p-Units in ray class fields of real quadratic number fields

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Abstract

Let K be a real quadratic number field and let p be a prime number which is inert in K. We denote the completion of K at the place p by K_p. We propose a p-adic construction of special elements in $K_p^\times$ and formulate the conjecture that they should be p-units lying in narrow ray class fields of K. The truth of this conjecture would entail an explicit class field theory for real quadratic number fields. This construction can be viewed as a natural generalization of a construction obtained by Darmon and Dasgupta who proposed a p-adic construction of p-units lying in narrow ring class fields of K.

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1. Introduction

Let K be a real quadratic number field and let p be a prime number which is inert in K. Darmon and Dasgupta proposed a p-adic construction of special elements $u \in K_p^\times$ where K_p stands for the completion of K at p. The two authors have conjectured that u is a p-unit in L, i.e., $u \in \mathcal{O}_L[\frac{1}{p}]^\times$, where L is a suitable *narrow ring class field* of K. Moreover they also predicted that for all infinite places ν of L, $|u|_\nu = 1$. Because of the last condition it is essential to assume beforehand that L is a totally complex field, otherwise $u = \pm 1$, thus the importance of working in the narrow sense. In fact it is not too hard to see that such a $u \neq \pm 1$ is necessarily contained in a CM-field. As is explained in the introduction of [DD06], those conjectural p-units can be thought of as analogues of classical elliptic units which are constructed by evaluating modular functions at imaginary quadratic

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numbers. Darmon and Dasgupta also constructed a p-adic L-function which interpolates $\mathbb{Z}$-linear combinations of special values of partial zeta functions attached to L/K and related it to their invariant u. This is the so-called p-adic Kronecker limit formula. The first goal of this paper is to extend their p-adic construction to the case where L is a *narrow ray class field of* K. The transition from a narrow ring class field situation to the case of a narrow ray class field is a natural interesting question raised by Darmon and other experts and requires some subtle refinements but the main central ideas come from [DD06]. The second goal is to prove a p-adic Kronecker limit formula which allows us to relate the first derivative of a certain p-adic zeta function to our p-adic invariant. The approach used to define our p-adic invariant is similar to the one developed in [DD06] but it is more direct since the p-adic measures appearing in our construction are known to be $\mathbb{Z}$-valued rather than just $\mathbb{Z}_p$-valued. The analogue of this result in the context of ring class fields was not available when the paper [DD06] was written, but it was later proved by Dasgupta (see Theorem 1.3 of [Das07a]) and proof could be adapted to the more general setting of ray class fields (see Theorem 13.1 of [Cha]).

We now describe the construction of our p-adic invariant and its appearance in a p-adic Kronecker limit formula. We need first to fix some notation and definitions. Let (p, N_0, f) be a triple of strictly positive integers which are pairwise coprime and where p is a prime number. Also, fix a pair $(K, \mathfrak{N})$ where K is real quadratic number field with ring of algebraic integers $\mathcal{O}_K$ and $\mathfrak{N}$ is an integral $\mathcal{O}_K$-ideal such that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N_0\mathbb{Z}$ ("Heegner hypothesis"). Finally, we also require the prime number p to be inert in K.

Definition 1.1 *We define $D(N_0, f)$ to be the free abelian group generated by the symbols $\{[d_0, r] : 0 < d_0|N_0, r \in \mathbb{Z}/f\mathbb{Z}\}$. If $\delta \in D(N_0, f)$ we call f the conductor of δ and N_0 the level of δ. A typical element $\delta \in D(N_0, f)$ will be denoted by*

$$\delta = \sum_{d_0|N_0,r} n(d_0, r)[d_0, r],$$

where the sum goes over $d_0|N_0$ ($d_0 > 0$) and $r \in \mathbb{Z}/f\mathbb{Z}$ with $n(d_0, r) \in \mathbb{Z}$. We have a natural action of $(\mathbb{Z}/f\mathbb{Z})^\times$ on $D(N_0, f)$ given by $j \star [d_0, r] := [d_0, jr]$ where $j \in (\mathbb{Z}/f\mathbb{Z})^\times$ and we extend this action $\mathbb{Z}$-linearly to all of $D(N_0, f)$. We will use the short hand notation

$$\delta_j := j \star \delta.$$

Let $\delta = \sum_{d_0|N_0,r\in\mathbb{Z}/f\mathbb{Z}} n(d_0, r)[d_0, r] \in D(N_0, f)$ be such that the integers $n(d_0, r)$ are subject to the following three conditions

1. *If $r \equiv 0 \pmod f$ then for all $d_0|N_0$ we have $n(d_0, r) = 0$,*
2. *For all $r \in \mathbb{Z}/f\mathbb{Z}$, $\sum_{d_0|N_0} n(d_0, r)d_0 = 0$,*
3. *For all $d_0|N_0$ and $r \in \mathbb{Z}/f\mathbb{Z}$, $n(d_0, pr) = n(d_0, r)$.*

An element $\delta \in D(N_0, f)$ satisfying (1) and (2) and (3) will be called a **good divisor** *for the triple (N_0, f, p).*

We want to associate Eisenstein series to any good divisor $\delta \in D(N_0, f)$. Let

$$E_k(r, \tau) := \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (0,0)\neq(m,n)}} \frac{e^{-2\pi i m \frac{r}{f}}}{(m + nf\tau)^k} \tag{1}$$

$$= \frac{-\widetilde{B}_k(-r/f)}{k} + \frac{1}{f^k}\sum_{b=0}^{f-1} e^{-2\pi i br/f} \sum_{m\geq1}\sum_{n\geq1} m^{k-1}(q^m_{n\tau+b/f} + (-1)^k q^m_{n\tau-b/f}),$$

where $\tau \in \mathcal{H} = \{x+iy \in \mathbb{C} : y > 0\}$ stands for the complex upper half plane, $r \in \mathbb{Z}/f\mathbb{Z}$, $q_{n\tau+b/f} = e^{2\pi i(n\tau+b/f)}$ and $\widetilde{B}_k(x) := B_k(\{x\})$ where $B_k(x)$ is the k-th Bernoulli polynomial and $0 \le \{x\} < 1$ is the fractional part of x. When $k \ge 3$ the convergence of the right hand side of (1) is absolute and therefore $E_k(r,\tau)$ is a modular form of weight k for the modular group $\Gamma_1(f)$. When $k = 2$ the convergence is not absolute. Nevertheless, the corresponding q-expansion of (1) still converges and therefore we take it as the definition of $E_2(r,\tau)$. In the case where $r \not\equiv 0 \pmod f$ and $k = 2$, one can show that $E_k(r,\tau)$ satisfies the correct transformation formula and therefore corresponds to a holomorphic modular form of weight 2 for the modular group $\Gamma_1(f)$. In a similar way, we also define

$$E_k^*(r,\tau) := \left(\frac{(-1)^k(2\pi i)^k}{(k-1)!}\right)^{-1} \sum_{\substack{(m,n)\in\mathbb{Z}^2 \\ (0,0)\neq(m,n)}} \frac{e^{2\pi i n\frac{r}{f}}}{(m+n\tau)^k}$$

$$= \frac{-\widetilde{B}_k(r/f)}{k} + \sum_{b=0}^{f-1} e^{2\pi i b r/f} \sum_{m\ge1}\sum_{n\ge1} m^{k-1}(q^m_{(fn+b)\tau} + (-1)^k q^m_{(fn-b)\tau}).$$

The Eisenstein series $E_k(r,\tau)$ and $E_k^*(r,\tau)$ are related by the formula

$$E_k^*(r,\tau) = det(W_f)^k E_k\left(r, W_f\tau\right)\left(W_f\tau\right)^k, \tag{2}$$

where $W_f = \begin{pmatrix} 0 & -1 \\ f & 0 \end{pmatrix}$.

Next we want to associate Eisenstein series to a good divisor $\delta \in D(N_0, f)$.

Definition 1.2 *Let* $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)$ *be a fixed good divisor. To any integer* $k \ge 2$ *we associate the Eisenstein series*

$$F_{k,\delta}(\tau) := \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} d_0 n(d_0,r) E_k(r, d_0\tau) \quad \text{and} \quad F^*_{k,\delta}(\tau) := \sum_{d_0,r} d_0^{k-1} n\left(\frac{N_0}{d_0}, r\right) E_k^*(r, d_0\tau),$$

and

$$F_{k,\delta,p}(\tau) := F_{k,\delta}(\tau) - p^{k-1}F_{k,\delta}(p\tau) \quad \text{and} \quad F^*_{k,\delta}(\tau) - p^{k-1}F^*_{k,\delta}(p\tau),$$

which are related by the formula

$$F_{k,\delta}(W_{fN_0}\tau) = (-1)^k\tau^k N_0 F^*_{k,\delta}(\tau), \quad \text{where } W_{fN_0} = \begin{pmatrix} 0 & -1 \\ fN_0 & 0 \end{pmatrix}. \tag{3}$$

For every $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ *we set*

$$\widetilde{F}_k(r,z) := -12fF_{k,\delta_r}(z) \quad \text{and} \quad \widetilde{F}_{k,p}(r,z) := -12fF_{k,\delta_r,p}(z),$$

and similarly we set

$$\widetilde{F}^*_k(r,z) := -12F^*_{k,\delta_r}(z) \quad \text{and} \quad \widetilde{F}^*_{k,p}(r,z) := -12F^*_{k,\delta_r,p}(z).$$

In the definition of a good divisor we have forced the condition $n(d_0,r) = 0$ for all $d_0|N_0$ when $r \equiv 0 \pmod f$, because we want the function $\widetilde{F}_2(r,\tau)$ to satisfy the correct transformation formula, i.e., we want $\widetilde{F}_2(r,\tau)$ to be a modular form of weight 2. For a fixed integer $k \ge 2$, we can think of $\{\widetilde{F}_k(r,z)\}_{(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ as a family of Eisenstein series indexed by elements of $(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$. For any $\gamma \in \Gamma_0(f)$ we have the transformation formula

$$\widetilde{F}_k(\gamma \star r, \gamma\tau)(c\tau+d)^{-k} = \widetilde{F}_k(r,\tau),$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star r := dr \pmod{f}$. A similar formula holds for $\widetilde{F}_k^*(r,\tau)$. Because the divisor δ satisfies the condition (2), the constant terms of the q-expansions of $\widetilde{F}_k(r,\tau)$ (resp. $\widetilde{F}_k^*(r,\tau)$) vanish at the cusps $\Gamma_0(fN_0)\{\infty\}$ (resp. $\Gamma_0(fN_0)\{0\}$) where ∞ stands for the cusp $\frac{1}{0}$. It is "well known" that the period integrals

$$\int_{c_1}^{c_2} z^n \widetilde{F}_k(r,z)dz \tag{4}$$

are rational numbers, for $c_1, c_2 \in \Gamma_0(fN_0)\{\infty\}$ and $0 \leq n < k$. For explicit formulas of these periods given in terms of Dedekind sums see Proposition 11.1 of [Cha].

We need to introduce some background about p-adic integration. Let

$$\mathbb{X} := (\mathbb{Z}_p \times \mathbb{Z}_p)\backslash(p\mathbb{Z}_p \times p\mathbb{Z}_p).$$

Definition 1.3 *Let A be an abelian group. An A-valued distribution on $\mathbb{X}$ is a map*

$$\mu : \{\textit{Compact open sets of } \mathbb{X}\} \to A$$

which is finitely additive, i.e., for any disjoint union, $\bigcup_{i=1}^n U_i$, of compact open sets of $\mathbb{X}$ we have

$$\mu\left(\bigcup_{i=1}^n U_i\right) = \sum_{i=1}^n \mu(U_i).$$

A distribution is said to be a **measure** *if A can be chosen to be a bounded subgroup of $\mathbb{Q}_p$.*

Let

$$\widetilde{\Gamma}_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \gamma \in GL_2(\mathbb{Z}[1/p]) : \; det(\gamma) > 0 \,, c \equiv 0 \pmod{fN_0} \right\}.$$

Note that the orbit $\widetilde{\Gamma}_0\{\infty\} = \Gamma_0(fN_0)\{\infty\}$.

The next theorem is the crucial technical ingredient for the definition of our p-adic invariant.

Theorem 1.1 *There exists a unique collection of p-adic measures $\widetilde{\mu}_r\{c_1 \to c_2\}$ on $(\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0)$ taking values in $\mathbb{Z}$ and indexed by triples*

$$(r, c_1, c_2) \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle \overline{p} \rangle \times \widetilde{\Gamma}_0\{\infty\} \times \widetilde{\Gamma}_0\{\infty\},$$

such that:

(1) *For every homogeneous polynomial $h(x,y) \in \mathbb{Z}_p[x,y]$ of degree $k-2$,*

$$\int_{\mathbb{X}} h(x,y) d\widetilde{\mu}_r\{c_1 \to c_2\}(x,y) = (1-p^{k-2}) \int_{c_1}^{c_2} h(z,1)\widetilde{F}_k(r,z)dz,$$

(2) *For all $\gamma \in \widetilde{\Gamma}_0$ and all compact open subset $U \subseteq \mathbb{Q}_p^2\backslash(0,0)$,*

$$\widetilde{\mu}_r\{c_1 \to c_2\}(U) = \widetilde{\mu}_{\gamma \star r}\{\gamma c_1 \to \gamma c_2\}(\gamma U),$$

(3) *For every homogeneous polynomial $h(x,y) \in \mathbb{Z}_p[x,y]$ of degree $k-2$,*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} h(x,y) d\widetilde{\mu}_r\{c_1 \to c_2\}(x,y) = \int_{c_1}^{c_2} h(z,1)\widetilde{F}_{k,p}(r,z)dz.$$

Proof See Section 3. $\square$

Remark 1.1 A similar statement is true if one replaces $\widetilde{F}_k(r,z)$ by $\widetilde{F}_k^*(r,z)$ and the orbit $\widetilde{\Gamma}_0\{\infty\}$ by $\widetilde{\Gamma}_0\{0\}$.

Now we need to introduce certain notions in order to give a precise definition of our p-adic invariant. Let $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$ be the so-called p-adic upper half plane endowed with its structure of rigid analytic space and let K_p be the completion of K at the prime p. Note that $\mathcal{H}_p \cap K \neq \emptyset$. For certain pairs $(r,\tau) \in \mathbb{Z}/f\mathbb{Z} \times (\mathcal{H}_p \cap K)$ we want to associate a p-adic invariant $u(r,\tau) \in K_p^\times$. Let us fix an embedding $K \subseteq \mathbb{R}$. For every $\tau \in K - \mathbb{Q}$ we define the order $\mathcal{O}_\tau$ as $End_K(\Lambda_\tau)$ where Λ_τ is the lattice $\mathbb{Z} + \tau\mathbb{Z}$. Let $\mathcal{O}$ be an order of K of conductor coprime to N_0 and let $\mathfrak{n} = \mathfrak{N} \cap \mathcal{O}$. Note that $\mathcal{O}/\mathfrak{n} \sim \mathbb{Z}/N_0\mathbb{Z}$. A pair $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times (\mathcal{H}_p \cap K)$ is said to be $(\mathcal{O},\mathfrak{n})$-*admissible* if $\mathcal{O} = \mathcal{O}_\tau = \mathcal{O}_{N_0\tau}$, $\Lambda_{N_0\tau} = \mathfrak{n}\Lambda_\tau$, and if $\tau - \tau^\sigma > 0$ where σ is the non trivial automorphism of K. When there is no need to specify the pair $(\mathcal{O},\mathfrak{n})$ we simply say that the pair (r,τ) is admissible. In Section 4 we give some motivation for the notion of admissibility which we develop further. We also introduce an important relation of equivalence on admissible pairs which we denote by $\sim$ (see Definition 4.4 and Remark 4.3). We are now ready to define our p-adic invariant.

Definition 1.4 *For every admissible pair $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times (\mathcal{H}_p \cap K)$ such that τ is reduced (see Definition 1.6), we define the p-adic invariant*

$$u(\delta_r,\tau) = u(r,\tau) := p^{\psi_r\{\infty\to\gamma_\tau\infty\}} \times\!\!\!\!\!\!\int_{\mathbb{X}} (x-\tau y)d\widetilde{\mu}_r\{\infty \to \gamma_\tau\infty\}(x,y) \in K_p^\times, \tag{5}$$

where γ_τ is an oriented generator of the stabilizer of τ under the action of Γ_1 (see Definition 1.5), i.e., γ_τ is chosen in such a way that it generates the quotient $Stab_{\Gamma_1}(\tau)/\langle\pm 1\rangle \simeq \mathbb{Z}$ and

$$\gamma_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}$$

with $\epsilon > 1$. For any pair of cusps $c_1, c_2 \in \Gamma_0(fN_0)\{\infty\}$, the quantity $\psi_r\{c_1 \to c_2\}$ is defined by the following integral

$$\psi_r\{c_1 \to c_2\} := \frac{1}{2\pi i}\int_{c_1}^{c_2} \widetilde{F}_2(r,\tau)d\tau, \tag{6}$$

where the complex line integral on the right hand side is taken along the unique geodesic C in the complex upper half plane $\mathcal{H}$ connecting the cusps c_1 and c_2.

Remark 1.2 One can define in an analoguous way a p-adic invariant $u^*(r,\tau)$ by replacing the Eisenstein series $\widetilde{F}_k(r,\tau)$ in the statement of Theorem 1.1 by the Eisenstein series $\widetilde{F}_k^*(r,\tau)$.

It is explained in Section 2 that the rational number $\psi_r\{c_1 \to c_2\}$ is in fact always an integer. Some explanations about the multiplicative integral appearing in (5) are in order. This p-adic integral is defined by

$$\times\!\!\!\!\!\!\int_{\mathbb{X}} (x-\tau y)d\widetilde{\mu}_r\{\infty \to \gamma_\tau\infty\}(x,y) := \lim_{||\mathcal{U}||\to 0} \prod_{U\in\mathcal{U}} (x_U - \tau y_U)^{\widetilde{\mu}_r\{\infty\to\gamma_\tau\infty\}(U)} \in K_p^\times, \tag{7}$$

where $\mathcal{U}$ is a cover of $\mathbb{X}$ by disjoint compact open sets, (x_U, y_U) is an arbitrary point of $U \in \mathcal{U}$, and the p-adic limit is taken over increasingly fine covers $\mathcal{U}$. The product in (7) makes sense since the measures $\widetilde{\mu}_r\{c_1 \to c_2\}$ are $\mathbb{Z}$-valued and not only $\mathbb{Z}_p$-valued

The appellation p-adic invariant for the quantity $u(r,\tau)$ is appropriate in light of the following theorem.

Theorem 1.2 *Let (r,τ) and (r',τ') be $(\mathcal{O},\mathfrak{n})$-admissible pairs such that τ and τ' are reduced and*

$$\gamma \star (r,\tau) := (\gamma \star r, \gamma\tau) = (r',\tau'), \quad \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \widetilde{\Gamma}_0.$$

Then

$$u(r,\tau) \equiv u(r',\tau') \pmod{(K_p^\times)_{tor}}, \tag{8}$$

where $(K_p^\times)_{tor} = \mu_{p^2-1}$.

It is a natural question to ask if (8) remains valid without the modulo $\pmod{\mu_{p^2-1}}$. The author did not attempt to prove it but numerical examples suggest that this refinement is true. Under a mild assumption on τ, one can show that $(r,\tau) \sim (r',\tau')$ if and only if there exists a $\gamma \in \widetilde{\Gamma}_0$ such that $\gamma \star (r,\tau) = (r',\tau')$ (see Section 4).

When $\mathcal{O} = \mathcal{O}_K$, we conjecture that the element $u(r,\tau)$ lies in the narrow ray class field K of conductor f which we denote by $K(f\infty)$. To be precise, assume that the minimal quadratic polynomial with integer coefficients satisfied by τ has the form

$$A\tau^2 + B\tau + C = 0, \quad (A,B,C) = 1, \quad A > 1$$

where $N_0|A$ and $B^2 - 4AC = disc(K)$.

Conjecture 1.1 *Let $L = K(f\infty)^{Frob(\mathfrak{p}/\wp)}$ where $\wp = p\mathcal{O}_K$ and $\mathfrak{p}$ is a prime ideal of $K(f\infty)$ above $\wp$. Then the element $u(r,\tau) \in K_p^\times$ is a "strong p-unit" in L i.e., an element of $\mathcal{O}_L[\frac{1}{p}]^\times$, such that $|u(r,\tau)|_\nu = 1$ for all infinite places ν of L.*

In Subsection 4.1 (see Conjecture 4.1), we propose a *conjectural Shimura reciprocity law* which describes the action of $Gal(\overline{K}/K)$ on $u(r,\tau)$.

In Section 5 we introduce the zeta functions $\zeta^*(\delta,(r,\tau),s)$ (resp. $\zeta(\delta,(r,\tau),s)$) which interpolates certain periods of $\widetilde{F}_k(r,\tau)$ with respect to the cusp ∞ (resp. certain periods of $\widetilde{F}_k^*(r,\tau)$ with respect to the cusp 0). The reader should keep in mind the following diagram:

$$\text{periods of } \widetilde{F}_k(r,z) \rightsquigarrow \zeta^*(\delta,(r,\tau),s) \text{ and } u(r,\tau),$$
$$\text{periods of } \widetilde{F}_k^*(r,z) \rightsquigarrow \zeta(\delta,(r,\tau),s) \text{ and } u^*(r,\tau).$$

Finally in Section 6, we prove a p-adic analogue of the Kronecker limit formula which relates our p-adic invariant $u(r,\tau)$ to the first derivative at $s=0$ of a certain p-adic zeta function. More precisely we prove that

(1) $3(\zeta_p^*)'(\delta,(r,\tau),0) = -\log_p N_{K_p/\mathbb{Q}_p}(u(r,\tau))$,

(2) $3\zeta^*(\delta,(r,\tau),0) = v_p(u(r,\tau))$,

where $\zeta_p^*(\delta,(r,\tau),s)$ is a p-adic zeta function interpolating the special values

$$(1-p^{-2n})\zeta^*(\delta,(r,\tau),n)$$

for even integer $n \leq 0$ such that $n \equiv 0 \pmod{p-1}$.

Here we want to point out that our choice of working with the periods of $\widetilde{F}_k(r,z)$ (with respect to the cusp ∞) rather than the one coming from $\widetilde{F}_k^*(r,z)$ (with respect to the cusp 0) is not necessarily the best choice. For example, the formulas which relate the special values of $\zeta(\delta,(r,\tau),s)$ to the special values of classical partial zeta functions K are much simpler than the one showing up in the case of $\zeta^*(\delta,(r,\tau),s)$. Nevertheless, we have decided to work with the periods of $\widetilde{F}_k(r,z)$ rather than the one coming from $\widetilde{F}_k^*(r,z)$ since this has the advantage of simplifying the formulas which relate the Darmon-Dasgupta invariant to the p-adic invariant $u(r,\tau)$, see [Cha07b]. Note that in the special case where $f=1$, which was the case considered in [DD06], $\zeta_p^*(\delta,(r,\tau),s) = \zeta_p(\delta,(r,\tau),s)$, which becomes false when $f>1$. This can be accounted by the fact that the two cusps 0 and ∞ are inequivalent modulo $\Gamma_0(f)$ when $f>1$.

In [Das07b], Dasgupta proposed a conjectural p-adic construction of p-units lying in narrow ray class fields of any totally real number field. In particular, his method allows him to construct p-units in narrow ray class fields of a real quadratic number field K. However, his new method is rather different from the modular symbols approach initiated in [DD06] and the one developed here. One special feature of our modular symbols approach is the possibility of computing in polynomial time the p-adic invariant $u(r,\tau) \in K_p^\times$. For numerical examples which support the conjectural algebraicity of $u(r,\tau)$ see [Das07a], [Cha] and [Cha07a].

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Notation

Let K be a real quadratic number field and $\mathcal{O}$ be a $\mathbb{Z}$-order of K. Let $\mathcal{O}_K = \mathbb{Z} + \omega\mathbb{Z}$ be the maximal $\mathbb{Z}$-order of K. Every $\mathbb{Z}$-order $\mathcal{O}$ of K can be written uniquely as $\mathcal{O} = \mathbb{Z} + n\omega\mathbb{Z}$ where $n \in \mathbb{Z}_{>0}$ is called the conductor of $\mathcal{O}$. An $\mathcal{O}$-module $\Lambda \subseteq K$ will be called an $\mathcal{O}$-ideal. An $\mathcal{O}$-ideal $\mathfrak{a}$ will be called integral if $\mathfrak{a} \subseteq \mathcal{O}$. By an invertible $\mathcal{O}$-ideal we mean an $\mathcal{O}$-ideal Λ such that $End_K(\Lambda) = \{\lambda \in K : \lambda\Lambda \subseteq \Lambda\} = \mathcal{O}$. Note that if $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals then $\mathfrak{a} \cap \mathfrak{b}$ is an invertible $\mathcal{O}$-ideal. However, if $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals, then in general $(\mathfrak{a}, \mathfrak{b}) := \mathfrak{a} + \mathfrak{b}$ is not an invertible $\mathcal{O}$-ideal. If $\mathfrak{c}$ is an invertible ideal and $\mathfrak{f}$ is an integral $\mathcal{O}$-ideal then we say that $(\mathfrak{c}, \mathfrak{f}) = 1$ if there exists two $\mathcal{O}$-invertible integral ideals $\mathfrak{a}, \mathfrak{b}$ such that $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$, $\mathfrak{a} + \mathfrak{f} = \mathcal{O}$ and $\mathfrak{b} + \mathfrak{f} = \mathcal{O}$. Given an integral $\mathcal{O}$-ideal $\mathfrak{f}$ we define the set

$$I_{\mathcal{O}}(\mathfrak{f}) := \{\mathfrak{b} \subseteq K : \mathfrak{b} \text{ is an invertible integral } \mathcal{O}\text{-ideal coprime to } \mathfrak{f}, \text{ i.e., } \mathfrak{f} + \mathfrak{b} = \mathcal{O}\}.$$

Consider the monoid $I_{\mathcal{O}}(1)$. For every integral $\mathcal{O}$-ideal $\mathfrak{f}$ we define an equivalence relation on the monoid $I_{\mathcal{O}}(1)$ which we denote by $\sim_{\mathfrak{f}}$. Let $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}}(1)$. We say that $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists an element $\lambda \in 1 + \mathfrak{f}\mathfrak{a}^{-1}$, $\lambda \gg 0$ (totally positive), such that $\lambda\mathfrak{a} = \mathfrak{b}$. Note that if $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ then $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f})$. The set $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is a finite monoid. The set of invertible elements of $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is exactly $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. By class field theory, the ideal class group $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ corresponds to an abelian extension of K which we denote by $K(\mathfrak{f}\infty)$ where $\infty = \infty_1\infty_2$ corresponds to the product of the two distinct real places of K. We call $K(\mathfrak{f}\infty)$ the narrow ray class field of K of conductor $\mathfrak{f}$.

We let

$$P_{\mathcal{O}}(\mathfrak{f}\infty) = \left\{ \frac{\alpha}{\beta} \in K : \alpha, \beta \in \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}}, \frac{\alpha}{\beta} \gg 0 \right\}. \tag{9}$$

It is easy to see that for $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}}(\mathfrak{f})$, $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists a $\lambda \in P_{\mathcal{O}}(\mathfrak{f}\infty)$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. We can thus think of $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ as $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O}}(\mathfrak{f}\infty)$.

Let p be a prime number which is inert in K. Instead of working with $\mathbb{Z}$-lattices and $\mathbb{Z}$-orders of K, one could well work with $\mathbb{Z}[\frac{1}{p}]$-lattices and $\mathbb{Z}[\frac{1}{p}]$-orders of K. It is an easy exercise to see that all the notions introduced previously are still valid in this setting. For any $\mathbb{Z}$-module $M \subseteq \mathbb{C}$ and a prime number p, we define $M^{(p)} := M[\frac{1}{p}] \simeq M \otimes_{\mathbb{Z}} \mathbb{Z}[\frac{1}{p}]$.

Definition 1.5 *For quantities p, f, N_0 fixed, we define*

(1) $\widetilde{\Gamma}_0 := \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[1/p]) : c \equiv 0 \pmod{fN_0} \right\}$,

(2) $\Gamma_0 = \{\gamma \in \widetilde{\Gamma}_0 : det(\gamma) = 1\}$,

(3) $\widetilde{\Gamma}_1 = \left\{\gamma \in \widetilde{\Gamma}_0 : a \equiv 1 \pmod{f}, c \equiv 0 \pmod{fN_0}\right\}$,

(4) $\Gamma_1 = \{\gamma \in \widetilde{\Gamma}_1 : det(\gamma) = 1\}$.

For a fixed prime number p we let $\mathcal{T} = \mathcal{T}_0 \cup \mathcal{T}_1$ be the Bruhat-Tits tree for $PGL_2(\mathbb{Q}_p)$ where $\mathcal{T}_0$ corresponds to its set of vertices and $\mathcal{T}_1$ corresponds to its set of edges. We let v_0 be the standard vertex of $\mathcal{T}$ which corresponds to the homothety class of $\mathbb{Z}_p \oplus \mathbb{Z}_p$. Finally we let $red : \mathcal{H}_p \to \mathcal{T}$ be the reduction map.

Definition 1.6 *A point $\tau \in \mathcal{H}_p$ is said to be reduced if $red(\tau) = v_0$. This is equivalent to say that $|\tau - t|_p \geq 1$ for $t = 0, 1, \ldots, p-1$ and $|\tau|_p \leq 1$ where $|\ |_p$ stands for the p-adic valuation on $\mathbb{C}_p$ normalized in a such a way that $|p|_p = \frac{1}{p}$.*

For a short introduction to the objects defined in the previous paragraph see chapter 5 of [Dar04].

2. Modular units and Eisenstein series

The results presented in this section relate periods of modular units with periods of Eisenstein series. This was the initial point of view that was taken in [Cha]. We also explain how the p-adic invariant $u(r,\tau)$ is related to a certain 2-cocycle $\kappa \in Z^2(\widetilde{\Gamma}_1, K_p^\times)$. All the results presented here can be found in [Cha]. The main result proved in this section is the proof of Theorem 1.2 which uses in an essential way the 2-cocycle κ.

Let $\mathcal{H}^* = \mathcal{H} \cup \mathbb{P}^1(\mathbb{Q})$, $f > 1$ be a positive integer and let $X(f)(\mathbb{C}) = \mathcal{H}^*/\Gamma(f)$ be the modular curve with full level f structure. For a pair $(\frac{r}{f}, \frac{s}{f}) \in (\frac{1}{f}\mathbb{Z})^2$ we associate the Siegel function

$$g_{(\frac{r}{f},\frac{s}{f})}(\tau) = -e^{2\pi i \frac{s}{f}(\frac{r}{f}-1)/2} q_\tau^{\frac{1}{2}\widetilde{B}_2(\frac{r}{f})}(1-q_z)\prod_{n\geq 1}(1-q_\tau^n q_z)(1-q_\tau^n q_{-z}), \tag{10}$$

where $\tau \in \mathcal{H}$, $z = \frac{r}{f}\tau + \frac{s}{f}$, $q_\tau = e^{2\pi i\tau}$, $q_z = e^{2\pi i z}$, $B_2(x) = x^2 - x + \frac{1}{6}$ is the second Bernoulli polynomial and $\widetilde{B}_2(x) := B_2(\{x\})$ with $0 \leq \{x\} < 1$ being the fractional part of x. The infinite product (10) converges whenever $Im(\tau) > 0$. On page 36 of [DK81] it is explained that the function $g_{(\frac{r}{f},\frac{s}{f})}(\tau)^{12f}$ is a modular unit on $X(f)(\mathbb{C})$, i.e., a meromorphic function on $X(f)(\mathbb{C})$ with its divisor supported on the set of cusps of $X(f)(\mathbb{C})$. Let $N_0 > 0$ be a positive integer prime to f and let $d_0|N_0$, $d_0 > 0$. Consider this special case of Siegel functions

$$g_{(\frac{r}{f},0)}(d_0 f\tau) = q_{fd_0\tau}^{\frac{1}{2}\widetilde{B}_2(\frac{r}{f})}(1-q_{rd_0\tau})\prod_{n\geq 1}(1-q_{d_0 f\tau}^n q_{rd_0\tau})(1-q_{d_0 f\tau}^n q_{-rd_0\tau}).$$

Let $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)$ be a good divisor. We define a family of modular functions, indexed by $j \in (\mathbb{Z}/f\mathbb{Z})^\times$, by the formula

$$\beta_{\delta_j}(\tau) := \prod_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} g_{(\frac{r}{f},0)}(d_0 f\tau)^{12n(d_0,jr)}. \tag{11}$$

Because of assumption (3) in Definition 1.1, $\beta_{\delta_{pj}}(\tau) = \beta_{\delta_j}(\tau)$. Therefore we can think of the functions $\beta_{\delta_j}(\tau)$ as being indexed by the cosets $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle$. Using assumption (2) of Definition 1.1, a direct calculation shows that the function $\beta_{\delta_j}(\tau)$ is invariant under the substitution $\tau \mapsto \gamma\tau$ for all $\gamma \in \Gamma_1(f) \cap \Gamma_0(fN_0)$. In particular, the function $\beta_{\delta_j}(\tau)$ may be viewed as a modular unit of level fN_0. Moreover, for all $c \in \Gamma_0(fN_0)\{\infty\}$ one has that $\beta_{\delta_j}(c) = 1$ (this uses assumption (2)), so that $\beta_{\delta_j}(\tau)$ is holomorphic on the set of cusps $\Gamma_0(fN_0)\{\infty\} = \{\frac{a}{c} : (a,c) = 1 \text{ and } (fN_0)|c\}$. Here, ∞ stands for the cusp $\frac{1}{0}$.

The definition of $\widetilde{F}_k(r,\tau)$ of the introduction was motivated by the following proposition.

Proposition 2.1 *Let* $\delta = \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r)[d_0,r] \in D(N_0,f)$ *be a good divisor. Then when the weight* k *is equal to* 2*, we have*

(1) $\mathrm{dlog}\,\beta_{\delta_r}(\tau) = 2\pi i \widetilde{F}_2(r,\tau)d\tau$,

(2) $\mathrm{dlog}\,\beta_{\delta_r,p}(\tau) = 2\pi i \widetilde{F}_{2,p}(r,\tau)d\tau$.

Proof This is a straightforward computation. □

2.1 Construction of a modular symbol

Let $\mathcal{M} = Div_0(\Gamma_0(fN_0)\{\infty\})$ denote the group of degree-zero divisors on the set $\Gamma_0(fN_0)\{\infty\}$. Note that $\mathcal{M}$ has a natural left action by $\Gamma_0(fN_0)$. A *partial modular symbol* with values in an abelian group A is simply a group homomorphism from $\mathcal{M}$ to A. If ψ is a partial modular symbol and $c_1, c_2 \in \Gamma_0(fN_0)\{\infty\}$, then we write

$$\psi\{c_1 \to c_2\} \text{ or } \psi[m] \text{ for } \psi([c_1]-[c_2]), \text{ where } m = [c_1]-[c_2] \in \mathcal{M}.$$

The assumption (2) of Definition 1.1 implies that the differential $\mathrm{dlog}\beta_{\delta_r}(\tau)$ on $\mathcal{H}^*$ is holomorphic at the points of the set $\Gamma_0(fN_0)\{\infty\}$. Thus, we may define a family of partial modular symbols ψ_r, indexed by $r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle$, by the rule

$$\psi_r\{c_1 \to c_2\} = \frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\,\beta_{\delta_r}(\tau), \tag{12}$$

where the complex line integral on the right hand side is taken along the unique geodesic C in $\mathcal{H}^*$ connecting the cusps c_1 and c_2. The rational integer $\psi_r\{c_1 \to c_2\}$ may be understood as the winding number of the closed loop $\beta_{\delta_r}(C)$ around the origin in the complex plane.

Remark 2.1 In light of Proposition 2.1 we see that (12) coincides with (6).

The partial modular symbol ψ_r is $\Gamma_0(fN_0)$-invariant in the sense that for all $\gamma \in \Gamma_0(fN_0)$ one has

$$\psi_r\{c_1 \to c_2\} = \psi_{\gamma\star r}\{\gamma c_1 \to \gamma c_2\}, \tag{13}$$

where, for $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $\gamma \star j \equiv dj \pmod f$. The identity (13) follows directly from the transformation formula $\beta_{\delta_{\gamma\star r}}(\gamma\tau) = \beta_{\delta_r}(\tau)$ where $\gamma \in \Gamma_0(fN_0)$.

We define the p-stabilization of $\beta_{\delta_r}(\tau)$ to be

$$\beta_{\delta_r,p}(\tau) := \frac{\beta_{\delta_r}(\tau)}{\beta_{\delta_r}(p\tau)}.$$

Using the infinite product of $\beta_{\delta_r,p}(\tau)$ and (3) of Definition of 1.1 one can show that $\beta_{\delta_r,p}(\tau)$ is $U_{p,m}$-invariant i.e.,

$$U_{p,m}(\beta_{\delta_r,p}(\tau)) := \prod_{k=0}^{p-1} \beta_{\delta_r,p}\left(\frac{\tau+k}{p}\right) = \beta_{\delta_r,p}(\tau). \tag{14}$$

(The index m of $U_{p,m}$ stands for multiplicative). For a proof of (14) see Proposition 3.5 and Remark 4.7 of [Cha]. Moreover, for all $c \in \Gamma_0(fN_0)\{\infty\}$ one also has $\beta_{\delta_r,p}(c) = 1$.

The family of p-stabilized modular units $\{\beta_{\delta_j,p}(\tau)\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle}$ gives rise naturally to a family of $\widetilde{\Gamma}_0$-invariant partial modular symbols with values in the abelian group of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$. In order to make the previous statement precise, we need to introduce some notation.

The group of matrices

$$GL_2^+(\mathbb{Z}[\tfrac{1}{p}]) = \{\gamma \in GL_2(\mathbb{Z}[\tfrac{1}{p}]) : det(\gamma) > 0\}$$

acts naturally on $\mathbb{P}^1(\mathbb{Q}_p)$ by the rule $x \mapsto \gamma x = \dfrac{ax+b}{cx+d}$, where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2^+(\mathbb{Z}[\frac{1}{p}])$ and $x \in \mathbb{P}^1(\mathbb{Q}_p)$. We define a ball in $\mathbb{P}^1(\mathbb{Q}_p)$ to be a translate of $\mathbb{Z}_p$ by some element of $GL_2^+(\mathbb{Z}[\frac{1}{p}])$, i.e., a ball in $\mathbb{P}^1(\mathbb{Q}_p)$ is a set of the form

$$\gamma\mathbb{Z}_p := \{\gamma x \in \mathbb{P}^1(\mathbb{Q}_p) : x \in \mathbb{Z}_p\},$$

where $\gamma \in GL_2^+(\mathbb{Z}[\frac{1}{p}])$. We denote the set of all such balls by $\mathcal{B}$.

Theorem 2.1 *There exists a unique system of $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$, indexed by triples*

$$(\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle \times \widetilde{\Gamma}_0\{\infty\} \times \widetilde{\Gamma}_0\{\infty\}$$

satisfying the following properties: for all $(r, c_1, c_2) \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle\overline{p}\rangle \times \widetilde{\Gamma}_0\{\infty\} \times \widetilde{\Gamma}_0\{\infty\}$

(1) $\mu_r\{c_1 \to c_2\}(\mathbb{P}^1(\mathbb{Q}_p)) = 0$,

(2) $\mu_r\{c_1 \to c_2\}(\mathbb{Z}_p) = \frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\beta_{\delta_r,p}(\tau)$,

(3) *($\widetilde{\Gamma}_0$-invariance property) For all $\gamma \in \widetilde{\Gamma}_0$ and all compact open $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ we have*

$$\mu_{\gamma\star r}\{\gamma c_1 \to \gamma c_2\}(\gamma U) = \mu_r\{c_1 \to c_2\}(U).$$

Proof The key idea is to use the $U_{p,m}$-invariance of the modular units $\beta_{\delta_r,p}(\tau)$. The latter property can be used to "package" these various winding numbers into a family of p-adic $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$. For a proof, see Theorem 5.1 of [Cha]. □

The following lemma states that the system of measures $\widetilde{\mu}$ appearing in Theorem 1.1 lifts the system of measures μ appearing in Theorem 2.1.

Lemma 2.1 *For all compact open $U \subseteq \mathbb{P}^1(\mathbb{Q}_p)$ we have*

$$\pi_*\widetilde{\mu}\{c_1 \to c_2\}(U) := \widetilde{\mu}_j\{c_1 \to c_2\}(\pi^{-1}(U)) = \mu_j\{c_1 \to c_2\}(U),$$

where $\pi : \mathbb{X} \to \mathbb{P}^1(\mathbb{Q}_p)$ is the $\mathbb{Z}_p^\times$-bundle given by $(x, y) \mapsto \frac{x}{y}$.

Proof See the proof of Lemma 6.1 in [Cha]. □

2.2 Construction of a 2-cocycle

The family of measures constructed in Theorem 2.1 will enable us to construct a 2-cocycle $\kappa \in Z^2(\widetilde{\Gamma}_1, K_p^\times)$.

Let $\delta \in D(N_0, f)$ be a fixed good divisor.

Definition 2.1 *Let $r \in (\mathbb{Z}/f\mathbb{Z})^\times$, $c_1, c_2 \in \widetilde{\Gamma}_0\{\infty\}$ and let $\tau_1, \tau_2 \in \mathcal{H}_p \cap K_p$. We define*

$$\int_{\tau_1}^{\tau_2}\int_{c_1}^{c_2} \mathrm{dlog}\beta_{\delta_r,p}(z) := \int_{\mathbb{P}^1(\mathbb{Q}_p)} \log_p\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_r\{c_1 \to c_2\}(t),$$

where $\mu_r\{c_1 \to c_2\}$ is the measure of Theorem 2.1 for the modular unit $\beta_{\delta,p}(\tau)$. Since the measures $\mu_r\{c_1 \to c_2\}$ are $\mathbb{Z}$-valued it makes sense also to define the multiplicative integral

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2}\int_{c_1}^{c_2} \beta_{\delta_r,p}(\tau) d\mu_r\{c_1 \to c_2\}(t) := \lim_{\mathcal{C}=\{U_i\}} \prod_i \left(\frac{t_i - \tau_2}{t_i - \tau_1}\right)^{\mu_r\{c_1\to c_2\}(U_i)},$$

where t_i is an arbitrary point of U_i and the limit goes over a set of covers that become finer and finer.

Definition 2.2 *Let $\tau \in \mathcal{H}_p \cap K_p$ and fix $c \in \widetilde{\Gamma}_0\{\infty\}$ and $r \in (\mathbb{Z}/f\mathbb{Z})^\times$. Then for all $\gamma_1, \gamma_2 \in \widetilde{\Gamma}_0$ we define*

$$\kappa_{c,(r,\tau)}(\gamma_1,\gamma_2) := \times\!\!\!\!\!\!\int_{\tau}^{\gamma_1\tau} \int_{\gamma_1 c}^{\gamma_1\gamma_2 c} \mathrm{dlog}\beta_{\delta_r,p}(z) \in K_p^\times.$$

We let the group $\widetilde{\Gamma}_0$ act trivially on $K_p^\times$.

Proposition 2.2 *The 2-cochain $\kappa_{c,(r,\tau)} \in C^2(\widetilde{\Gamma}_0, K_p^\times)$ is a "twisted" 2-cocycle satisfying the relation*

$$(d\kappa_{c,(r,\tau)})(\gamma_1,\gamma_2,\gamma_3) = \kappa_{c,(r,\tau)}(\gamma_2,\gamma_3) - \kappa_{c,(\gamma_1^{-1}\star r,\tau)}(\gamma_2,\gamma_3)$$

for all $\gamma_1,\gamma_2,\gamma_3 \in \widetilde{\Gamma}_0$.

In particular $(d\kappa_{c,(r,\tau)})|_{\widetilde{\Gamma}_1} = 0$, i.e., $\kappa_{c,(r,\tau)}|_{\widetilde{\Gamma}_1} \in Z^2(\widetilde{\Gamma}_1, K_p^\times)$.

Proof See Proposition 5.7 in [Cha]. □

2.3 Explicit splitting of a 2-cocycle

Definition 2.3 *To each $v \in \mathcal{T}_0$ (set of vertices of the Bruhat-Tits tree for $PGL_2(\mathbb{Q}_p)$) we associate a well defined partial modular symbol $m_v\{c_1 \to c_2\}$ on the set of cusps $\widetilde{\Gamma}_0\{\infty\}$ taking values in the set of $\widetilde{\Gamma}_0$-invariant $\mathbb{Z}$-valued measures on $\mathbb{P}^1(\mathbb{Q}_p)$. We define*

$$m_{v_0,r}\{c_1 \to c_2\} := \frac{1}{2\pi i}\int_{c_1}^{c_2} \mathrm{dlog}\beta_{\delta_r}(z), \qquad m_{\gamma v,\gamma\star r}\{\gamma c_1 \to \gamma c_2\} = m_{v,r}\{c_1 \to c_2\},$$

where $v \in \mathcal{V}(\mathcal{T})$, $\gamma \in \widetilde{\Gamma}_0$, $r \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ and $c_1, c_2 \in \Gamma_0(fN_0)\{\infty\}$.

Note that the assignment $v \mapsto m_{v,r}\{c_1 \to c_2\}$ satisfies the harmonicity property

$$\sum_{d(v',v)=1} m_{v',r}\{c_1 \to c_2\} = (p+1)m_{v,r}\{c_1 \to c_2\}.$$

The last equality comes from the fact that $\widetilde{F}_2(r,z)$ is an eigenvector with eigenvalue $(1+p)$ for the Hecke operator $T_2(p)$ (see equation (4.19) of [Cha]).

The next theorem gives an explicit splitting of the 2-cocycle appearing in Definition 2.2.

Theorem 2.2 *Let $\tau \in \mathcal{H}_p \cap K_p$, $r \in (\mathbb{Z}/f\mathbb{Z})^\times$, $\gamma \in \widetilde{\Gamma}_1$ and $v = red(\tau)$. Define*

$$\rho_{c,(r,\tau)}(\gamma) := p^{m_{v,r}\{c\to\gamma c)\}} \times\!\!\!\!\!\!\int_{\mathbb{X}} (x - \tau y)d\widetilde{\mu}_r\{c \to \gamma c\}(x,y). \tag{15}$$

Then $\rho_{c,(r,\tau)} \in C^1(\Gamma_1, K_p^\times)$ is a 1-cochain such that of $d\rho_{c,(r,\tau)} = \kappa_{c,(r,\tau)}$.

Proof The proof uses in an essential way Lemma 2.1 and property (3) of Theorem 1.1. For a detailed proof see Theorem 6.2 of [Cha]. □

Remark 2.2 For an admissible pair (r,τ) one has that $\rho_{\infty,(r,\tau)}(\gamma_\tau) = u(r,\tau)$.

Corollary 2.1 *Let $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p \cap K$ be an admissible pair such that $red(\tau) = v_0$. Let $\gamma \in \widetilde{\Gamma}_1$. Then*

$$ord_p(\rho_{c,(r,\tau)}(\gamma)) = m_{v_0,r}\{c \to \gamma c\}. \tag{16}$$

Proof This follows directly from the definiton of $\rho_{c,(r,\tau)}$. □

2.4 Proof of Theorem 1.2

Theorem 1.2 will be a direct consequence of the next two propositions.

Proposition 2.3 *For $\tau \in \mathcal{H}_p \cap K$ let*

$$\Gamma_{1,\tau} := \{\gamma \in \Gamma_1 : \gamma\tau = \tau\}.$$

Let $\rho_{c,(r,\tau)}$ be the 1-cochain appearing in Theorem 2.2 when viewed as an element of $Z^1(\Gamma_1, K_p^\times)$. Then $\rho_{c,(r,\tau)}|_{\Gamma_{1,\tau}}$ modulo $Hom(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}}$ does not depend on the base point $c \in \Gamma_0\{\infty\}$.

Proof Let $x, y \in \Gamma_1\{\infty\}$. We want to show that

$$\rho_{x,(r,\tau)}|_{\Gamma_{1,\tau}} - \rho_{y,(r,\tau)}|_{\Gamma_{1,\tau}} \in Hom(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}} = Z^1(\Gamma_1, K_p^\times)|_{\Gamma_{1,\tau}}.$$

This is equivalent to showing that $(d\rho_{x,(r,\tau)})|_{\Gamma_{1,\tau}} - (d\rho_{y,(r,\tau)})|_{\Gamma_{1,\tau}} = 0$. The previous equality means exactly that $(\kappa_{x,(r,\tau)} - \kappa_{y,(r,\tau)})|_{\Gamma_{1,\tau}} = 0$. Let us compute.

Let $\gamma_1, \gamma_2 \in \Gamma_1$. We have

$$\begin{aligned}
\kappa_{x,(r,\tau)}(\gamma_1,\gamma_2) - \kappa_{y,(r,\tau)}(\gamma_1,\gamma_2) &= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1\gamma_2 x} \mathrm{dlog}\beta_{\delta_r,p}(z) - \int_\tau^{\gamma_1\tau}\int_{\gamma_1 y}^{\gamma_1\gamma_2 y} \mathrm{dlog}\beta_{\delta_r,p}(z) \\
&= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1 y} \mathrm{dlog}\beta_{\delta_k,p}(z) - \int_\tau^{\gamma_1\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} \mathrm{dlog}\beta_{\delta_r,p}(z) \\
&= \int_\tau^{\gamma_1\tau}\int_{\gamma_1 x}^{\gamma_1 y} \mathrm{dlog}\beta_{\delta_r,p}(z) - \int_\tau^{\gamma_1\gamma_2\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} \mathrm{dlog}\beta_{\delta_r,p}(z) \\
&\quad + \int_{\gamma_1\tau}^{\gamma_1\gamma_2\tau}\int_{\gamma_1\gamma_2 x}^{\gamma_1\gamma_2 y} \mathrm{dlog}\beta_{\delta_r,p}(z).
\end{aligned}$$

Now applying γ_1^{-1} to the bounds of the third term of the last equality (note that $\gamma_1^{-1} \star r = r$) and setting

$$c_{x,y}(\gamma) := \int_\tau^{\gamma\tau}\int_{\gamma x}^{\gamma y} \mathrm{dlog}\beta_{\delta_r,p}(z) \in C^1(\Gamma_1, K_p^\times),$$

we get

$$\kappa_{x,(r,\tau)}(\gamma_1,\gamma_2) - \kappa_{y,(r,\tau)}(\gamma_1,\gamma_2) = c_{x,y}(\gamma_1) - c_{x,y}(\gamma_1\gamma_2) + c_{x,y}(\gamma_2) = (dc_{x,y})(\gamma_1,\gamma_2).$$

We thus have proved that $d(\rho_{x,(r,\tau)} - \rho_{y,(r,\tau)} - c_{x,y})|_{\Gamma_1} = 0$. So $\rho_{x,(r,\tau)} - \rho_{y,(r,\tau)} - c_{x,y} \in Hom(\Gamma_1, K_p^\times)$. Finally evaluating at γ_τ and using the fact that $c_{x,y}(\gamma_\tau) = 0$ proves the proposition. □

Proposition 2.4 *The abelianization of Γ_1, i.e., $(\Gamma_1)^{ab} = \Gamma_1/[\Gamma_1, \Gamma_1]$ is a finite group.*

Proof See [Men67] and [Ser70].

Corollary 2.2 *The group $Hom(\Gamma_1, K_p^\times)$ is a finite abelian group of exponent dividing $\#(K^\times)_{tor} = p^2 - 1$.*

Remark 2.3 Note that the Proposition 2.4 is obviously false if one replaces Γ_1 by the larger group $\widetilde{\Gamma}_1$.

Proof of Theorem 1.2 Let (r, τ) be an admissible pair. Let $U \subset (\mathbb{Q}_p \times \mathbb{Q}_p)\backslash(0,0)$ be a compact open subset, let $c_1, c_2 \in \widetilde{\Gamma}_0$ and let $\gamma = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \widetilde{\Gamma}_0$. A direct computation which uses property

(2) of Theorem 1.1 shows that

$$\int\!\!\!\!\!\!\times_U (x-\tau y)d\widetilde{\mu}_r\{c_1\to c_2\}(x,y)=\int\!\!\!\!\!\!\times_{\gamma U}(C\tau+D)(x-\gamma\tau y)d\widetilde{\mu}_{\gamma\star r}\{\gamma c_1\to\gamma c_2\}(x,y). \tag{17}$$

Now let (r,τ) and (r',τ') be admissible pairs as given in the statement of Theorem 1.2. By assumption there exists a $\eta\in\widetilde{\Gamma}_0$ such that $\eta\star(r,\tau)=(r',\tau')$. Since τ and τ' are reduced we see that $\eta\in\widetilde{\Gamma}_0\cap GL_2(\mathbb{Z}_p)=\Gamma_0(fN_0)$. Note that $\gamma_{\tau'}=\eta\gamma_\tau\eta^{-1}$ where γ_τ and $\gamma_{\tau'}$ are as in Definition 1.4. Let $\xi=\eta\infty$. We have

$$u(r',\tau')=\rho_{\infty,(r',\tau')}(\gamma_{\tau'})=\rho_{\xi,(r',\tau')}(\gamma_{\tau'})\pmod{\mu_{p^2-1}},$$

where the last equality follows from Proposition 2.3 ($\gamma_{\tau'}\in\Gamma_{1,\tau'}$) and Corollary 2.2. Now let us compute directly $\rho_{\xi,(r',\tau')}(\gamma_{\tau'})$. We have

$$\begin{aligned}&\rho_{\xi,(r',\tau')}(\gamma_{\tau'})\\&=p^{m_{v_0,r'}\{\xi\to\gamma_{\tau'}\xi\}}\int\!\!\!\!\!\!\times_{\mathbb{X}}(x-\tau'y)d\widetilde{\mu}_{r'}\{\xi\to\gamma_{\tau'}\xi\}(x,y)\\&=p^{m_{v_0,r}\{\infty\to\gamma_\tau\infty\}}\int\!\!\!\!\!\!\times_{\mathbb{X}}(x-\tau y)d\widetilde{\mu}_r\{\infty\to\gamma_\tau\infty\}(x,y),\end{aligned}$$

where the last equality uses (17) and the $\widetilde{\Gamma}_0$-equivariance of the modular symbol $m_{v_0,r'}\{\xi\to\gamma_{\tau'}\xi\}$. This concludes the proof. □

3. Proof of Theorem 1.1

This section is devoted to the existence of the system of measures which appear in Theorem 1.1.

The proof is technical and long but essentially it follows the same lines as the one given in [DD06]. We will only prove in details the new ingredients which are not straightforward adaptations of [DD06]; for a more detailed version of it see [Cha]. The uniqueness of the family of measures follows easily from properties (1) and (2). It remains to show the existence of such a family. We will first prove the existence of a family of measures which satisfy properties (1), (2) and (3) under the weaker assumption that they take values in $\mathbb{Z}_p$ rather than $\mathbb{Z}$. We break the proof in **five steps**. We let $\xi=\frac{a}{c}\in\widetilde{\Gamma}_0\{\infty\}$ where $p\nmid c$ and $j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ and write $\mu_{\xi,j}:=\widetilde{\mu}_j\{\xi\to\infty\}$.

First step *There exists a unique family of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p\times\mathbb{Z}_p$ which satisfies the property*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p}h(x,y)d\mu_{\xi,j}(x,y)=(1-p^{k-2})\int_\xi^{i\infty}h(z,1)\widetilde{F}_k(j,z)dz.$$

A direct computation shows that

$$\begin{aligned}I_{n,m}(j)&:=\int_{\mathbb{Z}_p\times\mathbb{Z}_p}x^ny^md\mu_{\xi,j}(x,y)=(1-p^{n+m})\int_\xi^{i\infty}z^n\widetilde{F}_{n+m+2}(j,z)dz\\&=-\frac{12}{f^{n+m}}(1-p^{n+m})\sum_{l=0}^n\binom{n}{l}\left(\frac{a}{c}\right)^{n-l}(-1)^l\sum_{d_0|N_0,r\in\mathbb{Z}/f\mathbb{Z}}n(d_0,r)d_0^{-l}D^{jr\pmod f}_{n+m-l+1,l+1}(a,c/d_0)\end{aligned}\tag{18}$$

for all integers $n,m\geq 0$, where

$$D^{r\pmod f}_{s,t}(a,c):=c^{s-1}\sum_{\substack{1\leq h\leq c\\ h\equiv r\pmod f}}\frac{\widetilde{B}_s(h/c)}{s}\frac{\widetilde{B}_t(ha/c)}{t}.$$

The second equality follows from Proposition 11.4 of [Cha] which provides explicit formulas for the rational periods of $\widetilde{F}_k(j,z)$.

The key tool in showing the existence and uniqueness of $\{\mu_{\xi,j}\}$ is the following result, which is a two variables version of a classical theorem of Mahler.

Lemma 3.1 *Let $b_{n,m} \in \mathbb{Z}_p$ be constants indexed by integers $n, m \geq 0$. There exists a unique measure μ on $\mathbb{Z}_p \times \mathbb{Z}_p$ such that*

$$\int_{\mathbb{Z}_p \times \mathbb{Z}_p} \binom{x}{n}\binom{y}{m} d\mu(x,y) = b_{n,m}.$$

For any $0 \leq n$ and $0 \leq i \leq n$, define the rational numbers $c_{n,i}$ via $\binom{x}{n} = \sum_{i=0}^{n} c_{n,i} x^i$. For $j \in (\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle$ we let

$$J_{n,m}(j) := \sum_{i=0}^{n}\sum_{i'=0}^{m} c_{n,i}c_{m,i'} I_{i,i'}(j).$$

So in order to show that the measures $\mu_{\xi,j}$ are $\mathbb{Z}_p$-valued, it is enough to show, by Lemma 3.1, that $J_{n,m}(j) \in \mathbb{Z}_p$. The way that this is proved is by interpreting the quantity $J_{n,m}(j)$ as the partial derivative of a certain rational function. More precisely,

$$J_{n,m}(j) = \binom{D_w}{m}\binom{D_z}{z} H_j^*(u,v)|_{(1,1)},$$

where $(z,w) = (\frac{1}{u}, u^{\frac{a}{c}} v^{\frac{1}{c}})$, $D_w = w\frac{\partial}{\partial w}$, $D_z = z\frac{\partial}{\partial z}$ and $H_j^*(u,v)$ is a rational function in $\mathbb{Z}_p(u^{1/c}, v^{1/c})$. For the exact definition of $H_j^*(u,v)$ see equations (12.9) and (12.10) of [Cha].

Now the p-integrality of $J_{n,m}(j)$ is a direct consequence of the following lemma.

Lemma 3.2 *Consider the subring R of $\mathbb{Z}_p(u^{1/c}, v^{1/c})$ defined by*

$$R := \left\{ \frac{P}{Q} : P, Q \in \mathbb{Z}_p[u^{1/c}, v^{1/c}] \text{ and } Q(1,1) \in \mathbb{Z}_p^\times \right\}.$$

Then R is a ring stable under the operators $\binom{D_w}{m}$ and $\binom{D_z}{n}$. Furthermore $H_j^*(u,v) \in R$.

The proof of Lemma 3.2 is identical to the proof of Lemma 4.11 of [DD06] (see also Lemma 12.2 of [Cha]). □

Second step *There exists a unique family of partial modular symbols $\{\nu_j\}_{j\in(\mathbb{Z}/f\mathbb{Z})^\times/\langle p\rangle}$ supported on the set of cusps $\Gamma_0(fN_0)\{\infty\}$ of $\mathbb{Z}_p$-valued measures on $\mathbb{Z}_p \times \mathbb{Z}_p$ such that*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_j\{r \to s\}(x,y) = (1-p^{k-2})\int_r^s h(z,1)\widetilde{F}_k(j,z)dz$$

for $r, s \in \Gamma_0(fN_0)\{\infty\}$ and for every homogeneous polynomial $h(x,y) \in \mathbb{Z}[x,y]$ of degree $k-2$. Furthermore, if $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$ then $\nu_j\{r \to s\}(U) = \nu_{\gamma\star j}\{\gamma r \to \gamma s\}(\gamma U)$, i.e., the system of measures is $\Gamma_0(fN_0)$-invariant.

The proof of this step uses Step 1. The argument is identical to the proof of Lemma 4.13 of [DD06] (see also Lemma 12.4 in [Cha]). Note that the $\Gamma_0(fN_0)$-invariance boils down basically to the transformation formula $E_k(\gamma\star r, \gamma\tau)(c\tau+d)^{-k} = E_k(r,\tau)$ where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN_0)$.

Third step *Let $r,s\in\Gamma\{\infty\}$. The measures $\nu_j\{r\to s\}$ constructed in* **Step** *2 satisfy the following formula*

$$\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(x,y)d\nu_j\{r\to s\}(x,y)=\int_r^s h(z,1)\widetilde{F}_{k,p}(j,z)dz,$$

for every homogeneous polynomial $h(x,y)\in\mathbb{Z}[x,y]$ of degree $k-2$.

The proof of this step uses Step 2 and follows the same lines as the proof of Lemma 4.14 of [DD06] where Lemma 4.15 of [DD06] is replaced by the following lemma

Lemma 3.3 *Let $s,t\geq 1$. For any rational number $\frac{a}{c}$ (p could divide c), we have inside $\mathbb{Q}_p$ the identity*

$$\lim_{j\to\infty} D^{r\pmod f}_{s+(p-1)p^j,t}(a,c)=D^{r\pmod f}_{s,t}(a,c)-p^{s-1}D^{p^{-1}r\pmod f}_{s,t}(pa,c).$$

The proof of Lemma 3.3 is different from the proof of Lemma 4.15 of [DD06] since we don't use reciprocity formulas for Dedekind sums. For this reason we have decided to include it.

Proof of Lemma 3.3 Let $x=\frac{a}{c}\in\mathbb{Q}$ with $(a,c)=1$ and assume first that $p\nmid c$. Let b be an integer such that $abp\equiv 1\pmod c$. Note that

$$D^{r\pmod f}_{s,t}(a,c)=c^{s-1}\sum_{\substack{1\leq l\leq c\\ l\equiv ar\pmod f}}\frac{\widetilde{B}_s(lbp/c)}{s}\frac{\widetilde{B}_t(l/c)}{t}.\tag{19}$$

Therefore

$$D^{r\pmod f}_{s+(p-1)p^j,t}(a,c)=c^{s-1+(p-1)p^j}\sum_{\substack{1\leq l\leq c\\ l\equiv ar\pmod f}}\frac{\widetilde{B}_{s+(p-1)p^j}(lbp/c)}{s}\frac{\widetilde{B}_t(l/c)}{t}\tag{20}$$

and similarly

$$D^{r\pmod f}_{s+(p-1)p^j,t}(pa,c)=c^{s-1}\sum_{\substack{1\leq l\leq c\\ l\equiv ar\pmod f}}\frac{\widetilde{B}_{s+(p-1)p^j}(lb/c)}{s}\frac{\widetilde{B}_t(l/c)}{t}.\tag{21}$$

Write $y=\{lbp/c\}$ and $y'=\{lb/c\}$. Since $c^{(p-1)p^j}\to 1$, then subtracting p^{s-1} times (21) to (20) we see that it suffices to prove that

$$\lim_{j\to\infty} B_{s+(p-1)p^j}(y)=B_s(y)-p^{s-1}B_s(y').\tag{22}$$

For $s>0$, this follows from the proof of Theorem 3.2 of [You01]. In the course of the proof of Theorem 3.2 of [You01] Young gets for any positive integer b coprime to p the congruence

$$\begin{aligned}(b^{s+(p-1)p^j}-1)&\frac{B_{s+(p-1)p^j}(x)-p^{s-1+(p-1)p^j}B_{s+(p-1)p^j}(x')}{s+(p-1)p^j}\\ -(b^s-1)&\frac{B_s(x)-p^{s-1}B_s(x')}{s}\equiv 0\pmod{p^{j+1}\mathbb{Z}_p}\end{aligned}\tag{23}$$

where x' is such that $px'-x\in\{0,1,\dots,p-1\}$ and $s\geq 1$. The denominator of $\frac{B_n}{n}$ at p is well behaved. If $(p-1)\nmid n$ then $\frac{B_n}{n}$ is p-integral. If $(p-1)|n$ then $v_p(\frac{B_n}{n})=-1-v_p(n)$. Using the previous observation it follows that $\lim_{j\to\infty}p^{(p-1)p^j}B_{s+(p-1)p^j}(x')=0$. Letting $j\to\infty$ in (23) we get that

$$(b^s-1)\lim_{j\to\infty}\frac{B_{s+(p-1)p^j}(x)}{s}=(b^s-1)\frac{B_s(x)-p^{s-1}B_s(x')}{s}.\tag{24}$$

When $s \geq 1$ we can always choose b such that $b^s - 1 \neq 0$. Therefore we can cancel the two factors $b^s - 1$ in (24) to get (22). It remains to treat the case where $s = 0$.

We have $v_p(y) \geq 1$. Let $g = (p-1)p^j$. Note that

$$\begin{aligned} B_g(y) &= \sum_{k=0}^{g} \binom{g}{k} B_k y^{g-k} \\ &= y^g + g\left(\sum_{k=1}^{g-1} \binom{g-1}{k-1} \frac{B_k}{k} y^{g-k}\right) + B_g. \end{aligned} \tag{25}$$

If $(p-1) \nmid k$ then $\frac{B_k}{k} \in \mathbb{Z}_p$. If $(p-1)|k$ then we can write $k = (p-1)p^u m$ with $(m,p) = 1$. So $v_p(\frac{B_k}{k} y^{g-k}) \geq -1 - u + (p-1)p^u \geq 0$ since $p^{j-u} - 1 \geq m$. We thus deduce from (25) that $\lim_{j\to\infty} B_{(p-1)p^j}(y) = B_{(p-1)p^j}$.

Let ω be the Teichmüller character at p. If we look at $L_p(s)$ the p-adic L-function twisted by the trivial character, we have the formula

$$L_p(1-n) = -(1 - \omega^{-n}(p)p^{n-1})\frac{B_{n,\omega^{-n}}}{n}.$$

Here ω^{-n} means the primitive character associated to ω^{-n} (so $\omega^{-n}(a)$ is not necessarily equal to $\omega(a)^{-n}$). So letting $n = (p-1)p^j$, we have $\omega^{-n}(p) = 1$ and we get

$$L_p(1-(p-1)p^j) = -(1 - p^{(p-1)p^j-1})\frac{B_{(p-1)p^j}}{(p-1)p^j}.$$

Now we know that $\lim_{s\to 1}(s-1)L_p(s) = 1 - \frac{1}{p}$. So letting $j \to \infty$ we get

$$\lim_{j\to\infty} B_{(p-1)p^j} = 1 - \frac{1}{p}.$$

This proves the claim for $s = 0$.

We need to treat now the case where $p|c$. This case turns out to be simple. Let us prove the following elementary lemma.

Lemma 3.4 *Let h be any integer and $0 \neq c \in \mathbb{Z}$ such that $p|c$. Then we have the following:*

(1) $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = c^s\widetilde{B}_s(\frac{h}{c})$, *if* $(h,p) = 1$,

(2) $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$, *if* $p|h$,

where $g = (p-1)p^j$.

Proof of Lemma 3.4 Let us prove the first case. We have

$$\begin{aligned} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) &= \sum_{k=0}^{s+g} \binom{s+g}{k} B_k h^{s+g-k} c^k \\ &= \sum_{k=0}^{s} \binom{s+g}{k} B_k h^{s+g-k} c^k + \sum_{k=s+1}^{s+g} \binom{s+g}{k} B_k h^{s+g-k} c^k. \end{aligned} \tag{26}$$

Now since $|c|_p < 1, |h|_p = 1, |\binom{m}{k}|_p \leq 1$ and $|B_k|_p \leq p$, the limit in (26) when $j \to \infty$ exists. Since $(h,p) = 1$ the limit of the first term is $c^s\widetilde{B}_s(\frac{h}{c})$ and the limit of the second term is 0. This proves the first part of the lemma.

Assume now that $p|h$. If $v_p(h) \geq v_p(c)$ then $\frac{h}{c} \in \mathbb{Z}_p$. In this case we know that $\lim_{j\to\infty} \widetilde{B}_{s+(p-1)p^j}(\frac{h}{c})$ exists by (22). Finally since $p|c$ it follows that $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$. Assume now that $v_p(c) >$

$v_p(h) = m \geq 1$. Then by the first part of the lemma 3.4 we know that $\lim_{j\to\infty}(\frac{c}{p^m})^{s+g}\widetilde{B}_{s+g}(\frac{h/p^m}{c/p^m})$ exists. It follows $\lim_{j\to\infty} c^{s+g}\widetilde{B}_{s+g}(\frac{h}{c}) = 0$ since $m \geq 1$. $\square$

With Lemma 3.4 it is now easy to prove Lemma 3.3 for the case where $p|c$. We have

$$\lim_{j\to\infty} D^{r \pmod f}_{s+g,t}(a,c) = \lim_{j\to\infty} \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f}} c^{s+g-1}\widetilde{B}_{s+g-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right)$$
$$= \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f\\ (p,h)=1}} c^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right) \tag{27}$$

On the other hand we have

$$\begin{aligned}
&D^{r \pmod f}_{s,t}(a,c) - p^{s-1}D^{p^{-1}r \pmod f}_{s,t}(pa,c)\\
&= D^{r \pmod f}_{s,t}(a,c) - p^{s-1}D^{p^{-1}r \pmod f}_{s,t}(a,c/p)\\
&= \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f}} c^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right) - \sum_{\substack{1\leq h\leq c/p\\ h\equiv p^{-1}r \pmod f}} p^{s-1}\left(\frac{c}{p}\right)^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c/p}\right)\widetilde{B}_t\left(\frac{ah}{c/p}\right)\\
&= \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f}} c^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right) - \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f\\ h\equiv 0 \pmod p}} c^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right)\\
&= \sum_{\substack{1\leq h\leq c\\ h\equiv r \pmod f\\ (p,h)=1}} c^{s-1}\widetilde{B}_{s-1}\left(\frac{h}{c}\right)\widetilde{B}_t\left(\frac{ah}{c}\right).
\end{aligned}$$

Compare with (27). This concludes the proof of Lemma 3.3. $\square$

Fourth step *Let $r, s \in \Gamma\{\infty\}$. The measures $\nu_j\{r \to s\}$ are supported on $\mathbb{X}$.*

Proof Let $\gamma = \begin{pmatrix} a & b\\ c & d\end{pmatrix} \in \Gamma_0(fN_0)$ and set $\mu(\gamma,z) := (cz+d)$. Let $h(x,y) \in \mathbb{Z}[x,y]$ be a homogeneous polynomial of degree $k-2 = m+n-2$. Then

$$\begin{aligned}
\int_{\gamma(\mathbb{Z}_p\times\mathbb{Z}_p^\times)} h(x,y)d\nu_j\{r\to s\}(x,y) &= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(\gamma(x,y))d\nu_j\{r\to s\}(\gamma(x,y))\\
&= \int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(\gamma(x,y))d\nu_{\gamma^{-1}\star j}\{\gamma^{-1}r\to\gamma^{-1}s\}(x,y)\\
&= \int_{\gamma^{-1}r}^{\gamma^{-1}s} h(\gamma z,1)\mu(\gamma,z)^{k-2}\widetilde{F}_{k,p}(\gamma^{-1}\star j,z)dz\\
&= \int_r^s h(z,1)\mu(\gamma^{-1},z)^{-(k-2)}\widetilde{F}_{k,p}(\gamma^{-1}\star j,\gamma^{-1}z)d(\gamma^{-1}z).
\end{aligned}$$

Let $M(p) \subset M_2(\mathbb{Z})$ be the set of primitive matrices of determinant p. Let

$$\left\{\eta_i = \begin{pmatrix} a_i & b_i\\ c_i & d_i\end{pmatrix}\right\}_{i=1}^{p+1}$$

be a complete set of representatives of $SL_2(\mathbb{Z})\backslash M(p)$. Then we have

$$T_k(p)E_k(j,z) = p^{k-1}\sum_{i=1}^{p+1} E_k(d_i j, \eta_i z)\mu(\eta_i, z)^{-k}, \tag{28}$$

where $T_k(p)$ stands for the Hecke operator at p. For some background about Hecke operators in this context see Section 4.8 of [Cha].

Let $P = \begin{pmatrix} p & 0 \\ 0 & 1 \end{pmatrix}$ and $\left\{\gamma_i = \begin{pmatrix} a_i & b_i \\ c_i & d_i \end{pmatrix}\right\}_{i=1}^{p+1}$ be a complete set of representatives of $\Gamma_0(pfN_0)\backslash\Gamma_0(fN_0)$. Note that the set $\{P\gamma_i^{-1}\}_{i=1}^{p+1}$ is a complete set of representatives of $SL_2(\mathbb{Z})\backslash M(p)$.

From (30) we deduce that

$$\begin{aligned}
&\frac{-1}{12f}\sum_{i=1}^{p+1} \widetilde{F}_{k,p}(\gamma_i^{-1}\star j, \gamma_i^{-1}z)\mu(\gamma_i^{-1},z)^{-(k-2)}d(\gamma_i^{-1}z) \\
&= \sum_{d_0,r} n(d_0,r)d_0\sum_{i=1}^{p+1} E_k(a_i rj, d_0\gamma_i^{-1}z)\mu(\gamma_i^{-1},z)^{-(k-2)}d(\gamma_i^{-1}z) - \\
&\qquad p^{k-1}\sum_{d_0,r} n(d_0,r)d_0\sum_{i=1}^{p+1} E_k(a_i rj, pd_0\gamma_i^{-1}z)\mu(\gamma_i^{-1},z)^{-(k-2)}d(\gamma_i^{-1}z).
\end{aligned} \tag{29}$$

Because

$$E_k(r,\gamma z)\mu(\gamma,z)^{-(k-2)}d(\gamma z) = E_k(\gamma^{-1}\star r, z)dz,$$

for any $\gamma \in \Gamma_0(f)$, we have that

$$\begin{aligned}
E_k(a_i rj, d_0\gamma_i^{-1}z)\mu(\gamma_i^{-1},z)^{-(k-2)}d(\gamma_i^{-1}z) &= E_k(\gamma_i\star(a_i rj), d_0 z)dz \\
&= E_k(jr, d_0 z)dz.
\end{aligned}$$

From equation (4.19) of [Cha] one may deduce that

$$T_k(p)E_k(j,z) = p^{k-1}E_k(j,z) + E_k(pj,z). \tag{30}$$

Using the fact that (28) is equal to (30), that $\mu(P\gamma,z) = \mu(\gamma,z)$ and $pd_0\gamma_i^{-1}z = d_0P\gamma_i^{-1}z$ we obtain

$$\begin{aligned}
p^{k-1}\sum_{i=1}^{p+1} E_k(a_i rj, d_0P\gamma_i^{-1}z)\mu(\gamma_i^{-1},z)d(\gamma_i^{-1}z) &= p^{k-1}\sum_{i=1}^{p+1} E_k(a_i rj, d_0P\gamma_i^{-1}z)\mu(P\gamma_i^{-1},z)^{-(k-2)}d(P\gamma_i^{-1}z) \\
&= (T_k(p)E_k(rj,d_0z))dz \\
&= (p^{k-1}E_k(rj,d_0z) + E_k(prj,d_0z))dz.
\end{aligned}$$

Now because $p\star\delta = \delta$ we find that

$$\sum_{d_0,r} n(d_0,r)d_0(p^{k-1}E_k(rj,d_0z) + E_k(prj,d_0z))dz = (p^{k-1}+1)\sum_{d_0,r} n(d_0,r)d_0E_k(rj,d_0z).$$

Substituting the last expression in (29) we find

$$-12f((p+1)-(p^{k-1}+1))\sum_{d_0,r} n(d_0,r)d_0E_k(rj,d_0) = (p-p^{k-1})\widetilde{F}_k(j,z).$$

Finally note that $\cup_{i=1}^{p+1}\gamma_i(\mathbb{Z}_p\times\mathbb{Z}_p^\times)$ is a degree p cover of $\mathbb{X}$. Hence we get

$$\begin{aligned}
p\int_{\mathbb{X}} h(x,y)d\nu_j\{r\to\}(x,y) &= \sum_{i=1}^{p+1}\int_{\gamma_i(\mathbb{Z}_p\times\mathbb{Z}_p^\times)} h(x,y)d\nu_j\{r\to\}(x,y)\\
&= \sum_{i=1}^{p+1}\int_r^s h(z,1)\widetilde{F}_{k,p}(\gamma_i^{-1}\star j,\gamma_i^{-1}z)\mu(\gamma_i^{-1},z)^{-(k-2)}d(\gamma^{-1}z)\\
&= (p-p^{k-1})\int_r^s h(z,1)\widetilde{F}_{k,p}(j,z)dz\\
&= (p-p^{k-1})\int_{\mathbb{Z}_p\times\mathbb{Z}_p^\times} h(x,y)d\nu_j\{r\to s\}(x,y)\\
&= p\int_{\mathbb{Z}_p\times\mathbb{Z}_p} h(x,y)d\nu_j\{r\to s\}(x,y).
\end{aligned}$$

Since this holds for any h homogeneous of degree k we get that the support of $\nu_j\{r\to s\}$ is included in $\mathbb{X}$. □

Fifth step Now we want to extend the measures $\nu_j\{r\to s\}$ to the space $\mathbb{Q}_p^2\backslash\{(0,0)\}$. The compact open set $\mathbb{X}$ is a fundamental domain for the action of multiplication by p on $\mathbb{Q}_p^2\backslash\{(0,0)\}$ where by multiplication by p we mean $\begin{pmatrix} p & 0\\ 0 & p\end{pmatrix}(x,y)=(px,py)$. Hence if for a compact open $U\subseteq\mathbb{X}$ we define

$$\widetilde{\mu}_j\{r\to s\}(U):=\nu_j\{r\to s\}(U),$$

then $\widetilde{\mu}_j\{r\to s\}$ extends uniquely to a $\Gamma_0(fN_0)$-invariant partial modular symbol of $\mathbb{Z}_p$-valued measures on $\mathbb{Q}_p^2\backslash\{0\}$ which is invariant under the action of multiplication by p:

$$\widetilde{\mu}_j\{r\to s\}(pU)=\widetilde{\mu}_j\{r\to s\}(U),$$

for all compact open $U\subseteq\mathbb{Q}_p^2\backslash\{(0,0)\}$. This almost proves Theorem 1.1. It remains to show that the modular symbol $\widetilde{\mu}_j$ is "$\widetilde{\Gamma}_0$-invariant", i.e., for all compact open set $U\subseteq\mathbb{Q}_p^2\backslash\{(0,0)\}$ and all pair of cusps $r,s\in\widetilde{\Gamma}_0\{\infty\}$,

$$\widetilde{\mu}_{\gamma\star j}\{\gamma r\to\gamma s\}(\gamma U)=\widetilde{\mu}_j\{r\to s\}(U). \tag{31}$$

Note that $\widetilde{\Gamma}_0=\langle\Gamma_0(fN_0),P\rangle$ where $P=\begin{pmatrix} p & 0\\ 0 & 1\end{pmatrix}$. By construction the modular symbol $\widetilde{\mu}_j$ is $\Gamma_0(fN_0)$-invariant therefore in order to show (31) it is enough prove it for the matrix P. This is proved in the exact same way as the proof of Lemma 4.17 of [DD06].

Finally, it remains to show that our measures $\nu_j\{r\to s\}$ are $\mathbb{Z}$-valued.

Theorem 3.1 *The measures* $\widetilde{\mu}_j\{\infty\to\frac{a}{c}\}$ *take values in* $\mathbb{Z}$.

Proof This result is an easy adaptation of the proof of Theorem 1.3 of [Das07a]. The interested reader may find all the details in Section 13 of [Cha]. □

This concludes the proof of Theorem 1.1. □

4. From $\mathcal{H}$ to $\mathbb{Z}/f\mathbb{Z}\times\mathcal{H}_p^{\mathcal{O}}(N_0)$ and the Shimura reciprocity law

Let K be a real quadratic field and let p be a prime number inert in K. Let us fix an embedding $K\subseteq\mathbb{R}$ and let $G_{K/\mathbb{Q}}=\{1,\sigma\}$. Choose a $\mathbb{Z}$-order $\mathcal{O}\subseteq K$ and fix a positive integer N coprime to p.

In [DD06] the authors associate to such data the set

$$\mathcal{H}_p^{\mathcal{O}}(N) := \mathcal{H}_p^{\mathcal{O}} = \{\tau \in \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N\tau}^{(p)} = \mathcal{O}^{(p)}, \tau - \tau^\sigma > 0\}, \tag{32}$$

where $\mathcal{O}_\tau = End_K(\Lambda_\tau)$ and $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$.

Remark 4.1 Note that the notion involved in (32) differs slightly from the one in [DD06] since in their setting $\mathcal{O}$ was assumed to be $\mathbb{Z}[\frac{1}{p}]$-orders instead of $\mathbb{Z}$-orders. Therefore there is no need to tensor over $\mathbb{Z}[\frac{1}{p}]$. One can verify that the set $\mathcal{H}_p^{\mathcal{O}}(N)$ is nonempty if and only if there exists an $\mathcal{O}$-ideal $\mathfrak{N}$ such that $\mathcal{O}/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$; this is the so-called Heegner hypothesis.

We propose the following generalization of $\mathcal{H}_p^{\mathcal{O}}(N)$.

Definition 4.1 *Let (N_0, f, p) and $(K, \mathfrak{N})$ be as in the introduction. Let $\mathcal{O}$ be an order of K of conductor coprime to N_0 and let $\mathfrak{n} = \mathcal{O} \cap \mathfrak{N}$. To such data we associate the following sets*

(1) $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$,
(2) $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(\mathfrak{n})$,
(3) $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$,
(4) $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(\mathfrak{n}, f)$,

where

$$\mathcal{H}_p^{\mathcal{O}}(\mathfrak{n}) := \{\tau \in \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N_0\tau}^{(p)} = \mathcal{O}^{(p)}, \mathfrak{n}\Lambda_\tau^{(p)} = \Lambda_{N_0\tau}^{(p)}, \tau - \tau^\sigma > 0\},$$

$$\mathcal{H}_p^{\mathcal{O}}(N_0, f) := \{\tau \in \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N_0\tau}^{(p)} = \mathcal{O}^{(p)}, (\Lambda_\tau^{(p)}, f\mathcal{O}^{(p)}) = 1, \tau - \tau^\sigma > 0\},$$

and

$$\mathcal{H}_p^{\mathcal{O}}(\mathfrak{n}, f) := \{\tau \in \mathcal{H}_p : \mathcal{O}_\tau^{(p)} = \mathcal{O}_{N_0\tau}^{(p)} = \mathcal{O}^{(p)}, \mathfrak{n}\Lambda_\tau^{(p)} = \Lambda_{N_0\tau}^{(p)}, (\Lambda_\tau^{(p)}, f\mathcal{O}^{(p)}) = 1, \tau - \tau^\sigma > 0\}.$$

Note that the notation $(\Lambda_\tau^{(p)}, f) = 1$ is equivalent to say that $(A, f) = 1$ where $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$. One has the two "stratifications"

(1) $\coprod_{\mathfrak{n}} \big(\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(\mathfrak{n})\big) = \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$,
(2) $\coprod_{\mathfrak{n}} \big(\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(\mathfrak{n}, f)\big) = \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$,

where the two disjoint unions run over the elements of the set

$$\{\mathfrak{n} \trianglelefteq \mathcal{O} : \mathfrak{n} \text{ is an invertible } \mathcal{O}\text{-ideal and } \mathcal{O}/\mathfrak{n} \simeq \mathbb{Z}/N_0\mathbb{Z}\}. \tag{33}$$

Definition 4.2 *We have a natural left action of $\widetilde{\Gamma}_0$ on the set $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$ given by*

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} \star (r, \tau) := \left(dr, \frac{a\tau + b}{c\tau + d}\right).$$

If there exists a $\gamma \in \widetilde{\Gamma}_0$ such that $\gamma \star (r, \tau) = (r', \tau')$ then we simply write $(r, \tau) \approx (r', \tau')$.

We now define a map that allows us to go from the set $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ to the set of integral $\mathcal{O}^{(p)}$-ideals.

Definition 4.3 *We define a map Ω (which depends on p and f)*

$$\Omega : \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p(N_0, f) \to \left\{\mathbb{Z}[\tfrac{1}{p}]\text{-modules contained in } K \text{ of rank } 2\right\},$$

by the rule

$$(r, \tau) \mapsto A_r \Lambda_\tau^{(p)},$$

where $0 \neq A_r \in \mathbb{Z}_{>0}$ *is the smallest integer such that the following properties hold:*

(1) $A_r \equiv r \pmod{f}$,

(2) $A_r \Lambda_\tau^{(p)}$ *is* $\mathcal{O}^{(p)}$*-integral,*

where $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ *and* $\mathcal{O}^{(p)} = End_K(\Lambda_\tau^{(p)})$.

Remark 4.2 In the definition of the map Ω, it is important to assume that $\tau \in \mathcal{H}_p(N_0, f)$ otherwise the integer A_r doesn't always exist.

Now we introduce another equivalence relation, denoted by $\sim$, on $\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$.

Definition 4.4 *Let* $(r, \tau), (r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p(N_0)$. *We say that* $(r, \tau) \sim (r', \tau')$ *if and only if there exists a totally positive element*

$$\lambda \in 1 + f \left(\widetilde{r'} \Lambda_{\tau'}^{(p)}\right)^{-1}$$

such that $\left(\widetilde{r} \Lambda_\tau^{(p)}, \widetilde{r} \Lambda_{N_0\tau}^{(p)}\right) = \left(\lambda \widetilde{r'} \Lambda_{\tau'}^{(p)}, \lambda \widetilde{r'} \Lambda_{N_0\tau'}^{(p)}\right)$ *where* $\widetilde{r}, \widetilde{r'}$ *are the unique integers such that* $1 \leq \widetilde{r}, \widetilde{r'} \leq f$, $\widetilde{r} \equiv r \pmod{f}$ *and* $\widetilde{r'} \equiv r' \pmod{f}$.

Remark 4.3 It is an easy exercise to see that the two stratifications (1) and (2) appearing at the bottom of Definition 4.1 are preserved under the equivalence relation $\sim$. In the case where $(r, \tau), (r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$ it is easy to see that $(r, \tau) \sim (r', \tau')$ if and only if there exists a totally positive element $\lambda \in 1 + f\Omega(r', \tau')^{-1}$ such that

$$(A_r \Lambda_\tau^{(p)}, A_r \Lambda_{N_0\tau}^{(p)}) = (\lambda A'_{r'} \Lambda_{\tau'}^{(p)}, \lambda A'_{r'} \Lambda_{N_0\tau'}^{(p)}).$$

where $A_r \Lambda_\tau^{(p)} = \Omega(r, \tau)$ and $A'_{r'} \Lambda_{\tau'}^{(p)} = \Omega(r', \tau')$.

Lemma 4.1 *The equivalence relations induced by* $\sim$ *and* $\approx$*, when restricted to the distinguished subset* $(\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0, f) \subseteq \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$*, are the same.*

Proof See section 5 of [Cha]. $\square$

Corollary 4.1 *The stratification*

$$(\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0, f) = \coprod_{\mathfrak{n}} \left((\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(\mathfrak{n}, f)\right) \tag{34}$$

is preserved under $\approx$.

Define the set $\mathcal{M}_K(N_0, f, p)$ to be

$$\{(L, M) : \text{ pairs of } \mathbb{Z}[\tfrac{1}{p}]\text{-modules of rank 2 in } K, End_K(L) = End_K(M) = \mathcal{O}^{(p)},$$
$$(L, f\mathcal{O}^{(p)}) = (M, f\mathcal{O}^{(p)}) = 1 \text{ and } L/M \simeq \mathbb{Z}/N_0\mathbb{Z}\}.$$

We have a natural equivalence relation on $\mathcal{M}(N_0, f, p)$ which we denote again by $\sim$, where $(L, M) \sim (L', M')$ if and only if there exists a totally positive element $\lambda \in 1 + fL'^{-1}$ such that $(L, M) = (\lambda L', \lambda M')$.

Proposition 4.1 *There exists a natural bijection of sets, which we denote by ψ, between*

$$\psi : \left((\mathbb{Z}/f\mathbb{Z})^{\times} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)\right) / \sim \longrightarrow \mathcal{M}_K(N_0, f, p)/ \sim,$$

where $\psi([(r,\tau)]) := \left[(A_r\Lambda_\tau^{(p)}, A_r\Lambda_{N_0\tau}^{(p)})\right]$ *and* $A_r\Lambda_\tau^{(p)} = \Omega(r,\tau)$. *(the brackets denote the class modulo $\sim$)*

Proof First define the map $\widetilde{\psi} : (\mathbb{Z}/f\mathbb{Z})^{\times} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f) \to \mathcal{M}_K(N_0, f, p)$, by

$$\widetilde{\psi}(r,\tau) = (A_r\Lambda_\tau^{(p)}, A_r\Lambda_{N_0\tau}^{(p)}),$$

where $A_r\Lambda_\tau^{(p)} = \Omega(r,\tau)$. A direct calculation shows that the map $\widetilde{\psi}$ descends to a well defined map when one goes to the quotient on both sides; we denote this new map by ψ. Now let us construct a "map"going in the the other direction. Let $(L, M) \in \mathcal{M}_K(N_0, f, p)$. Because $(L, f\mathcal{O}^{(p)}) = 1$, there exists an integer $a \in \mathbb{Z}_{>0}$ such that $a \equiv 1 \pmod f$ and aL is $\mathcal{O}^{(p)}$-integral. We can thus assume beforehand that L is $\mathcal{O}^{(p)}$-integral without changing the class of modulo $\sim$. Let $\mathcal{O}^{(p)} = \mathbb{Z}[\frac{1}{p}] + \mathbb{Z}[\frac{1}{p}]\omega$. Because $L/M \simeq \mathbb{Z}/N_0\mathbb{Z}$, there exists an ordered $\mathbb{Z}[\frac{1}{p}]$-basis (ω_1, ω_2) of L such that $L = \mathbb{Z}[\frac{1}{p}]\omega_1 + \mathbb{Z}[\frac{1}{p}]\omega_2$ and $M = \mathbb{Z}[\frac{1}{p}]\omega_1 + \mathbb{Z}[\frac{1}{p}]N_0\omega_2$. We claim that we can always choose ω_1 in such a way that $\omega_1 \equiv \textit{integer} \pmod f$. Let us prove it:

If $\omega_1 \equiv \textit{integer} \pmod f$ then we are done. Let us suppose that $\omega_1 \not\equiv \textit{integer} \pmod f$. In this case one can assume without lost of generality that $\omega_2 = a + b\omega$ where $a, b \in \mathbb{Z}[\frac{1}{p}]$ and $b \not\equiv 0 \pmod f$, otherwise replace ω_2 by $\omega_2 + \omega_1$. Now since N_0 is coprime to f one can find an integer k such that $\omega_1 - N_0k\omega_2 \equiv \textit{integer} \pmod f$. Then the new basis $\{\widetilde{\omega}_1, \widetilde{\omega}_2\}$ where $\widetilde{\omega}_1 = (\omega_1 - N_0k\omega_2)$ and $\widetilde{\omega}_2 = \omega_2$ satisfies the required property.

Let $\omega_1 \equiv u \pmod f$ where $u \in \mathbb{Z}$. Because $(L, f\mathcal{O}) = 1$ we have $(u, f) = 1$. Therefore there exists an $\alpha \in \mathcal{O}^{(p)}$ such that $\alpha\omega_1 \equiv 1 \pmod f$ and $\alpha\omega_1 \gg 0$. Note that $\alpha \equiv u^{-1} \pmod f$. Now we can write the pair (L, M) as

$$\begin{aligned}(L, M) &= \left(\alpha^{-1}\left(\mathbb{Z}[\tfrac{1}{p}]\alpha\omega_1 + \mathbb{Z}[\tfrac{1}{p}]\alpha\omega_2\right), \alpha^{-1}\left(\mathbb{Z}[\tfrac{1}{p}] + \mathbb{Z}[\tfrac{1}{p}]N_0\frac{\omega_2}{\omega_1}\right)\right) \\ &\sim \left(\alpha^{-1}\left(\mathbb{Z}[\tfrac{1}{p}] + \mathbb{Z}[\tfrac{1}{p}]\frac{\omega_2}{\omega_1}\right), \alpha^{-1}\left(\mathbb{Z}[\tfrac{1}{p}] + \mathbb{Z}[\tfrac{1}{p}]N_0\frac{\omega_2}{\omega_1}\right)\right).\end{aligned}$$

Now set $\tau = \frac{\omega_2}{\omega_1}$. Without lost of generality we can assume that $\tau > \tau^{\sigma}$ otherwise replace τ by $-\tau$. Finally, we send the pair (L, M) on the pair $(u, \tau) \in (\mathbb{Z}/f\mathbb{Z})^{\times} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$. One can check that this construction gives a well defined map (when one descends to the quotient on both sides) which is an inverse of ψ. $\square$

Let $\mathfrak{f}^{(p)} = f\mathcal{O}^{(p)}$. Class field theory gives an isomorphism

$$I_{\mathcal{O}^{(p)}}(\mathfrak{f}^{(p)})/\sim_{\mathfrak{f}^{(p)}} \overset{rec^{-1}}{\to} Gal(K(\mathfrak{f}^{(p)}\infty)/K),$$

where $K(\mathfrak{f}^{(p)}\infty)$ is the abelian extension of K which corresponds by class field theory to the ideal class group $I_{\mathcal{O}^{(p)}}(\mathfrak{f}^{(p)})/\sim_{\mathfrak{f}^{(p)}}$. Note that $K(\mathfrak{f}^{(p)}\infty) = K(\mathfrak{f}\infty)^{\langle Fr_{\wp}\rangle}$ where $p\mathcal{O} = \wp$, $K(\mathfrak{f}\infty)$ is the abelian extension corresponding to the ideal class group $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ and $K(\mathfrak{f}\infty)^{\langle Fr_{\wp}\rangle}$ is the subfield of $K(\mathfrak{f}\infty)$ fixed by the Frobenius at $\wp$. We set $L := K(\mathfrak{f}\infty)^{\langle Fr_{\wp}\rangle}$.

Using Proposition 4.1 we see that there is a natural action of $G_{L/K}$ on $((\mathbb{Z}/f\mathbb{Z})^{\times} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f))/\sim$ given by the following rule: Let $[(r,\tau)] \in ((\mathbb{Z}/f\mathbb{Z})^{\times} \times \mathcal{H}_p^{\mathcal{O}}(N_0, f))/\sim$ and $\psi[(r,\tau)] = [(L, M)]$. Now define

$$rec^{-1}(\mathfrak{b}) \star [(r,\tau)] := \psi^{-1}[(\mathfrak{b}L, \mathfrak{b}M)].$$

Note that this Galois action preserves the stratification (34). From this one sees directly that this action is simple but in general not transitive since the indexing set of the stratification might be of size larger than one.

4.1 Shimura reciprocity law

We are now ready to formulate the Shimura reciprocity law which describes the action of G_K on $u(r,\tau)$. We assume in the next conjecture that the number field L is totally complex otherwise the conjecture says nothing interesting.

Conjecture 4.1 *Let $(r,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0, f)$. Then*

$$u(r,\tau) \in \mathcal{O}_L[\tfrac{1}{p}]^\times,$$

where $L = K(f\mathcal{O}\infty)^{\langle Fr_\wp \rangle}$, $\wp = p\mathcal{O}$. Moreover, we have a Shimura reciprocity law. Let

$$rec : G_{L/K} \to I_{\mathcal{O}}(\mathfrak{f})/\langle Q_{\mathcal{O}}(\mathfrak{f}\infty), p\rangle.$$

where $\mathfrak{f} = f\mathcal{O}$. Then for $\sigma \in G_{L/K}$ we have

$$u(k,\tau)^{\sigma^{-1}} = u(k',\tau') \pmod{\mu_{p^2-1}},$$

where $\sigma \star [(k,\tau)] = [(k',\tau')]$. Furthermore, if we let c_∞ denotes the complex conjugation in $G_{L/K}$, then

$$u(r,\tau)^{c_\infty} = u(r,\tau)^{-1} \pmod{\mu_{p^2-1}}.$$

Remark 4.4 In [DD06], since the conductor $f = 1$, one is led to consider various orders of K. However in our case, since f can vary, it is sufficient to consider only the case, where $\mathcal{O} = \mathcal{O}_K$. As explained before the statement of the theorem, if we want our construction to be interesting, it is essential to assume beforehand that L is totally complex. Let $L = K(\mathfrak{f}\infty)^{\langle Fr_\wp \rangle}$ where $\mathfrak{f} = f\mathcal{O}_K$. Then Class field theory implies that L is totally complex if and only if the index

$$n := [\mathcal{O}_K[\tfrac{1}{p}](f)^\times : \mathcal{O}_K[\tfrac{1}{p}](f\infty)^\times],$$

is equal to 1 or 2, where $\mathcal{O}_K[\frac{1}{p}](f\infty)^\times$ corresponds to the group of totally positive units of $\mathcal{O}_K[\frac{1}{p}]$ congruent to 1 modulo f and $\mathcal{O}_K[\frac{1}{p}](f)^\times$ corresponds to the group of units of $\mathcal{O}_K[\frac{1}{p}]$ congruent to 1 modulo f. We expect $u(r,\tau)$ to be contained in the largest CM subfield contained in L which we denote by L_{CM}. In general the field L_{CM} can be a proper subfield of L of index 2 (see Proposition 7.1 of [Cha07a]).

5. Special values of zeta functions and periods of Eisenstein series

In this section we introduce various zeta functions and we show how their special values are related to certain periods on Eisenstein series.

5.1 The zeta function twisted by an additive character

Let K be a real quadratic field with discriminant D and fix a positive integer f coprime to D. We let $\mathcal{O}_K(\infty)^\times$ stand for the group of totally positive units of $\mathcal{O}_K$.

Definition 5.1 *Let $\mathfrak{a}$ be an integral $\mathcal{O}_K$-ideal. We define*

$$\Psi(\mathfrak{a}, f, w_1, s) := \mathbf{N}_{K/\mathbb{Q}}\left(\frac{\mathfrak{a}}{f\sqrt{D}}\right)^s \sum_{\Gamma_{\mathfrak{a}} \backslash \left\{0 \neq \mu \in \frac{\mathfrak{a}}{f\sqrt{D}}\right\}} \frac{w_1(\mu) e^{2\pi i Tr_{K/\mathbb{Q}}(\mu)}}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s},$$

where $\Gamma_{\mathfrak{a}} = \mathcal{O}_K(\infty)^\times \cap (1 + f\mathfrak{a}^{-1})$ and w_1 is the sign character given by $sign \circ N_{K/\mathbb{Q}}$.

It is easy to see that the first entry of Ψ depends only on the the narrow ray class modulo f, i.e., for $\mathfrak{a},\mathfrak{b}\in I_{\mathcal{O}_K}(1)$ if $\mathfrak{a}\sim_f\mathfrak{b}$ then $\Psi(\mathfrak{a},f,w_1,s)=\Psi(\mathfrak{b},f,w_1,s)$.

For any point $\tau\in\mathcal{H}_p\cap K$ we let $Q_\tau(x,y)=A(x-\tau y)(x-\tau^\sigma y)=Ax^2+Bxy+Cy^2$ ($A,B,C\in\mathbb{Z}$, $A>0$ and $(A,B,C)=1$) be the primitive quadratic form associated to τ. We always have the formulas $\mathbf{N}_{K/\mathbb{Q}}(\Lambda_\tau)=\frac{1}{A}$ and $cond(\mathcal{O}_\tau)^2D=B^2-4AC$ where $Q_\tau(x,y)=Ax^2+Bxy+Cy^2$ and $cond(\mathcal{O}_\tau)$ is the conductor of the order $\mathcal{O}_\tau$.

Let $\widetilde{A}\Lambda_\tau$ be an integral $\mathcal{O}_K$-ideal where $\widetilde{A}\in\mathbb{Z}_{>0}$ and $\tau\in\mathcal{H}_p\cap K$. Note that $A|\widetilde{A}$ where $(\frac{1}{A})=\mathbf{N}_{K/\mathbb{Q}}(\Lambda_\tau)$. A direct calculation shows that

$$\Psi(\widetilde{A}\Lambda_\tau,f,w_1,s)=-\sum_{\langle\eta_\tau\rangle\backslash\{(m,n)\in\mathbb{Z}^2\backslash(0,0)\}}\frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}e^{\frac{-2\pi i\frac{\widetilde{A}}{A}n}{f}},\quad Re(s)>1, \tag{35}$$

where η_τ is the matrix corresponding to the action of the generator $\epsilon>1$ of $\Gamma_{\widetilde{A}\Lambda_\tau}=\mathcal{O}_K(\infty)^\times\cap f(\widetilde{A}\Lambda_\tau)^{-1}$ on the lattice Λ_τ with respect to the ordered basis $\{\tau,1\}$. The action of a matrix $\begin{pmatrix} a & b\\ c & d\end{pmatrix}$ acting on the vector (x,y) is given by $(ax+by,cx+dy)$. In the case where $(\widetilde{A},f)=1$ one has that $c\equiv 0\pmod f$ and $d\equiv 1\pmod f$. In fact one can show that $\langle\pm\eta_\tau\rangle=Stab_{\Gamma_1(f)}(\tau)$. Based on the previous discussion we introduce the following zeta function.

Definition 5.2 *Let $(r,\tau)\in\mathbb{Z}/f\mathbb{Z}\times(\mathcal{H}_p\cap K)$. We define*

$$\zeta((r,\tau),s)=-\sum_{\langle\eta_\tau\rangle\backslash\{(m,n)\in\mathbb{Z}^2\backslash(0,0)\}}\frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}e^{\frac{-2\pi irn}{f}},\quad Re(s)>1,$$

where $Q_\tau(x,y)=Ax^2+Bxy+Cy^2$ and $\langle\pm\eta_\tau\rangle=Stab_{\Gamma_1(f)}(\tau)$.

A direct calculation shows that if (r,τ) is equivalent to (r',τ') under the action of $\Gamma_0(f)$ then $\zeta((r,\tau),s)=\zeta((r',\tau'),s)$. Let $\widetilde{A}\in\mathbb{Z}$ and $\widetilde{A}\Lambda_\tau$ be an integral $\mathcal{O}_K$-ideal. Let r be the image of the integer $\widetilde{A}/A$ inside $\mathbb{Z}/f\mathbb{Z}$. Then from (35) one readily sees that

$$\Psi(\widetilde{A}\Lambda_\tau,f,w_1,s)=\zeta((r,\tau),s).$$

Now we want to define a dual zeta function to $\zeta((r,\tau),s)$ (dual in the sense of the functional equation).

Definition 5.3 *Let $(r,\tau)\in\mathbb{Z}/f\mathbb{Z}\times(\mathcal{H}_p\cap K)$. We define*

$$\widehat{\zeta}((r,\tau),s):=f^{2s}\sum_{\langle\eta_\tau\rangle\backslash(0\neq(m,n)\equiv(r,0)\pmod f)}\frac{sign(Q_\tau(m,n))}{|Q_\tau(m,n)|^s},\quad Re(s)>1, \tag{36}$$

where $\langle\pm\eta_\tau\rangle=Stab_{\Gamma_1(f)}(\tau)$.

Note that the matrix η_τ preserves the congruence $(r,0)\pmod f$.

There is a functional equation which relates $\zeta((r,\tau),s)$ to $\widehat{\zeta}((r,\tau),s)$.

Theorem 5.1 *Let $(r,\tau)\in\mathbb{Z}/f\mathbb{Z}\times(\mathcal{H}_p\cap K)$. Then we have*

$$-F_{w_1}(s)\zeta((r,\tau),s)=F_{w_1}(1-s)\widehat{\zeta}((r,\tau),1-s),\quad Re(s)<-1, \tag{37}$$

where $F_{w_1}(s)=disc(Q_\tau)^{s/2}\pi^{-s}\Gamma\left(\frac{s+1}{2}\right)^2$.

Proof Note that the left hand side of (37) makes sense when $Re(s)<-1$, since ζ admits a meromorphic continuation to $\mathbb{C}$ (see Corollary 8.1 of [Cha]). All the essential ingredients in the proof of the previous theorem can be found in [Sie68]. For a detailed proof, see Section 8.3 of [Cha]. □

By the previous theorem we can now evaluate the zeta function $\widehat{\zeta}((r,\tau),s)$ at negative integers. The key result concerning these special values is due to Siegel.

Theorem 5.2 (**Siegel**) *For any even integer $n \leq 0$ we have that $\widehat{\zeta}((r,\tau),n)$ is a rational number.*

Proof In [Sie68], Siegel gives explicit formulas for the values $\zeta((r,\tau),n)$ for odd integers $n \geq 1$. These values are equal to a certain power of π times a rational number for which he gives an explicit formula involving Bernoulli polynomials. Using the functional equation of $\zeta((r,\tau),s)$ and Siegel's explicit formulas for the values of $\zeta((r,\tau),n)$, where $n \geq 1$ is an odd integer, we deduce the rationality of $\widehat{\zeta}((r,\tau),n)$ for even integers $n \leq 0$. □

5.2 Archimedean zeta function associated to a class in $(\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0))/\widetilde{\Gamma}_0$

In this subsection we want to associate to any class in $(\mathbb{Z}/f\mathbb{Z}\times\mathcal{H}_p^{\mathcal{O}}(N_0))/\widetilde{\Gamma}_0$ a well defined Archimedean zeta function. We first start by stating a useful elementary lemma.

Lemma 5.1 *Let $\tau \in \mathcal{H}_p \cap K$ such that $(disc(Q_\tau),p) = 1$ and let*

$$\langle \pm\gamma_\tau \rangle = Stab_{SL_2(\mathbb{Z}[\frac{1}{p}])}(\tau).$$

Then $\gamma_\tau \in SL_2(\mathbb{Z})$ where γ_τ is well defined up to ± 1.

Proof See the proof of Lemma 8.1 in [Cha]. □

Remark 5.1 It is easy to show that if τ is reduced, i.e., if $red(\tau) = v_0$ where v_0 is the standard vertex on the Bruhat-Tits tree and red is the reduction map, then $(disc(Q_\tau),p) = 1$. However the converse is false. We can therefore think of the reduced requirement as a finer notion compared to the more naive condition $(disc(Q_\tau),p) = 1$.

Proposition 5.1 *Let $(r,\tau),(r',\tau') \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$ and assume that $red(\tau) = red(\tau') = v_0$. Then if $(r',\tau') \approx (r,\tau)$, i.e., if there exists a $\gamma \in \widetilde{\Gamma}_0$ such that $(r',\tau') = \gamma \star (r,\tau)$ then we have*

$$\zeta((r,\tau),s) = \zeta((r',\tau'),s) \text{ and } \zeta((r,\tau^*),s) = \zeta((r',\tau'^*),s),$$

where $\tau^* = \frac{1}{fN_0\tau}$ and $\tau'^* = \frac{1}{fN_0\tau'}$.

Proof Since τ and τ' are reduced we have $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$. It thus follows that $\zeta((r,\tau),s) = \zeta((r',\tau'),s)$. Let us show the other equality. A direct calculation reveals that

$$\begin{pmatrix} d & c/fN_0 \\ bfN_0 & a \end{pmatrix}\begin{pmatrix} \tau^* \\ 1 \end{pmatrix} = \begin{pmatrix} \tau'^* \\ 1 \end{pmatrix}.$$

It follows that (r,τ^*) and (r',τ'^*) are $\Gamma_0(f)$-equivalent and thus $\zeta((r,\tau^*),s) = \zeta((r',\tau'^*),s)$. □

We have thus succeeded to attach well defined Archimedean zeta functions to any class of $(\mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0))/\widetilde{\Gamma}_0$. So far we have not used the level N_0-structure built in $\mathcal{H}_p^{\mathcal{O}}(N_0)$. The next object we define is a zeta function attached to a good divisor $\delta \in D(N_0,f)$ and a pair $(r,\tau) \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$.

Definition 5.4 *Let* $\delta = \sum_{d_0,r} n(d_0,r)[d_0,r] \in D(N_0,f)$ *be a good divisor and* $(j,\tau) \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$ *with* $red(\tau) = v_0$ *and* $j \in (\mathbb{Z}/f\mathbb{Z})^\times$*. Then we define*

(1) $\zeta(\delta_j,(1,\tau),s) := \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(d_0,r) d_0^s \widehat{\zeta}((rj,d_0\tau),s),$

(2) $\zeta^*(\delta_j,(1,\tau),s) := \sum_{d_0|N_0, r\in\mathbb{Z}/f\mathbb{Z}} n(\frac{N_0}{d_0},r) d_0^s \widehat{\zeta}((-rj,d_0\tau^*),s)$ *where* $\tau^* = \frac{1}{fN_0\tau}$.

With the help of Proposition 5.1 it is an easy exercise to show that $\zeta(\delta_j,(1,\tau),s)$ and $\zeta^*(\delta_j,(1,\tau),s)$ depend only on the class of $(1,\tau)$ modulo $\approx$ when τ is reduced. We have the formulas $\zeta(\delta_{aj},(1,\tau),s) = \zeta(\delta_j,(a,\tau),s)$ and $\zeta^*(\delta_{aj},(1,\tau),s) = \zeta^*(\delta_j,(a,\tau),s)$ for all $a \in \mathbb{Z}/f\mathbb{Z}$.

Remark 5.2 Note that there is a hat on zeta functions appearing on the right hand side of (1) and (2). One can think of $*$ as an involution on the set $\mathbb{Z}/f\mathbb{Z} \times (\mathcal{H}_p \cap K)$ given by $(r,\tau) \mapsto (-r,\tau^*)$ where $\tau^* = \frac{1}{fN_0\tau}$. This involution $*$ allows us to relate our p-adic invariant $u(r,\tau)$ to the Gross-Stark p-units; see [Cha07b]. If we restrict this involution to the distinguished subset $(\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0,f)$ then we obtain a map

$$* : (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0,f) \to (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}^*}(N_0)$$
$$(r,\tau) \mapsto (-r,\tau^*),$$

where $\tau^* = \frac{1}{fN_0}$ and $cond(\mathcal{O}^*) = f \cdot cond(\mathcal{O})$. Note that if $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$, then $Q_{\tau^*}(x,y) = sign(C)(CN_0f^2x^2 + Bfxy + \frac{A}{N_0}y^2)$.

5.3 The special values $\zeta^*(\delta,(r,\tau),1-k)$ as integrals of Eisenstein series $\widetilde{F}_{2k}(r,z)$

We would like to state a result which relates periods of Eisenstein series to special values of the Archimedean zeta functions $\zeta(\delta_r,(1,\tau),s)$ and $\zeta^*(\delta_r,(1,\tau),s)$.

Proposition 5.2 *Let* $(j,\tau) \in \mathbb{Z}/f\mathbb{Z} \times \mathcal{H}_p^{\mathcal{O}}(N_0)$ *where* $j \in (\mathbb{Z}/f\mathbb{Z})^\times$ *and* $red(\tau) = v_0$*. Then for all odd integers* $k \geq 1$ *we have*

(1) $3\zeta^*(\delta_j,(1,\tau),1-k) = \int_{\xi_2}^{\gamma_{\tau^*}\xi_2} Q_{\tau^*}(z,1)^{k-1}\widetilde{F}^*_{2k}(j,z)dz$

$\qquad = f^{2k-2}\int_{\xi_1}^{\gamma_\tau\xi_1} Q_\tau(z,1)^{k-1}\widetilde{F}_{2k}(j,z)dz,$

(2) $3\zeta(\delta_j,(1,\tau),1-k) = \int_{\xi_2}^{\gamma_\tau\xi_2} Q_\tau(z,1)^{k-1}\widetilde{F}^*_{2k}(j,z)dz,$

where $\tau^* = \frac{1}{fN_0\tau}$, $\xi_1 = \infty$, $\xi_2 = 0$, $\langle \pm\gamma_\tau \rangle = Stab_{\Gamma_1}(\tau)$ *with* $c\tau + d > 1$ *for* $\gamma_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$.

Proof The second equality of (1) follows from (3). The proof of (1) and (2) is similar to the proof of Proposition 3.2 of [DD06]. For a detailed proof see Lemma 9.2 of [Cha]. □

6. P-adic zeta functions and p-adic Kronecker limit formula

Definition 6.1 *Let* $\delta \in D(N_0,f)$ *be a good divisor and let* $(j,\tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times \mathcal{H}_p^{\mathcal{O}}(N_0)$ *such that* $(disc(Q_\tau),p) = 1$*. We define the p-adic zeta function*

$$\zeta_p^*(\delta_j,(1,\tau),s) := \frac{1}{3}\int_{\mathbb{X}} \langle Q_{f\tau}(fx,y)\rangle^{-s} d\widetilde{\mu}_j\{\infty \to \gamma_\tau\infty\}(x,y) \tag{38}$$
$$= \frac{1}{3}\langle f\rangle^{-2s}\int_{\mathbb{X}} \langle Q_\tau(x,y)\rangle^{-s} d\widetilde{\mu}_j\{\infty \to \gamma_\tau\infty\}(x,y),$$

where $\langle x\rangle$ denotes the unique element in $1+p\mathbb{Z}_p$ that differs from x by a $(p-1)$-th root of unity.

This zeta function makes sense for any $s\in\mathbb{Z}_p$ and as usual $\langle\pm\gamma_\tau\rangle = Stab_{\Gamma_1}(\tau)$.

Corollary 6.1 *For an even integer $n\leq 0$ congruent to 0 modulo $p-1$, we have*

$$(1-p^{-2n})\zeta^*(\delta_j,(1,\tau),n)=\zeta_p^*(\delta_j,(1,\tau),n).$$

Proof Combine (1) of Proposition 5.2 with (1) of Theorem 1.1. □

Remark 6.1 We thus see that our p-adic zeta function interpolates rational values of the Archimedean zeta function $\zeta^*(\delta_j,(1,\tau),s)$ at negative integers.

Lemma 6.1 *The derivative $(\zeta_p^*)'(\delta_j,(1,\tau),0)$ at $s=0$ is given by*

$$(\zeta_p^*)'(\delta_j,(1,\tau),0)=-\frac{1}{3}\int_{\mathbb{X}}\log_p(Q_\tau(x,y))d\widetilde{\mu}_j\{\xi\to\gamma_\tau\xi\}(x,y)\ \ where\ \ \xi=\infty.$$

Proof This is a direct calculation using equation (38). Note that the integral over $\mathbb{X}$ of $\log_p(Q_{f\tau}(fx,y)) = \log_p f^2+\log_p Q_\tau(x,y)$ is the same as $\log_p Q_\tau(x,y)$ since the total measure is zero so that the constant term $\log_p f^2$ vanishes. □

We can now deduce a p-adic Kronecker limit formula.

Theorem 6.1 *Let $(r,\tau)\in(\mathbb{Z}/f\mathbb{Z})^\times\times\mathcal{H}_p^{\mathcal{O}}(N_0)$ with τ reduced, i.e., $red(\tau)=v_0$. Then*

$$3(\zeta_p^*)'(\delta_r,(1,\tau),0)=-\log_p \mathbf{N}_{K_p/\mathbb{Q}_p}(u(\delta_r,\tau)). \tag{39}$$

Proof From Theorem 2.2 we have

$$\log_p u(\delta_r,\tau)=\int_{\mathbb{X}}\log_p(x-\tau y)d\widetilde{\mu}_r\{\infty\to\gamma_\tau\infty\}(x,y). \tag{40}$$

Replacing τ by τ^σ in the previous identity gives us

$$\log_p u(\delta_r,\tau^\sigma)=\int_{\mathbb{X}}\log_p(x-\tau^\sigma y)d\widetilde{\mu}_r\{\infty\to\gamma_{\tau^\sigma}(\infty)\}(x,y). \tag{41}$$

But $\gamma_\tau=\gamma_{\tau^\sigma}$. Therefore adding (40) and (41) and using Lemma 6.1 gives (39). □

Proposition 6.1 *We have $3\zeta^*(\delta_r,(1,\tau),0)=ord_p(u(\delta_r,\tau))$.*

Proof Combine Corollary 2.1 with (1) of Proposition 5.2 after having set $k=1$. □

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