascribes the volume 1 to the cube spanned by an orthonormal basis.

The key ingredient to prove the functional equation appearing in Theorem 1.1 is the following transformation formula for the theta function.

Theorem 3.1 *(theta transformation formula) Let $a, b \in \mathbf{R}$ and let $p \in \prod_\tau \mathbb{Z}$ be admissible. Then one has*

$$\theta_V^p(a, b, -1/z) = (i^{Tr(p)} e^{-2\pi i \langle a,b \rangle} \sqrt{d_V})^{-1} N((z/i)^{p+\frac{1}{2}\cdot\mathbf{1}}) \theta_{V^*}^p(-b, a, z) \quad \text{for all } z \in \mathbf{H},$$

where $\mathbf{1}$ is the unit element in $\mathbf{R}$ and V^* is the dual lattice of V i.e.,

$$V^* = \{v' \in \mathbf{R} : < v', v > \in \mathbb{Z} \text{ for all } v \in V\}.$$

Proof See equation (**19**) on p. 264 of [Hec59a] or (**3**.**6**) on page 454 of [Neu99]. □

We have

$$\mathbf{R}_+^\times = \prod_{\nu|\infty} \mathbf{R}_{+,\nu}^\times \quad \text{and} \quad \mathbf{R}_\pm^\times = \prod_{\nu|\infty} \mathbf{R}_{\pm,\nu}^\times \tag{3.1}$$

where $\mathbf{R}_{+,\nu}^\times = \mathbb{R}_+^\times$ (resp. $\mathbf{R}_{\pm,\nu}^\times = \mathbb{R}_\pm^\times$) if ν is real and $\mathbf{R}_{+,\nu}^\times = \{(y, y) : y \in \mathbb{R}_+^\times\}$ (resp. $\mathbf{R}_{\pm,\nu}^\times = \{(y, y) : y \in \mathbb{R}_\pm^\times\}$) if ν is complex. We define isomorphisms $\mathbf{R}_{+,\nu}^\times \to \mathbb{R}_+^\times$ given by $y \mapsto y$ if ν is real and $(y, y) \mapsto y^2$ is ν is complex. We thus obtain an isomorphism $\mathbf{R}_+^\times \to \prod_{\nu|\infty} \mathbb{R}_+^\times$. We denote by $\frac{dy}{y}$ the Haar measure on $\mathbf{R}_+^\times$ which corresponds to the pull back of the product measure $\prod_\nu \frac{dt}{t}$ where $\frac{dt}{t}$ is the usual Haar measure on $\mathbb{R}_+^\times$. The Haar measure thus defined is called the canonical measure on $\mathbf{R}_+^\times$. Consider the isomorphism

$$\mathbf{R}_+^\times \overset{\log}{\to} \mathbf{R}_\pm \overset{j}{\to} \prod_{\nu|\infty} \mathbb{R}$$

where $j : \mathbf{R}_{\pm,\nu} \to \mathbb{R}$ is given by $x_\nu \mapsto x_\nu$ (resp. $(x_\nu, x_\nu) \mapsto 2x_\nu$) if ν is real (resp. if ν is complex). Then the canonical Haar measure $\frac{dy}{y}$ pushes forward to the Lebesgue measure on $\prod_{\nu|\infty} \mathbb{R}$.

Definition 3.4 *For* $\mathbf{s} = (s_\tau)_\tau \in \mathbf{C}$ *such that* $\Re(s_\tau) > 0$ *and* $p = (p_\tau)_\tau$ *an admissible tuple we define the gamma function associated to the* $Gal(\mathbb{C}/\mathbb{R})$*-set* X *as*

$$\Gamma_X^p(\mathbf{s}) = \int_{\mathbf{R}_+^\times} N(e^{-y} y^{\mathbf{s}+\frac{1}{2}p}) \frac{dy}{y},$$

where $y = (y_\tau)_\tau \in \mathbf{R}_+^\times$, $e^{-y} = (e^{-y_\tau})_\tau$ *and* $y^{\mathbf{s}+\frac{1}{2}p} = (e^{(s_\tau + \frac{1}{2}p_\tau)\log y_\tau})_\tau$.

Using (3.1) we can write $\Gamma_X^p(\mathbf{s})$ as

$$\Gamma_X^p(\mathbf{s}) = \prod_{\nu|\infty} \Gamma_\nu^{p_\nu}(\mathbf{s}_\nu)$$

where $\mathbf{s}_\nu = s_{\sigma_i}$ $(p_\nu = p_{\sigma_i})$ if ν is the real place corresponding to σ_i and $\mathbf{s}_\nu = (s_{\rho_i}, s_{\overline{\rho_i}})$ $(p_\nu = (p_{\rho_i}, p_{\overline{\rho}_i}))$ if ν is the complex place corresponding to ρ_i. The factors are given explicitely by

$$\Gamma_\nu^{p_\nu}(\mathbf{s}_\nu) = \begin{cases} \Gamma(\mathbf{s}_\nu + \frac{1}{2}p_\nu), & \text{if } \nu \text{ is real,} \\ 2^{1-Tr(\mathbf{s}_\nu + \frac{1}{2}p_\nu)}\Gamma(Tr(\mathbf{s}_\nu + \frac{1}{2}p_\nu)) & \text{if } \nu \text{ is complex,} \end{cases}$$

where $\Gamma(x)$ is the usual one variable gamma function.

4 Proof of the functional equation

Consider the multivariable gamma function

$$\Gamma_X^p(s) = \int_{\mathbf{R}_+^\times} N(e^{-y} y^{s\mathbf{1}+\frac{1}{2}p}) \frac{dy}{y}, \tag{4.1}$$

where $s \in \mathbb{C}, \Re(s) > 0$, $\mathbf{1}$ is the unit of $\mathbf{C}$ and $p = (p_\tau)_\tau$ is an admissible tuple. Let V be a lattice of maximal rank in K and let $a, b \in \mathbf{R}$. In the integral of (4.1) we substitute

$$y \mapsto \pi |a+v|^2 y / d_V^{1/n}$$

where $|\,|$ denotes the map $\mathbf{R}^\times \to \mathbf{R}_+^\times$, $(x_\tau)_\tau \mapsto (|x_\tau|)_\tau$. We then obtain

(4.2)

$$\int_{\mathbf{R}_+^\times} e^{-\pi\langle (a+v)y/d_V^{1/n}, (a+v)\rangle} N(y^{s\mathbf{1}+\frac{1}{2}p}) \frac{dy}{y}$$
$$= \pi^{-Tr(\frac{1}{2}p)} (|d_V|^{1/n})^{Tr(\frac{1}{2}p)} |d_K|^s \pi^{-ns} \Gamma_X^p(s) \frac{[\mathcal{O}_K : V]^{2s}}{|N((a+v)^p)| N(a+v)|^{2s}}, \quad \Re(s) > 1.$$

Remember that $\mathcal{O}_V = \{\alpha \in K : \alpha V \subseteq V\}$. Let us denote by $\mathfrak{d}_K$ the discriminant ideal of K i.e., $\mathfrak{d}_K^{-1} = \{x \in K : Tr_{K/\mathbb{Q}}(xy) \in \mathbb{Z}$, for all $y \in \mathcal{O}_K\}$. From now on we identify V with $\iota(V) \subseteq \mathbf{R}$. We define

(4.3)
$$\Gamma_{a,b,V} = \{\epsilon \in \mathcal{O}_V : \sigma_i(\epsilon) > 0\ \forall i, (\epsilon - 1)a \in \iota(V), (\epsilon-1)b \in \iota(V^*), (\epsilon-1)ab \in \iota(\mathfrak{d}_K^{-1})\}.$$

One can verify that the subgroup $\Gamma_{a,b,V}$ has finite index in $\mathcal{O}_K^\times$. Let $\epsilon \in \Gamma_{a,b,V}$ and let $v \in V$ be such that $a+v \neq 0$. Since $\epsilon \in \Gamma_{a,b,V}$ we have $\epsilon(a+v) = a+v'$ for (a unique) $v' \in V$. Assume furthermore that the tuple $p = (p_\tau)_\tau$ is *strictly admissible*. Then a direct computation shows that

$$\frac{N((a+v)^p)}{|N((a+v)^p)|}\frac{e^{2\pi i\langle b,a+v\rangle}}{|N(a+v)|^{2s}} = \frac{N((a+v')^p)}{|N((a+v')^p)|}\frac{e^{2\pi i\langle b,a+v'\rangle}}{|N(a+v')|^{2s}}. \tag{4.4}$$

Note that $x \mapsto \frac{\mathbf{N}(x^p)}{|\mathbf{N}(x^p)|}$ is nothing else than a sign character of K i.e., a group homomorphism $\omega_p : (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \to \{\pm 1\}$.

Let $\mathfrak{R} = \{v_i \in V\}_{i\in I}$ be a complete set of representatives of $\{a + V\}/\Gamma_{a,b,V}$ in the sense that every element $0 \neq a+v \in a+V$ can be written uniquely as $\epsilon(a+v_i)$ for some $v_i \in \mathfrak{R}$ and $\epsilon \in \Gamma_{a,b,V}$. Let ω_p be the sign character associated to the *strictly admissible* tuple $p = (p_\tau)_\tau$. Using (4.2) and (4.4) we deduce that

$$\begin{aligned}
&\frac{1}{C}\int_{\mathbf{R}_+^\times}(\widetilde{\theta}_V^p(a,b,iy/d_V^{1/n}) - c_V^p(a,b))N(y^{s+\frac{1}{2}p})\frac{dy}{y} \\
&= |d_K|^s\pi^{-ns}[\mathcal{O}_K : V]^{2s}\Gamma_X^p(s)\sum_{\substack{v\in\mathfrak{R}\\ a+v\neq 0}}\omega_p(a+v)\frac{e^{2\pi i\langle b,a+v\rangle}}{|N(a+v)|^{2s}} \\
&= |d_K|^s\pi^{-ns}\Gamma_X^p(s)\Psi_V(a,b,\omega_p,2s).
\end{aligned}\tag{4.5}$$

where

$$\widetilde{\theta}_V^p(a,b,z/d_V^{1/n}) = \sum_{\substack{v\in\mathfrak{R}\\ a+v\neq 0}} N((a+v)^p)e^{\pi i\langle (a+v)z/d_V^{1/n},(a+v)\rangle}e^{2\pi i\langle b,a+v\rangle} \quad \text{for } z \in \mathbf{H},$$

$$c_V^p(a,b) = \lim_{z\to i\infty}\widetilde{\theta}_V^p(a,b,z/d_V^{1/n}) \quad \text{and} \quad C = \pi^{-Tr(\frac{1}{2}p)}(|d_V|^{\frac{1}{n}})^{Tr(\frac{1}{2}p)}.$$

Note that $c_V^p(a,b) = 0$ unless $p_\tau = 0$ for all $\tau \in X$ and $a \in V$. In the latter case we have $c_V^p(a,b) = 1$.

Definition 4.1 *We define the completed zeta function $Z_V(a,b,\omega_p,2s)$ to be*

$$Z_V(a,b,\omega_p,2s) := F_V^p(s)\Psi(a,b,\omega_p,2s).$$

where $F_V^p(s) = |d_K|^s\pi^{-ns}\Gamma_X^p(s)$.

The image of $\Gamma_{a,b,V}$ under the mapping $|\,| : \mathbf{R}^\times \to \mathbf{R}_+^\times$ is contained in the norm-one hypersurface

$$\mathbf{S} = \{x \in \mathbf{R}_+^\times : N(x) = 1\}.$$

We can write every $y \in \mathbf{R}_+^\times$ in the form

$$y = xt^{1/n}, \quad , x = \frac{y}{N(y)^{1/n}}, \quad t = N(y).$$

We thus obtain a direct decomposition

$$\mathbf{R}_+^\times = \mathbf{S} \times \mathbb{R}_+^\times.$$

We let d^*x be the unique Haar measure on the multiplicative group $\mathbf{S}$ such that the canonical Haar measure $\frac{dy}{y}$ on $\mathbf{R}_+^\times$ becomes the product measure

$$\frac{dy}{y} = d^*x \times \frac{dt}{t}.$$

Proposition 4.1 *The completed Zeta function $Z_V(a, b, \omega_p, 2s)$ is the Mellin transform*

$$\begin{aligned} Z_V(a, b, \omega_p, 2s) &= L(f, s) \\ &= \int_0^\infty (f(t) - f(\infty))t^s \frac{dt}{t}, \quad \Re(s) > 1, \end{aligned}$$

of the function

$$(4.6) \qquad f(t) = \frac{1}{C} \int_{\mathfrak{F}} \theta_V^p(a, b, \omega_p, ixt^{1/n}/d_V^{1/n}) N\left((ixt^{1/n}/d_V^{1/n})^{\frac{1}{2}p}\right) d^*x$$

where $\mathfrak{F}$ is a fondamental domain for the action of $\iota(\Gamma_{a,b,V}) \subseteq \mathbf{S}$ on $\mathbf{S}$,

$$C = \pi^{-Tr(\frac{1}{2}p)}(|d_V|^{\frac{1}{n}})^{Tr(\frac{1}{2}p)} \quad \text{and} \quad f(\infty) = \frac{c_V^p(a, b)}{C} vol(\mathfrak{F}).$$

Proof This is the same argument as the proof of Proposition (5.5) of [Neu99]. □

Lemma 4.1 *The fundamental domain $\mathfrak{F}$ of $\mathbf{S}$ has the following volume with respect to the measure d^*x:*

$$vol(\mathfrak{F}) = [\mathcal{O}_K^\times : \Gamma_{a,b,V}] R_K,$$

where R_K is the regulator of K.

Proof This is the same argument as Lemma (5.6) of [Neu99]. □

Proposition 4.2 *The function*

$$t \mapsto f(t) = \frac{1}{C} \int_{\mathfrak{F}} \theta_V^p(a, b, \omega_p, ixt^{1/n}/d_V^{1/n}) N\left((ixt^{1/n}/d_V^{1/n})^{\frac{1}{2}p} \right) d^*x$$

for $t \in \mathbb{R}_+$ satisfies the following functional equation

$$f\left(\frac{1}{t}\right) = (i^{Tr(p)} e^{-2\pi i \langle a,b \rangle})^{-1} \sqrt{t} g(t), \tag{4.7}$$

where

$$g(t) = \frac{1}{C} \int_{\mathfrak{F}'} \theta_{V^*}^p(-b, a, \omega_p, ixt^{1/n}/d_{V^*}^{1/n}) N\left((ixt^{1/n}/d_V^{1/n})^{\frac{1}{2}p} \right) d^*x,$$

and $\mathfrak{F}'$ is a fundamental domain for the action of $\iota(\Gamma_{-b,a,V^*})$ on $\mathbf{S}$ and $\Gamma = \Gamma_{a,b,V} \cap \Gamma_{-b,a,V^*}$. Morevoer, if we let

$$A_0 = \frac{c_V^p(a,b)}{C} \int_{\mathfrak{F}} N\left((ixt^{1/n}/d_V^{1/n})^{\frac{1}{2}p} \right) d^*x$$

which is equal to 0 unless $c_V^p(a,b) = 1$, in which case by Lemma 4.1 it is equal to $\frac{vol(\mathfrak{F})}{C}$, and similarly if we let

$$B_0 = \frac{c_{V^*}^p(-b,a)}{C} \int_{\mathfrak{F}'} N\left((ixt^{1/n}/d_V^{1/n})^{\frac{1}{2}p} \right) d^*x,$$

then

(4.8)
$$f(t) = A_0 + O(e^{-\alpha t^{1/n}}) \text{ and } g(t) = B_0 + O(e^{-\beta t^{1/n}}) \text{ for } t \to \infty \text{ and suitable } \alpha, \beta > 0.$$

Proof The proof of (4.8) follows from easy estimates. Let us prove (4.7). We have

$$\begin{aligned}
f(1/t) &= \frac{1}{C} \int_{\mathfrak{F}} \theta_V^p(a, b, \omega_p, ixt^{-1/n}/d_V^{1/n}) N((ixt^{-1/n}/d_V^{1/n})^{\frac{1}{2}p}) d^*x \\
&= \frac{1}{C} \int_{\mathfrak{F}} \theta_V^p(a, b, \omega_p, -1/(ix^{-1}(d_V t)^{1/n})) N((-1/(ix^{-1}(d_V t)^{1/n})^{\frac{1}{2}p}) d^*(x^{-1}) \\
&= \frac{1}{C} \int_{\mathfrak{F}^{-1}} \theta_V^p(a, b, \omega_p, -1/(ix(d_V t)^{1/n})) N((-1/(ix(d_V t)^{1/n})^{\frac{1}{2}p}) d^*x \\
&= \frac{1}{d_V t C} \int_{(d_V t)^{1/n} \mathfrak{F}^{-1}} \theta_V^p(a, b, \omega_p, -1/(iy)) N\left((-1/iy)^{\frac{1}{2}p} \right) d^*y \text{ where } y = x(d_V t)^{1/n}.
\end{aligned}$$

The second equality uses the fact that the transformation $x \mapsto x^{-1}$ fixes the Haar measure d^*x. The third equality uses the fact that $\frac{1}{d_V t} d^*y = d^*x$ where d^*y is the corresponding measure on $(d_V t)^{1/n} \mathfrak{F}^{-1}$ where the set $\mathfrak{F}^{-1} =$

$\{x^{-1} \in \mathbf{S} : x \in \mathfrak{F}\}$. Now applying Theorem 3.1 to the last equality we find that

$$f(1/t) = \frac{(i^{Tr(p)} e^{-2\pi i \langle a,b\rangle} \sqrt{d}_V)^{-1}}{d_V t C} \int_{(d_V t)^{1/n}\mathfrak{F}^{-1}} N(y^{p+\frac{1}{2}\mathbf{1}}) \theta^p_{V^*}(-b, a, \omega_p, iy) N\left((-1/iy)^{\frac{1}{2}p}\right) d^*y$$

$$= \frac{(i^{Tr(p)} e^{-2\pi i \langle a,b\rangle} \sqrt{d}_V)^{-1}}{d_V t C} \int_{(d_V t)^{1/n}\mathfrak{F}^{-1}} N(y^{\frac{1}{2}\mathbf{1}}) \theta^p_{V^*}(-b, a, \omega_p, iy) N\left((iy)^{\frac{1}{2}p}\right) d^*y$$

$$= (-i)^{Tr(p)} \frac{(e^{-2\pi i \langle a,b\rangle} \sqrt{d_V d_{V^*}})^{-1}}{d_V d_{V^*} C} \sqrt{t}$$

$$\cdot \int_{(d_V d_{V^*})^{1/n}\mathfrak{F}^{-1}} \theta^p_{V^*}(-b, a, \omega_p, iut^{1/n}/d_{V^*}^{1/n}) N\left((iut^{1/n}/d_{V^*}^{1/n})^{\frac{1}{2}p}\right) d^*u$$

where $y = ut^{1/n}/d_{V^*}^{1/n}$ and $dy^* = \frac{t}{d_{V^*}} d^*u$ where d^*u is the corresponding measure on $(d_V/d_{V^*})^{1/n}\mathfrak{F}^{-1}$. A direct compuation shows that $d_V d_{V^*} = 1$ and that $\mathfrak{F}^{-1}$ is a fundamental domain for the action of $\iota(\Gamma_{a,b,V})$ on $\mathbf{S}$. We can thus rewrite the last equality as

(4.9)

$$f(1/t) = \frac{(-i)^{Tr(p)} e^{2\pi i \langle a,b\rangle}}{C} \sqrt{t} \int_{\mathfrak{F}^{-1}} \theta^p_{V^*}(-b, a, \omega_p, iut^{1/n}/d_{V^*}^{1/n}) N\left((iut^{1/n}/d_{V^*}^{1/n})^{\frac{1}{2}p}\right) d^*u.$$

Note that $\mathfrak{F}^{-1}$ is also a fundamental domain for the action of $\iota(\Gamma_{-b,a,V^*}) = \iota(\Gamma_{a,b,V})$ on $\mathbf{S}$. Therefore using (4.9) we deduce that

$$f(1/t)$$

$$= \frac{(i^{Tr(p)} e^{-2\pi i \langle a,b\rangle})^{-1}}{C} \sqrt{t} \int_{\mathfrak{F}'} \theta^p_{V^*}(-b, a, \omega_p, iut^{1/n}/d_{V^*}^{1/n}) N\left((iut^{1/n}/d_{V^*}^{1/n})^{\frac{1}{2}p}\right) d^*u.$$

We are now ready to prove Theorem 1.1.

Proof of Theorem 1.1 Assume that $\Re(s) > 1$. From Proposition 4.1 we get

$$\begin{aligned} Z_V(a, b, \omega_p, 2s) &= \int_0^\infty (f(t) - A_0) t^s \frac{dt}{t} \\ &= \int_0^1 (f(t) - A_0) t^s \frac{dt}{t} + \int_1^\infty (f(t) - A_0) t^s \frac{dt}{t} \\ &= \int_1^\infty f(1/t) t^{-s} \frac{dt}{t} + \int_1^\infty (f(t) - A_0) t^s \frac{dt}{t} - \frac{A_0}{s}. \end{aligned}$$

Substituting (4.7) in the first integral of the last equality we get

(4.10)

$$\begin{aligned} Z_V(a, b, \omega_p, 2s) = (-i)^{Tr(p)} e^{2\pi i \langle a,b\rangle} &\int_1^\infty (g(t) - B_0) t^{-s+1/2} \frac{dt}{t} \\ + \int_1^\infty (f(t) - A_0) t^s \frac{dt}{t} - \frac{A_0}{s} &- \frac{B_0}{-s + \frac{1}{2}} (-i)^{Tr(p)} e^{2\pi i \langle a,b\rangle}. \end{aligned}$$

The two integrals of the last equality converge for all complex number $s \in \mathbb{C}$. It follows from this that $Z_V(a, b, \omega_p, 2s)$ admits an analytic continuation to all $s \in \mathbb{C} \backslash \{0, \frac{1}{2}\}$. This proves the first part of Theorem 1.1. The equality (4.10) can be rewritten as

$$\begin{aligned} &i^{Tr(p)} e^{-2\pi i \langle a,b \rangle} Z_V(a, b, \omega_p, 2s) = \\ (4.11) \qquad &\int_1^\infty (g(t) - B_0) t^{-s+1/2} \frac{dt}{t} + e^{-2\pi i \langle a,b \rangle} \int_1^\infty (f(t) - A_0) t^s \frac{dt}{t} \\ &- \frac{B_0}{-s + \frac{1}{2}} - \frac{A_0}{s} i^{Tr(p)} e^{-2\pi i \langle a,b \rangle}. \end{aligned}$$

In the exact same way as we obtained (4.10) one has that

$$\begin{aligned} (4.12) \qquad Z_{V^*}(-b, a, \omega_p, 2s) &= (-i)^{Tr(p)} e^{-2\pi i \langle a,b \rangle} \int_1^\infty (f(t) - A_0) t^{-s+1/2} \frac{dt}{t} \\ &+ \int_1^\infty (g(t) - B_0) t^s \frac{dt}{t} - \frac{A_0}{-s + \frac{1}{2}} (-i)^{Tr(p)} e^{-2\pi i \langle a,b \rangle} - \frac{B_0}{s} \end{aligned}$$

Substituting s by $-s + 1/2$ in (4.11) and comparing it with (4.12) reveals that

$$(-i)^{Tr(p)} e^{-2\pi i \langle a,b \rangle} Z_V(a, b, \omega_p, 2(-s + 1/2)) = Z_{V^*}(-b, a, \omega_p, 2s).$$

This concludes the proof. □

5 Partial zeta functions twisted by sign characters

Let K be a number field of degree $n = r_1 + 2r_2$ over $\mathbb{Q}$ with different ideal $\mathfrak{d}_K$. Let $\mathfrak{f}$ be an integral ideal of $\mathcal{O}_K$. Choose an integral ideal $\mathfrak{b}$ which is prime to $\mathfrak{f}$ and consider the lattice $V = \frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}_K}$. Note that $V^* = \mathfrak{f}\mathfrak{b}^{-1}$. Let $a = 0$, $b = -1$ and $p = (p_\tau)_\tau$ be a strictly admissible tuple corresponding to a sign character $\omega_p : (K \otimes_{\mathbb{Q}} \mathbb{R})^\times \to \{\pm 1\}$. By definition we have

$$\Gamma_{0,-1,V} = \mathcal{O}_K(\infty)^\times \cap (1 + \mathfrak{f}\mathfrak{b}^{-1}),$$

where $\mathcal{O}_K(\infty)^\times$ corresponds to the group of totally positive units of $\mathcal{O}_K$. A direct computation shows that

$$\Psi_V(0, -1, \omega_p, s) = \mathbf{N}_{K/\mathbb{Q}} \left(\frac{\mathfrak{b}}{\mathfrak{d}_K \mathfrak{f}} \right)^s \sum_{\left\{0 \neq \mu \in \frac{\mathfrak{b}}{\mathfrak{f}\mathfrak{d}_K}\right\}/\Gamma} \omega_p(\mu) \frac{e^{2\pi i Tr_{K/\mathbb{Q}}(\mu)}}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s}, \quad \Re(s) > 1,$$

where $\Gamma = \Gamma_{0,-1,V}$. When ω_p is trivial, the right hand side of the last equality is an example of what we call a zeta function twisted by an additive character. Similarly one has

$$\Psi_{V^*}(1,0,\omega_p,s) = \mathbf{N}_{K/\mathbb{Q}}\left(\mathfrak{f}\mathfrak{b}^{-1}\right)^s \sum_{\{0\neq\mu\in 1+\mathfrak{f}\mathfrak{b}^{-1}\}/\Gamma} \frac{\omega_p(\mu)}{|\mathbf{N}(\mu)|^s}$$

$$(5.1) \qquad = \mathbf{N}_{K/\mathbb{Q}}(\mathfrak{f})^s \left(\sum_{\{0\neq\mu\in 1+\mathfrak{f}\mathfrak{b}^{-1}\}/\Gamma} \frac{\omega_p(\mu)}{|\mathbf{N}_{K/\mathbb{Q}}(\mu\mathfrak{b})|^s} \right) \quad \Re(s) > 1.$$

The right hand side of the last equality is an example of what we call a partial zeta function twisted by a sign character. From Theorem 1.1 we deduce that

$$(5.2) \qquad (-i)^{Tr(p)} \frac{F_V^p(s/2)}{F_{V^*}^p((1-s)/2)} \Psi_V(0,-1,\omega_p,s) = \Psi_{V^*}(1,0,\omega_p,1-s),$$

where $\frac{F_V^p(s/2)}{F_{V^*}^p((1-s)/2)}$ is given explicitely by

(5.3)

$$\frac{F_V^p(s/2)}{F_{V^*}^p((1-s)/2)} = \frac{|d_K|^{s/2}\pi^{-ns/2}}{|d_K|^{(1-s)/2}\pi^{-n(1-s)/2}} \frac{\prod_{i=1}^{r_1}\Gamma\left(\frac{s+p_i}{2}\right)\prod_{i=1}^{r_2}(2^{1-s}\Gamma(s))}{\prod_{i=1}^{r_1}\Gamma\left(\frac{1-s+p_i}{2}\right)\prod_{i=1}^{r_2}(2^{s}\Gamma(1-s))}.$$

For any $s \in \mathbb{Z}_{\geq 1}$ the function $\Psi_V(0,-1,\omega_p,s)$ is holomorphic at s (for $s = 1$ this uses the fact that $\mathfrak{f} \nmid \mathfrak{b}$). From (5.2) and (5.3) we deduce that for $s \in \mathbb{Z}_{\geq 1}$ the value

$$\Psi_{V^*}(1,0,\omega_p,1-s) \neq 0$$

only when

(1) $r_2 = 0$, $s \equiv 0 \pmod 2$ and $p_i = 0$ for all i

or

(2) $r_2 = 0$, $s \equiv 1 \pmod 2$ and $p_i = 1$ for all i,

holds true.

For a given totally real number field K, the relationships between special values at negative integers of $\Psi_{V^*}(1,0,\omega_p,s)$ and classical partial zeta functions as in (1.8) was treated in section 2 of [Cha07].

Remark 5.1 In [Del79], Deligne introduced the notion of critical integers for an L-function $L(M,s)$ attached to a motive M. In the absence of a Motive associated to the zeta function $\Psi_V(a,b,\omega,s)$, it seems to be still useful to introduce the notion of critical integer. We will say that an integer n is critical for the zeta function $\Psi_V(a,b,\omega_p,s)$ if and only if

$$F_V^p(n/2) \neq \infty \text{ and } F_{V^*}^p((1-n)/2) \neq \infty \Longleftrightarrow \frac{F_V^p(n/2)}{F_{V^*}^p((1-n)/2)} \neq 0, \infty.$$

Let S be the set of critical integers of $\Psi_V(a, b, \omega_p, s)$. A direct calculation, similar to the one we did previously, shows that

$$S = \begin{cases} 2\mathbb{Z}_{\geq 1} \cup (1 + 2\mathbb{Z}_{\leq -1}) & \text{if} \quad r_2 = 0, \omega_p = 1, \\ (1 + 2\mathbb{Z}_{\geq 0}) \cup 2\mathbb{Z}_{\leq 0} & \text{if} \quad r_2 = 0, \omega_p = sign \circ \mathbf{N}_{K/\mathbb{Q}}, \\ \emptyset & \quad\quad \text{otherwise} \end{cases}$$

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