Relationships between p-unit constructions for real quadratic fields

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Abstract

Let K be a real quadratic field and let p be a prime number which is inert in K. Let K_p be the completion of K at p. In a previous paper, we constructed a p-adic invariant $u_C \in K_p^\times$, and we proved a p-adic Kronecker limit formula relating u_C to the first derivative at $s = 0$ of a certain p-adic zeta function. By analogy with the p-adic Gross-Stark conjectures, we conjectured that u_C is a p-unit in a suitable narrow ray class field of K. Recently, Dasgupta has proposed an exact p-adic formula for the Gross-Stark units of an arbitrary totally real number field. In our special setting, i.e., where one deals with a real quadratic number field, his construction produces a p-adic invariant $u_D \in K_p^\times$. In this paper we show precise relationships between the p-adic invariants u_C and u_D. In order to do so, we extend Dasgupta's construction of u_D to a broader setting.

Contents

1 Introduction

Let K be a real quadratic number field of discriminant d_K. Let p be a prime number which is inert in K and denote by K_p the completion of K at p. Let $\mathcal{H}_p = \mathbb{P}^1(\mathbb{C}_p)\backslash\mathbb{P}^1(\mathbb{Q}_p)$ be the p-adic upper half plane endowed with the structure of a rigid analytic space. Note that $\mathcal{H}_p \cap K = K\backslash\mathbb{Q}$ since p was assumed to be inert in K. We denote the maximal order of K by $\mathcal{O}_K = \mathbb{Z}+\omega\mathbb{Z}$ where $\omega = \frac{d_K+\sqrt{d_K}}{2}$ and its unique order of conductor $n \geq 1$ by $\mathcal{O}_n = \mathbb{Z}+n\omega\mathbb{Z}$. For $\tau \in \mathcal{H}_p \cap K$ we let $\Lambda_\tau = \mathbb{Z} + \tau\mathbb{Z}$ and $\mathcal{O}_\tau = \mathrm{End}_K(\Lambda_\tau) = \{\lambda \in K : \lambda\Lambda_\tau \subseteq \Lambda_\tau\}$. Let N be a positive square-free integer coprime to d_K. The square-free condition holds only for the introduction since it simplifies the presentation. Now make the following crucial assumption: there exists an ideal $\mathfrak{N}$ such that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$. The last condition is equivalent to saying that $N = \prod_{i=1}^r l_i$ where each l_i is a prime number which splits in K, the l_i's being distinct. The latter assumption is often called the *Heegner hypothesis* by analogy with the construction of Heegner points on modular elliptic curves (see for example Chap. 3 of [Dar04]). Now choose a positive integer n which is coprime to pNd_K. Let $\alpha(z)$ be a modular unit for the congruence group $\Gamma_0(N)$ which has no zeros nor poles on the set $\Gamma_0(N)\{\infty\}$ where ∞ corresponds to the cusp $\frac{1}{0}$ (for an example of such modular unit see

equation (B.3) in Appendix B). Let $\tau \in \mathcal{H}_p \cap K$ be such that $\mathcal{O}_\tau = \mathcal{O}_{N\tau} = \mathcal{O}_n$. In [DD06], Darmon and Dasgupta constructed a p-adic invariant $u_{DD}(\alpha, \tau) \in K_p^\times$. They conjectured that $u_{DD}(\alpha, \tau)$ is a global p-unit in $K(\mathcal{O}_n\infty)$, the *narrow ring class field* of K of conductor n. From now on we assume that the modular unit $\alpha(z)$ is fixed so we omit it from the notation and therefore we write $u_{DD}(\tau)$ instead of $u_{DD}(\alpha, \tau)$.

Conjecturally, the construction mentioned above produces p-units in the *ring class fields* of K. It is then natural to try to extend it to the larger setting of *ray class fields* of K. This was accomplished in the PhD thesis of the author, see [Cha] and [Cha09c]. Let us explain very briefly what kind of units we expect to construct in this more general setting. Let f and n be fixed positive integers which are prime to pNd_K. Let $\{\beta_r(z)\}_{r \in \mathbb{Z}/f\mathbb{Z}}$ be a family of modular units associated to $\alpha(z)$ (see Appendix B for more details). By definition, $\beta_r(z)$ has no zeros nor poles on the set $\Gamma_0(fN)\{\infty\}$ and moreover one has that

$$(1.1) \qquad \prod_{r=0}^{f-1} \beta_r(z) = \alpha(z).$$

In [Cha09c], we proposed a construction of a p-adic invariant $u_C(\beta_r, \tau) \in K_p^\times$ for certain pairs $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times (\mathcal{H}_p \cap K)$ such that $\mathcal{O}_\tau = \mathcal{O}_{N\tau} = \mathcal{O}_n$ (See Section 9 for the definition of $u_C(r, \tau)$). From now on we simply denote $u_C(\beta_r, \tau)$ by $u_C(r, \tau)$. The elements $u_C(r, \tau)$ are conjectured to be global p-units in $K(\mathfrak{f}_n\infty)^{\langle \sigma_{\wp_n} \rangle}$ where $\mathfrak{f}_n = f\mathcal{O}_n$, $\wp_n = p\mathcal{O}_n$, $K(\mathfrak{f}_n\infty)$ is the *narrow extended class field of conductor* $\mathfrak{f}_n$ *of* K, and $K(\mathfrak{f}_n\infty)^{\langle \sigma_{\wp_n} \rangle}$ is the subfield of $K(\mathfrak{f}_n\infty)$ which is fixed by the Frobenius $\sigma_{\wp_n}$ at $\wp_n$. (For the precise definition of $K(\mathfrak{f}_n\infty)$ see Definition 2.2). In the case where $n = 1$, so that $\mathfrak{f}_1 = f\mathcal{O}_K$, the abelian extension $K(\mathfrak{f}_1\infty)$ is nothing else than the usual narrow ray class field of K of conductor f. Note that $K(\mathcal{O}_n\infty) \subseteq K(\mathfrak{f}_n\infty)^{\langle \sigma_{\wp_n} \rangle}$ and that $K(\mathfrak{f}_n\infty)^{\langle \sigma_{\wp_n} \rangle}$ corresponds to the largest subfield of $K(\mathfrak{f}_n\infty)$ for which $\wp_n$ splits completely. Our construction of $u_C(r, \tau)$ may be viewed as a natural generalization of $u_{DD}(\tau)$ in the following sense:

(1) If $f = 1$ then $u_C(0, \tau) = u_{DD}(\tau)$.

(2) Moreover, if one sets $n = f$ then

$$(1.2) \qquad \prod_{r=0}^{f-1} u_C(r, \tau) = u_{DD}(\tau).$$

Note that (2) is equivalent to (1) in the case where $f = n = 1$. The identity (1.2) is a direct consequence of the identity (1.1). See Appendix B where this is explained in greater details.

We want to say here a few words about the method that was used to construct the p-adic invariants u_C and u_{DD}. One of the key features of the method is to use periods of a family of Eisenstein series which vary in the weight. If one renormalizes these Eisenstein series in order to clear the "transcendental period" (a certain power of $2\pi i$ which depends on the weight) one obtains rational numbers. One of the breakthrough that appeared in [DD06] was to realize that these rational periods can be "packaged" in a certain way

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in order to construct a partial modular symbol of $\mathbb{Z}$-valued measures on $\mathbb{X}$. Here $\mathbb{X} = (\mathbb{Z}_p \times \mathbb{Z}_p) \backslash (p\mathbb{Z}_p \times p\mathbb{Z}_p)$ is the set of primitive vectors of $\mathbb{Z}_p^2$. The invariants $u_{DD}(\tau)$ and $u_C(r, \tau)$ are then defined as a certain p-adic integral of such a measure on the space $\mathbb{X}$. Many numerical examples which support the algebraicity of $u_{DD}(\tau)$ and $u_C(r, \tau)$ can be found in [Das07] and [Cha09a]. Note that it is desirable to have a theory which allows N to be non square-free since in all the numerical examples which appear in the two papers above one has $N = 4$.

On the other hand well known conjectures of Gross and Stark relate special values of abelian L-functions to global units. Let F be a totally real number field. In the seventies, Stark formulated a series of conjectures relating the special values at $s = 0$ of Archimedean abelian L-functions of F to global units in F^{ab}. Then in the early eighties (see [Gro81]), Gross formulated a p-adic analogue of Stark's conjecture by replacing Archimedean abelian L-functions of F by p-adic abelian L-functions of F, and global units in F^{ab} by global p-units in F^{ab}. A few years later, Gross made a refinement of the previous conjecture. This refined conjecture will be referred as the "Strong Gross conjecture"(for a precise statement see Conjecture 1.1 below). It turns out that the Strong Gross Conjecture is closely related to the Darmon-Dasgupta construction explained above and therefore we would like to recall it. In order to do so we need to set up some notations.

Let $\mathfrak{f}$ be an integral ideal of F and let $M = F(\mathfrak{f}\infty)$ be the narrow ray class field of F of conductor $\mathfrak{f}$. Let $\mathfrak{p}$ be a prime of F above p that does not ramify in M. Let S be a finite set of places of F containing all the infinite places of F, the prime $\mathfrak{p}$ and the primes which ramify in M/F. Now we make the following crucial assumption on S:

Assumption 1.1 The only place of S (Archimedean or non-Archimedean) that splits completely in M is $\mathfrak{p}$.

In particular, because of this assumption on S, M has to be a totally imaginary number field. For an ideal $\mathfrak{a}$ of F we denote by $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(M/F)$ the Frobenius at $\mathfrak{a}$. For any $\sigma \in \mathrm{Gal}(M/F)$ let

$$(1.3) \qquad \zeta_S(M/F, \sigma, s) := \sum_{\substack{(\mathfrak{a}, S) = 1 \\ \sigma_{\mathfrak{a}} = \sigma}} \frac{1}{\mathbf{N}(\mathfrak{a})^s} = \left(1 - \frac{1}{\mathbf{N}(\mathfrak{p})^s}\right) \zeta_R(M/F, \sigma, s),$$

where $R = S \backslash \{\mathfrak{p}\}$. The first summations (resp. the summation which defines $\zeta_R(\sigma, s)$) is taken over all integral ideals of F coprime to S (resp. coprime to R). The second equality follows from Assumption 1.1 and implies that $ord_{s=0}\zeta_S(\sigma, s) = 1$. It was proved by Siegel and Klingen that the special values at negative integers of $\zeta_R(\sigma, s)$ are *rational numbers* (see [Kli62] and [Sie69]). This key fact is the starting point for the p-adic Gross-Stark conjecture.

In order to state the version of Gross's conjecture which is convenient in our context, we need to introduce one more set of places of F in order to "regularize"the special values at negative integers of $\zeta_R(M/F, \sigma, s)$. Let T be a finite set of places of F which is disjoint from S. The role played by the set T here is similar to the role played by the integer N in the Darmon-Dasgupta construction. For each complex number s, consider the group ring

element

$$(1.4) \qquad \prod_{\eta_i \in T}(1 - (\mathbf{N}\eta_i)^{1-s}[\sigma_{\eta_i}]) = \sum_{\mathfrak{d}|\mathfrak{N}}(-1)^{\widetilde{\tau}(\mathfrak{d})}\mathbf{N}(\mathfrak{d})^{1-s}[\sigma_{\mathfrak{d}}] \in \mathbb{C}[\mathrm{Gal}(M/F)],$$

where $\mathfrak{N} = \prod_{\eta_i \in T} \eta_i$ and $\widetilde{\tau}(\mathfrak{d}) = \#$ of prime divisors of $\mathfrak{d}$. For $\sigma \in \mathrm{Gal}(M/F)$, define the partial zeta function associated to the sets S and T by the group ring equation

$$(1.5) \qquad \zeta_{S,T}(M/F, \sigma, s) = \sum_{\mathfrak{d}|\mathfrak{N}}(-1)^{\widetilde{\tau}(\mathfrak{d})}\mathbf{N}(\mathfrak{d})^{1-s}\zeta_S(M/F, \sigma\sigma_{\mathfrak{d}}^{-1}, s).$$

In order to force the integrality at $s = 0$, Gross introduced the following assumption.

Assumption 1.2 The set T contains at least two primes of different residue characteristic or at least one prime η ($\mathbf{N}\eta = l$) with absolute ramification of degree at most $l - 2$.

In Appendix A the interested reader may find some key properties of special values of abelian L-functions at negative integers attached to F which motivate Assumption 1.2.

The Strong Gross Conjecture predicts the existence of a special kind of $\mathfrak{p}$-units in M for which their $\mathfrak{p}$-adic valuation can be related to the value $\zeta_{R,T}(M/F, \sigma, 0) \in \mathbb{Z}$. Define the following subgroup of the group of $\mathfrak{p}$-units of M:

$$U_{\mathfrak{p}} := \{x \in M^{\times} : |x|_{\nu} = 1 \text{ for all } \nu \nmid \mathfrak{p} \}.$$

Here ν ranges over all *finite and infinite* places of M. An element of $U_{\mathfrak{p}}$ will be called a *strong $\mathfrak{p}$-unit*. This terminology is justified for the following two reasons: First of all, an element $u \in U_{\mathfrak{p}}$ is necessarily a global p-unit, i.e., $u \in \mathcal{O}_M[\frac{1}{p}]^{\times}$. But it is more (stronger) than just being a p-unit since for all complex embeddings $\tau : M \to \mathbb{C}$ one has that $|u^{\tau}| = 1$, i.e., the image of u by any complex embedding lies on the unit circle. There is also a linguistic reason behind this terminology: If one translates from English to German the word "strong", then one gets the word "stark" and it is expected that these strong p-units are related to the Gross-Stark p-units.

For any finite abelian extension L of F which contains M and which is unramified outside S we let

$$rec_{\mathfrak{p}}^{L} : F_{\mathfrak{p}}^{\times} \to \mathbb{A}_F^{\times} \to \mathrm{Gal}(L/F),$$

denote the reciprocity map given by local class field theory. From $M \subseteq M_{\mathfrak{P}} \simeq F_{\mathfrak{p}}$ we may evaluate $rec_{\mathfrak{p}}^{L}$ on any element of $M^{\times}$; the image will be contained in $\mathrm{Gal}(L/M)$.

We can now give the precise statement of the strong Gross conjecture (see Conjectures 7.4 and 7.6 of [Gro88]). Here the use of the adjective strong emphasizes the fact Conjecture 1.1 below is a strengthening of Conjecture 3.13 of [Gro81].

Conjecture 1.1 *(Strong Gross Conjecture) Let S and T be two finite sets of places of F which satisfy Assumptions 1.1 and 1.2 and let $R = S\backslash\{\mathfrak{p}\}$. For every prime $\mathfrak{P}$ of M*

above $\mathfrak{p}$, there exists a unique strong $\mathfrak{p}$-unit $u_T \in U_\mathfrak{p}$ such that $u_T \equiv 1 \pmod{T}$ and such that for all $\sigma \in \mathrm{Gal}(M/F)$ one has

$$(1.6) \qquad \zeta_{R,T}(M/F, \sigma, 0) = \mathrm{ord}_\mathfrak{P}(u_T^\sigma).$$

Moreover, for all finite abelian extension L of F which contains M and which is unramified outside S one has

$$(1.7) \qquad rec_\mathfrak{p}^L(u_T^\sigma) = \prod_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_H = \sigma^{-1}}} \tau^{\zeta_{S,T}(L/F, \tau^{-1}, 0)} \in \mathrm{Gal}(L/M).$$

We remark that

$$\sum_{\substack{\tau \in \mathrm{Gal}(L/F) \\ \tau|_H = \sigma^{-1}}} \zeta_{S,T}(L/F, \tau, 0) = \zeta_{S,T}(M/F, \sigma^{-1}, 0) = 0,$$

where the second equality follows from the presence of the Euler factor in (1.3). So indeed the right hand side of (1.7) lies in $\mathrm{Gal}(L/M)$.

The condition $u_T \in U_\mathfrak{p}$ together with (1.6) specify the valuation of u_T at all places of F and therefore determine u_T uniquely up to a root of unity in L. Thanks to Assumption 1.2 on the set T, we see that the additional condition $u_T \equiv 1 \pmod{T}$ guarantees the uniqueness of u_T. We call u_T the *Gross-Stark unit* for the data $(S, T, M, \mathfrak{P})$.

Recently, in [Das08], Dasgupta has proposed an exact $\mathfrak{p}$-adic formula for the Gross-Stark unit u_T which can be viewed as a refinement of the identity (1.7) (See Appendix A where this is explained in greater details). One of the key ideas of his approach is to replace the special values of partial zeta functions of F by special values of *Shintani Zeta functions*. In order to control the denominator of these special values at $s = 0$ he makes an additional assumption on the set T:

Assumption 1.3 Assume that no prime of S has the same residue characteristic as any prime of T and that no two primes in T have the same residue characteristic.

Dasgupta considers special values at $s = 0$ of "Shintani zeta functions" which depend on the sets S, T and on a choice of a fundamental domain for the action of $\mathcal{O}_K(\mathfrak{f}\infty)^\times$ on the totally positive quadrant $\mathbb{R}_{>0}^n$. Here $\mathcal{O}_K(\mathfrak{f}\infty)^\times$ corresponds to the group of totally positive units of $\mathcal{O}_K$ which are congruent to 1 modulo $\mathfrak{f}$. For every ideal $\mathfrak{a}$ of F which is coprime to S he uses these special values to construct a $\mathbb{Z}$-valued measure $\mu(\mathcal{D}, \mathfrak{a})$ on $\mathcal{O}_{F_\mathfrak{p}}$. He then defines a p-adic invariant

$$(1.8) \qquad u_D(\mathfrak{a}, \mathfrak{f}) \in F_\mathfrak{p}^\times$$

as a certain multiplicative p-adic integral (defined as a limit of Riemann products) over the space $O := \mathcal{O}_{F_\mathfrak{p}} \backslash \mathfrak{p}^e \mathcal{O}_{F_\mathfrak{p}}$. Here $e \geq 1$ is the smallest integer such that $\mathfrak{p}^e = (\pi)$ where π is a totally positive element of F such that $\pi \equiv 1 \pmod{\mathfrak{f}}$. Roughly, one integrates the measure $\mu(\mathcal{D}, \mathfrak{a})$ against the identity function on O. He then shows that under some

technical conditions (which are always satisfied when F is a real quadratic field!) that his p-adic invariant $u_D(\mathfrak{a}, \mathfrak{f})$ does not depend on the choice of the Shintani domain $\mathcal{D}$ and that it only depends on $\sigma_{\mathfrak{a}}$ rather than $\mathfrak{a}$ itself. The previous fact is rather remarkable since the measure $\mu(\mathcal{D}, \mathfrak{a})$ really depends on the Shintani domain and the ideal itself. So the process of integrating the measure $\mu(\mathcal{D}, \mathfrak{a})$ may be viewed as a way of removing the dependence on the fundamental domain and relaxing the dependence on $\mathfrak{a}$. Let us choose a prime $\mathfrak{P}$ of M above $\mathfrak{p}$ so that $u_D(\mathfrak{a}, \mathfrak{f})$ may be viewed as an element of $M_{\mathfrak{P}} \simeq F_{\mathfrak{p}}$. Then Dasgupta conjectures that $u_D(\mathfrak{a}, \mathfrak{f})$ is equal to the Gross-Stark unit $u_T^{\sigma_{\mathfrak{a}}} \in M_{\mathfrak{P}}$ for the set of data $(S, T, M, \mathfrak{P})$. We will not say more about his construction and we encourage the interested reader to look at [Das08] for more details.

Let us go back to our original setting where $F = K = \mathbb{Q}(\sqrt{d_K})$ is a real quadratic field. Let (p, f, N) be a triple as in the first paragraph of the introduction and let $\wp = p\mathcal{O}_K$ and $\mathfrak{f} = f\mathcal{O}_K$. Choose a *splitting of* N, i.e., a factorization of the form $N\mathcal{O}_K = \mathfrak{N}\mathfrak{N}^{\sigma}$ where $\mathfrak{N}$ is an $\mathcal{O}_K$-ideal. Such a splitting exists because of the Heegner hypothesis. Let

(i) $S = \{\nu : \nu | \mathfrak{f} \text{ or } \nu | \infty\} \cup \{\wp\}$,

(ii) $T = \{\nu : \nu | \mathfrak{N}\}$,

where ν ranges over all places of K. We thus see that the triple $(p, f, \mathfrak{N})$ encodes the same data as the triple $(\wp, S, T)$. Note that the set S and T satisfy the previous three assumptions. Let $L = K(\mathfrak{f}\infty)^{\langle \sigma_{\wp} \rangle}$ and let $\mathfrak{a}$ be an integral ideal coprime to S. Let us fix a prime $\mathfrak{P}$ of L above $\wp$. Then Dasgupta's construction produces a p-adic invariant $u_D(\mathfrak{a}, \mathfrak{f})$ which is conjectured to be equal to $u_T^{\sigma_{\mathfrak{a}}}$ where u_T is the Gross-Stark unit for the set of data $(S, T, L, \mathfrak{P})$. On the other hand, the author's construction provides a p-adic invariant $u_C(r, \tau)$ which is also conjectured to be a strong p-unit in L.

We thus have two p-adic invariants u_C and u_D which are conjectured to be strong p-units in L. Moreover, in each case, one has a conjectural "analytic" description of how $\mathrm{Gal}(L/K)$ acts on u_C and u_D. In each case, this analytic description will be referred as the Shimura reciprocity law. (For the precise definitions of u_C and u_D and their respective Shimura reciprocity law, see Section 9). The main goal of this paper is to give precise relations between the p-adic invariants u_C and u_D. Our main theorem is the following:

Theorem 1.1 *The invariant $u_C^{\varphi(f)}$ (relative to the order $\mathcal{O}_K$) belongs to the group generated by the u_D's (relative to the order $\mathcal{O}_K$) where φ corresponds to the Euler function. The invariant u_D^{12} (relative to the order $\mathcal{O}_K$) belongs to the group generated by the u_C's (relative to the order $\mathcal{O}_K$). Moreover, all the previous relations between the u_D's and u_C's are compatible with the Shimura reciprocity laws.*

The meaning of "relative to the order $\mathcal{O}_K$" is important and will be explained below (see Definition 1.1). We will give in the text precise formulas that describe all these relations (see Theorem 10.1 and Corollary 10.1). In order to prove Theorem 1.1 it is desirable to work in greater generality and extend the definition of Dasgupta's invariant $u_D(\mathfrak{a}, \mathfrak{f})$ to invertible $\mathcal{O}$-ideals $\mathfrak{a}$ where $\mathcal{O}$ is allowed to be an *arbitrary* order of K (not just the maximal one) of conductor coprime to N and where $\mathfrak{a}$ is no longer assumed to be coprime to $\mathfrak{f} = f\mathcal{O}$. Let us explain the motivation behind this level of generality since it introduces

some technicalities. Let $\tau \in K \backslash \mathbb{Q}$ be such that $\mathcal{O}_\tau = \mathcal{O}_{N\tau} = \mathcal{O}$ where $\mathcal{O}$ is a fixed *ambient order* of conductor coprime to N. Note that we do not assume that that $\mathrm{cond}(\mathcal{O})$ is corpime to f. In fact, in our applications it will be important to allow $\mathrm{cond}(\mathcal{O})$ to be divisible by f. To fix the idea, suppose that $\mathrm{cond}(\mathcal{O}) = n$ so that $\mathcal{O} = \mathcal{O}_n$, $(N, n) = 1$. Let $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ be the unique primitive quadratic form associated to τ for which $A = A_\tau > 0$ and let $\mathfrak{a} = A_\tau \Lambda_\tau$ be the associated integral invertible $\mathcal{O}$-ideal. We note that $N|A$ and that $\mathrm{disc}(Q_\tau) = B^2 - 4AC = n^2 d_K$. The Atkin-Lehner involution $\tau \mapsto \tau^* = \frac{-1}{fN\tau}$ is the key ingredient which allows us to relate the invariant u_C to the invariant u_D (see identity (1.9) below). Now a direct computation shows that

$$Q_{\tau^*}(x, y) = \mathrm{sign}(C) \left(\frac{f^2}{d} CN x^2 - B\frac{f}{d} xy + \frac{A}{dN} y^2 \right),$$

where $0 < d = (A, f)$. In particular, we have $\mathrm{disc}(Q_{\tau^*}) = \left(\frac{f}{d} \right)^2 n^2 d_K$. Thus, if $d \neq f$, the Atkin Lehner involution changes the conductor of the order in the sense that $\mathrm{cond}(\mathcal{O}_{\tau^*}) = \frac{f}{d} \mathrm{cond}(\mathcal{O}_\tau)$. Moreover, in general, the invertible integral $\mathcal{O}_{\tau^*}$-ideal $\mathfrak{a}^* := A_{\tau^*} \Lambda_{\tau^*} = C\frac{f^2}{d} N \Lambda_{\tau^*}$ is no longer coprime to $\mathfrak{f}^* = f\mathcal{O}_{\tau^*}$ in the sense that $(\mathfrak{f}^*, \mathfrak{a}^*) := \mathfrak{f}^* + \mathfrak{a}^* \neq \mathcal{O}_{\tau^*}$. This explains in part the need of a theory which deals with arbitrary orders (not just the maximal one) and arbitrary ideals (not just the ones coprime to f). Having extended the definition of u_D to this broader setting it is useful to make the following definition:

Definition 1.1 *Let $\tau \in K \backslash \mathbb{Q}$ and let $\mathfrak{b} = A_\tau \Lambda_\tau \subseteq K$ be the associated normalized discrete $\mathbb{Z}$-module of rank 2. Assume furthermore that $\mathrm{End}_K(\mathfrak{b}) = \mathcal{O}$ where $\mathrm{cond}(\mathcal{O}, N) = 1$. Let $\mathfrak{g} \subseteq \mathcal{O}$ be an $\mathcal{O}$-ideal (not necessarily invertible). In this case we say that the p-adic invariant $u_D(\mathfrak{b}, \mathfrak{g})$ (resp. $(u_C(r, \tau))$) is **relative to the order** $\mathcal{O}$.*

We keep the same notation as the paragraph above Definition 1.1. One of the key steps in the proof of Theorem 1.1 will be to show the following relation (see Proposition 9.3)

$$(1.9) \qquad\qquad u_D(\mathfrak{a}^*, \mathfrak{f}^*)^{12} = u_C(1, \tau)^e,$$

where $u_C(1, \tau)$ is the p-adic invariant associated to a suitable choice of a family of modular units $\{\beta_r(z)\}_{r \in \mathbb{Z}/f\mathbb{Z}}$ and e is a certain rational number that can be computed explicitly. We note that the construction of the p-adic invariants u_D and u_C are different in nature and therefore the identity (1.9) is far from trivial. Roughly, the strategy that we use to prove (1.9) consists in relating the underlying p-adic measure of u_D (say μ_D), to the underlying p-adic measure of u_C (say μ_C). Both of μ_D and μ_C may be viewed as $\mathbb{Z}$-valued measures on the space $\mathbb{Z}_p^2$. In some suitable sense one can show that μ_D is equal to μ_C (when evaluated on a ball of $\mathbb{Z}_p^2$) up to some error term which disappears when one integrates over the space $\mathbb{X}$ (see Proposition 8.3 for a precise statement involving the error term and the measures μ_D and μ_C). However, we would like to emphasize here that the p-adic invariant on the left hand side of (1.9) is relative to the order $\mathcal{O}_{\tau^*}$ and the one on the right hand side is relative to the order $\mathcal{O}_\tau$. As pointed out earlier, in general these two orders differ. It is precisely the discrepancy of these two orders which makes Theorem 1.1 deeper than identity (1.9).

Let us explain in greater details why this discrepancy is problematic regarding Theorem 1.1. In [Das08], Dasgupta considered p-adic invariants of the form $u_D(\mathfrak{b}, \mathfrak{f})$ where $\mathfrak{f} = f\mathcal{O}_K$

and $\mathfrak{b} = A_\tau \Lambda_\tau$ where $\tau \in K\backslash\mathbb{Q}$ is such that $End_K(\Lambda_\tau) = \mathcal{O}_K$ and $(A_\tau, f) = 1$. Set $\rho = \tau^*$ where $\tau^* = \frac{-1}{fN\tau}$. Note that $\rho^* := \frac{-1}{fN\rho} = \tau$. Using (1.9), one may deduce that

$$(1.10) \qquad u_D(A_{\rho^*}\Lambda_{\rho^*}, f\mathcal{O}_{\rho^*})^{12} = u_D(\mathfrak{b}, f\mathcal{O}_K)^{12} = u_C(1, \rho)^e,$$

where e is a certain integer such that $e|\varphi(f)$. The first equality in (1.10) follows from the definition of ρ and the second equality follows from (1.9). One immediately sees that (1.10) gives a precise relation between Dasgupta's p-adic invariant u_D (as considered in [Das08]) and a special case of the p-adic invariant u_C considered in [Cha09c]. However, the left hand side of (1.10) is relative to the order $\mathcal{O}_K$ and the one on the right hand side is relative to the order $\mathcal{O}_f$. Therefore, identity (1.10) does not quite fulfill the statement of Theorem 1.1. In order to prove Theorem 1.1, one needs to work harder and prove some "distribution relations" which are satisfied by the p-adic invariants u_D's *relative to different orders*. Our Proposition 10.1 gives an explicit example of such distribution relations. This proposition is the key tool which is used in the proof of Theorem 1.1.

Finally, for the end of the introduction, we would like to introduce some new notions that emerged naturally from our work. Let $\mathcal{O}$ be an arbitrary order of K of conductor coprime to N. Let $\mathfrak{b} \subseteq K$ be an arbitrary $\mathcal{O}$-invertible ideal, i.e., $End_K(\mathfrak{b}) = End_K(\mathfrak{g}) = \mathcal{O}$ and let $\mathfrak{g} \subseteq \mathcal{O}$ be an arbitrary $\mathcal{O}$-module which we assume to be invertible only for the sake of simplicity. We say that $\mathfrak{b}$ is $\mathfrak{g}$-*int* if $\mathfrak{b}$ may be written as $\mathfrak{b} = \mathfrak{c}\mathfrak{d}^{-1}$ where $\mathfrak{c}, \mathfrak{d} \subseteq \mathcal{O}$ are invertible $\mathcal{O}$-ideals and where $(\mathfrak{d}, \mathfrak{g}) = \mathfrak{d} + \mathfrak{g} = \mathcal{O}$. Note in particular that if $\mathfrak{b} \subseteq \mathcal{O}$ then $\mathfrak{b}$ is automatically $\mathfrak{g}$-int. A pair $[\mathfrak{b}, \mathfrak{g}]$ is said to be *primitive* of conductor $[\mathcal{O}, \lambda\mathfrak{g}]$ if there exists $\lambda \in K^\times$ such that $\lambda\mathfrak{b}, \lambda\mathfrak{g} \subseteq \mathcal{O}$ and such that $(\lambda\mathfrak{b}, \lambda\mathfrak{g}) = \mathcal{O}$. We have put brackets $[\mathfrak{b}, \mathfrak{g}]$ in order to distinguish the "ordered pair $(\mathfrak{b}, \mathfrak{g})$" from the standard notation $(\mathfrak{b}, \mathfrak{g}) = \mathfrak{b} + \mathfrak{g}$. From now on we drop the $\mathcal{O}$ in the notation $[\mathcal{O}, \lambda\mathfrak{g}]$ since it was assumed from the outset that $\mathcal{O}$ was a fixed *ambient* order. Note that even though λ is not unique, the $\mathbb{Z}$-lattice $\lambda\mathfrak{g}$ is well defined. If no such λ exists then we say that the pair $[\mathfrak{b}, \mathfrak{g}]$ is *non-primitive*. For example, let $\mathfrak{a} = A_\tau\Lambda_\tau$ where $\mathcal{O}_\tau = \mathcal{O}_K$ and let $\mathfrak{f} = f\mathcal{O}_K$. Furthermore, assume that $(\mathfrak{a}, \mathfrak{f}) = \mathcal{O}_K$ which is equivalent to $(A_\tau, f) = 1$. Then one may check that $[\mathfrak{a}, \mathfrak{f}]$ is primitive of conductor $\mathfrak{f}$. Now consider the pair $[\mathfrak{a}^*, \mathfrak{f}^*]$ where $\tau^* = \frac{-1}{fN\tau}$, $\mathfrak{a}^* = A_{\tau^*}\Lambda_{\tau^*}$ and $\mathfrak{f}^* = f\mathcal{O}_{\tau^*}$. Then one may check that $[\mathfrak{a}^*, \mathfrak{f}^*]$ is a non-primitive pair if $f > 1$.

Let us explain how this notion of primitive pairs intervenes in the context of the p-adic invariant u_D that was introduced earlier. Let $[\mathfrak{b}, \mathfrak{g}]$ be a pair as in the paragraph above of conductor $\mathfrak{g}$. Then Conjecture 9.2 predicts that $u_D(\mathfrak{b}, \mathfrak{g})$ is a strong p-unit in $K(\mathfrak{g}\infty)^{\langle\sigma_\wp\rangle}$. The author expects, that for such a fixed primitive pair $[\mathfrak{b}, \mathfrak{g}]$ and for a "generic divisor $\widetilde{\delta}$" (see Section 5.2 for the definition of $\widetilde{\delta}$) that the p-adic invariant $u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{g})$ generates the maximal CM subfield of $K(\mathfrak{g}\infty)^{\langle\sigma_\wp\rangle}$. Thus we expect that (for a generic divisor $\widetilde{\delta}$) $u_{D,\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f})$ generates the maximal CM subfield of $K(\mathfrak{f}\infty)^{\langle\sigma_\wp\rangle}$, say L_{CM}. It is important here to use the word generic since explicit computations done in [Das07] and [Cha09a] reveal that it may happen that for some divisor $\widetilde{\delta}$, $u_{D,\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f})$ lies in a proper subfield of L_{CM}. On the other hand, the author expects things to be different for a non-primitive pair. For example, we may consider the non-primitive pair $[\mathfrak{a}^*, \mathfrak{f}^*]$. A priori, Conjecture 9.2, only predicts that for a divisor $\widetilde{\delta}$ one has that $u_{D,\delta}(\mathfrak{a}^*, \mathfrak{f}^*) \in K(\mathfrak{f}^*\infty)^{\langle\sigma_\wp\rangle}$. Note that $K(\mathfrak{f}\infty) \subsetneq K(\mathfrak{f}^*\infty)$. However, a more careful analysis of Conjecture 9.2 (see Section 9.4), implies that $u_{D,\widetilde{\delta}}(\mathfrak{a}^*, \mathfrak{f}^*) \in$

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$K(\mathfrak{f}\infty)^{\langle\sigma_\wp\rangle}$. This strange phenomenon can be accounted by the observation that the pair $[\mathfrak{a}^*, \mathfrak{f}^*]$ is non-primitive! The interested reader may also find many numerical evidence in [Cha09a] supporting the conjectural relation $u_{D,\widetilde{\delta}}(\mathfrak{a}^*, \mathfrak{f}^*)^{12} = u_C(1, \tau) \in K(\mathfrak{f}\infty)^{\langle\sigma_\wp\rangle}$.

One of the main goal of this paper is to extend Dasgupta's construction in the following broader setting: to an arbitrary order $\mathcal{O} \subseteq K$ (not necessarily the maximal one) of conductor coprime to N and to an arbitrary pair $[\mathfrak{b}, \mathfrak{g}]$ (not necessarily primitive) where $\mathfrak{g} \subset \mathcal{O}$ is an $\mathcal{O}$-ideal and where $\mathfrak{b}$ is $\mathcal{O}$-invertible and $\mathfrak{g}$-int, we construct a p-adic invariant $u_D(\mathfrak{b}, \mathfrak{g}) \in K_p^\times$.

2 Notation and some basic notions about orders

Let K be a real quadratic number field and let $\mathcal{O}_K = \mathbb{Z} + \omega\mathbb{Z}$ be its maximal $\mathbb{Z}$-order. An arbitrary $\mathbb{Z}$-order of K will be denoted by the letter $\mathcal{O}$. For every positive integer $n \geq 1$ there exists a unique order of K with *conductor n* which we denote by $\mathcal{O}_n = \mathbb{Z} + n\omega\mathbb{Z}$. Let $\mathcal{O}$ be a fixed $\mathbb{Z}$-order of K. A discrete $\mathcal{O}$-module $\mathfrak{a} \subseteq K$ will be called an $\mathcal{O}$-ideal. If $\mathfrak{a}, \mathfrak{b} \subseteq K$ are $\mathbb{Z}$-lattices (always assumed of rank 2) we denote by $(\mathfrak{a}, \mathfrak{b}) := \mathfrak{a} + \mathfrak{b}$. We remark here that if $\mathfrak{a}, \mathfrak{b} \subseteq K$ are $\mathcal{O}$-ideals then $(\mathfrak{a}, \mathfrak{b})$ is the smallest $\mathcal{O}$-ideal of K which contains $\mathfrak{a}$ and $\mathfrak{b}$ (note however that the $\mathbb{Z}$-module $(\mathfrak{a}, \mathfrak{b})$ is completely independent of the ambient order $\mathcal{O}$!). If $\mathfrak{a}, \mathfrak{b} \subseteq K$ are $\mathbb{Z}$-lattices we define the product of $\mathfrak{a}$ with $\mathfrak{b}$ to be

$$\mathfrak{a}\mathfrak{b} = \left\{ \sum_{finite} a_i b_i : a_i \in \mathfrak{a}, b_i \in \mathfrak{b} \right\}.$$

Since there exists an integer $n \in \mathbb{Z}_{\geq 1}$ such that $n\mathfrak{b} \subseteq End_K(\mathfrak{a})$ one has that $\mathfrak{a}\mathfrak{b}$ is again a lattice. By an *invertible $\mathcal{O}$-ideal* (or an $\mathcal{O}$-invertible ideal) we mean an $\mathcal{O}$-ideal $\mathfrak{a}$ such that

$$End_K(\mathfrak{a}) := \{\lambda \in K : \lambda\mathfrak{a} \subseteq \mathfrak{a}\} = \mathcal{O}.$$

Note that every $\mathbb{Z}$-lattice $\Lambda \subseteq K$ is an invertible $\mathcal{O}$-ideal for $\mathcal{O} = End_K(\Lambda)$. For an arbitrary $\mathbb{Z}$-lattice $\Lambda \subseteq K$ we define

$$(2.1) \qquad \Lambda^{-1} := \{\lambda \in K : \lambda\Lambda \subseteq End_K(\Lambda)\}.$$

A $\mathbb{Z}$-lattice $\Lambda \subseteq K$ will be called *integral* if $\Lambda \subseteq End_K(\Lambda)$. A $\mathbb{Z}$-lattice Λ is said to be $\mathcal{O}$-*integral* if Λ is an $\mathcal{O}$-ideal such that $\Lambda \subseteq \mathcal{O}$. We note that the notion of integrality is absolute since it does not depend on the choice of an ambient order while the notion of $\mathcal{O}$-integrality is relative since it depends on the choice of an ambient order $\mathcal{O}$. In general, if $\mathfrak{a}, \mathfrak{b}, \mathfrak{c}$ are lattices then one has the inclusion

$$(2.2) \qquad (\mathfrak{a} \cap \mathfrak{b})\mathfrak{c} \subseteq (\mathfrak{a}\mathfrak{c}) \cap (\mathfrak{b}\mathfrak{c}),$$

which in general could be strict. If moreover, $\mathfrak{a}, \mathfrak{b}$ and $\mathfrak{c}$ are $\mathcal{O}$-ideals and $\mathfrak{c}$ is $\mathcal{O}$-invertible then one has that $((\mathfrak{a}\mathfrak{c}) \cap (\mathfrak{b}\mathfrak{c}))\mathfrak{c}^{-1} \subseteq \mathfrak{a} \cap \mathfrak{b}$ which implies that $(\mathfrak{a}\mathfrak{c}) \cap (\mathfrak{b}\mathfrak{c}) \subseteq (\mathfrak{a} \cap \mathfrak{b})\mathfrak{c}$. Thus in this case the inclusion (2.2) is an equality.

We recall some facts about invertible $\mathcal{O}$-ideals. It is well known that an $\mathcal{O}$-ideal $\mathfrak{a}$ is invertible if and only if for every prime ideal $\mathfrak{p} \neq 0$ of $\mathcal{O}$ one has $\mathfrak{a}_{\mathfrak{p}}$ is a principal $\mathcal{O}_{\mathfrak{p}}$-ideal (here $\mathfrak{a}_{\mathfrak{p}}$ denotes the localization $\mathfrak{a}_{\mathfrak{p}}$ of $\mathfrak{a}$ at $\mathfrak{p}$). For a proof see Proposition 12.4 of [Neu99]). From this we may deduce that a prime $\mathfrak{p}$ of $\mathcal{O}$ is invertible if and only if $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring. We have the following criterion for invertible prime ideals of $\mathcal{O}$:

$$\mathfrak{p} \nmid \mathrm{cond}(\mathcal{O}) \Longleftrightarrow \mathfrak{p} \text{ is invertible,}$$

where $\mathrm{cond}(\mathcal{O})$ denotes the conductor of $\mathcal{O}$. For a proof of this fact see Proposition 12.10 of [Neu99].

If Λ and Λ' are two lattices in K then one always has that

$$(2.3) \qquad \mathrm{End}_K(\Lambda) \cap \mathrm{End}_K(\Lambda') \subseteq \mathrm{End}_K(\Lambda \cap \Lambda').$$

In general the inclusion in (2.3) could be strict. If $\mathfrak{a}$ and $\mathfrak{b}$ are $\mathcal{O}$-ideals the two lattices $\mathfrak{a} \cap \mathfrak{b}$ and $(\mathfrak{a}, \mathfrak{b})$ are again $\mathcal{O}$-ideals and one always has that

$$\mathcal{O} \subseteq \mathrm{End}_K(\mathfrak{a} \cap \mathfrak{b}) \quad \text{and} \quad \mathcal{O} \subseteq \mathrm{End}_K((\mathfrak{a}, \mathfrak{b})).$$

In general, the two inclusions above could be strict, even when restricted to invertible $\mathcal{O}$-ideals. However, in the special case where $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals supported only on invertible prime ideals of $\mathcal{O}$, then the two inclusions above become equalities.

Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathcal{O}$-integral ideals. Then we have the following short exact sequence

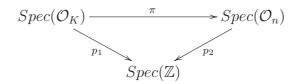$$0 \to \mathfrak{a}/\mathfrak{a}\mathfrak{b} \to \mathcal{O}/\mathfrak{a}\mathfrak{b} \to \mathcal{O}/\mathfrak{a} \to 0.$$

From this we may deduce that

$$(2.4) \qquad [\mathcal{O} : \mathfrak{a}\mathfrak{b}] = [\mathfrak{a} : \mathfrak{a}\mathfrak{b}][\mathcal{O} : \mathfrak{a}].$$

Under the additional assumption that $\mathfrak{a}$ is an invertible $\mathcal{O}$-ideal, an easy localization argument shows that $\mathfrak{a}/\mathfrak{a}\mathfrak{b}$ is isomorphic (non-canonically!) to $\mathcal{O}/\mathfrak{b}$ as an abelian group. Combining the previous observation with (2.4) we may conclude that if either $\mathfrak{a}$ or $\mathfrak{b}$ is $\mathcal{O}$-invertible then

$$(2.5) \qquad [\mathcal{O} : \mathfrak{a}\mathfrak{b}] = [\mathcal{O} : \mathfrak{a}][\mathcal{O} : \mathfrak{b}].$$

We have the following diagram

Since $\mathcal{O}_K[\frac{1}{n}] = \mathcal{O}_n[\frac{1}{n}]$ we see that the map π, when restricted to $p_1^{-1}(Spec(\mathbb{Z}[\frac{1}{n}]))$, is an isomorphism. Let l be a prime divisor of n and let $\mathfrak{I}_l := l\mathbb{Z} + n\omega\mathbb{Z}$. Note that $\mathfrak{I}_l$ is a prime ideal of $\mathcal{O}_n$ such that $\mathcal{O}_n/\mathfrak{I}_l \simeq \mathbb{Z}/l\mathbb{Z}$. The fiber above $l\mathbb{Z}$ (always for $l|n$) is given by one of the following two possibilities:

(1) if l is inert or ramified in K: $p_2^{-1}(l\mathbb{Z}) = \{\mathfrak{I}_l\}$,

(2) if l splits in K: $p_2^{-1}(l\mathbb{Z}) = \{\mathfrak{I}_l, \eta', \eta'^\sigma\}$ where $l\mathcal{O}_K = \eta\eta^\sigma$ and $\eta' = \eta \cap \mathcal{O}_n$,

In general, for a fixed integral $\mathcal{O}_K$-ideal $\mathfrak{N}$ and an arbitrary order $\mathcal{O}$ (we think of $\mathcal{O}$ here as varying), one has a natural inclusion

$$(2.6) \qquad\qquad \mathcal{O}/(\mathfrak{N} \cap \mathcal{O}) \hookrightarrow \mathcal{O}_K/\mathfrak{N},$$

which may fail to be onto. However, there is a special type of $\mathcal{O}_K$-ideal $\mathfrak{N}$ for which (2.6) is an isomorphism for *all* orders $\mathcal{O}$ of K, namely the case where $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$ is a cyclic abelian group. We leave the proof of this elementary fact to the reader. In particular, let $\{l_i\}_{i=1}^r$ be a set of prime numbers that split in K and let $l_i\mathcal{O}_K = \eta_i\eta_i^\sigma$. Then if one sets $\mathfrak{N} = \prod_{i=1}^r \eta_i^{e_i}$, for some arbitrary positive integers e_i, one has that $\mathcal{O}_K/\mathfrak{N} \simeq \mathbb{Z}/N\mathbb{Z}$ where $N = \prod_{i=1}^r l_i^{e_i}$. Therefore the map in (2.6) is onto. Always in this special case, one may furthermore check that for all orders $\mathcal{O}$, one has that $\mathrm{End}_K(\mathcal{O} \cap \mathfrak{N}) = \mathcal{O}$, so that (2.3) becomes an equality with $\Lambda = \mathcal{O}$ and $\Lambda' = \mathfrak{N}$.

We let $\mathcal{L}$ be the set of all free $\mathbb{Z}$-modules contained in K. The set $\mathcal{L}$ comes equipped with a natural stratification

$$(2.7) \qquad\qquad \mathcal{L} = \coprod_{\mathcal{O}} \mathcal{L}^{\mathcal{O}},$$

where $\mathcal{L}^{\mathcal{O}} = \{L \in \mathcal{L} : \mathrm{End}_K(L) = \mathcal{O}\}$ and the disjoint union is taken over all orders of K.

We now introduce equivalence relations on $\mathcal{L}$ and on a stratum $\mathcal{L}^{\mathcal{O}}$. These equivalence relations may be viewed as natural generalizations of the usual equivalence relation on the set of ideals of a Dedekind domain which gives rise to the ideal class group. Let $\lambda \in K^\times$. The notation $\lambda \gg 0$ is taken to mean that λ is a totally positive element, i.e., $\lambda, \lambda^\sigma > 0$.

Definition 2.1 *Let $f \in \mathbb{Z}_{>0}$ be a fixed integer and let $L_1, L_2 \in \mathcal{L}$. We say that $L_1 \sim_f L_2$ if and only if there exists $\lambda \in L_1^{-1}f + 1$, $\lambda \gg 0$, such that $\lambda L_1 = L_2$. (see (2.1) for the definition of L_1^{-1}). Let $\mathcal{O}$ be a fixed order and let $\mathfrak{f}$ be an $\mathcal{O}$-integral ideal. For $L_1, L_2 \in \mathcal{L}^{\mathcal{O}}$ we say that $L_1 \sim_{\mathfrak{f}} L_2$ if and only if there exists $\lambda \in L_1^{-1}\mathfrak{f} + 1$, $\lambda \gg 0$, such that $\lambda L_1 = L_2$.*

It is easy to see that $\sim_f$ is reflexive, transitive and symmetric. Therefore $\sim_f$ gives rise to an equivalence relation on $\mathcal{L}$. A similar statement holds for $\sim_{\mathfrak{f}}$ if we replace $\mathcal{L}$ by $\mathcal{L}^{\mathcal{O}}$. We note that $\sim_f$ preserves the stratification given by (2.7) and that if $L_1, L_2 \in \mathcal{L}^{\mathcal{O}}$ and $L_1 \sim_f L_2$ then L_1 is $\mathcal{O}$-integral if and only if L_2 is $\mathcal{O}$-integral.

Let $L_1, L_2, \mathfrak{a} \in \mathcal{L}^{\mathcal{O}}$ and assume that $L_1 \sim_f L_2$. Then if $\mathfrak{a}$ is $\mathcal{O}$-integral one can show that

$$(2.8) \qquad\qquad L_1\mathfrak{a} \sim_f L_2\mathfrak{a}.$$

Now suppose that $L_1, L_2, \mathfrak{a}, \mathfrak{b} \in \mathcal{L}^{\mathcal{O}}$, $L_1 \sim_f L_2$ and $\mathfrak{a} \sim_f \mathfrak{b}$. If L_1 and $\mathfrak{b}$ are $\mathcal{O}$-integral then applying (2.8) twice we find that $\mathfrak{a}L_1 \sim_f \mathfrak{b}L_1$ and $\mathfrak{b}L_1 \sim_f \mathfrak{b}L_2$. Thus by transitivity of $\sim_f$ we find that

$$(2.9) \qquad\qquad \mathfrak{a}L_1 \sim_f \mathfrak{b}L_2.$$

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Note that in general, if $L_1, L_2 \in \mathcal{L}^{\mathcal{O}}$ and $L_1 \sim_f L_2$, it is not necessarily true that $L_1^{-1} \sim_f L_2^{-1}$ (unless $f = 1$).

Let $\mathcal{O}$ be a fixed order. Let $\mathfrak{f}$ be an $\mathcal{O}$-integral ideal which is not necessarily $\mathcal{O}$-invertible. We define the set

$$I_{\mathcal{O}}(\mathfrak{f}) := \{\mathfrak{b} \subseteq \mathcal{O} : \mathfrak{b} \text{ is an invertible integral } \mathcal{O}\text{-ideal coprime to } \mathfrak{f}, \text{ i.e., } (\mathfrak{f}, \mathfrak{b}) = \mathcal{O}\}.$$

For $m \in \mathbb{Z}_{\geq 1}$ we also let $I_{\mathcal{O}}(m) = I_{\mathcal{O}}(m\mathcal{O})$. If $\mathfrak{a}, \mathfrak{b} \subseteq \mathcal{O}$ are comaximal $\mathcal{O}$-ideals, i.e., $(\mathfrak{a}, \mathfrak{b}) = \mathcal{O}$ then for any pair of integers $n, m \in \mathbb{Z}_{\geq 1}$ one has that $(\mathfrak{a}^n, \mathfrak{b}^m) = \mathcal{O}$. From this it follows that for every integer $n \in \mathbb{Z}_{\geq 1}$ one has that $I_{\mathcal{O}}(\mathfrak{f}^n) = I_{\mathcal{O}}(\mathfrak{f})$. Consider the monoid $I_{\mathcal{O}}(1)$ where the multiplication is given by the usual multiplication of ideals and where the identity element is given by $\mathcal{O}$. For every $\mathcal{O}$-integral ideal $\mathfrak{f}$ we have the equivalence relation $\sim_{\mathfrak{f}}$ on $I_{\mathcal{O}}(1)$. Note that if $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ then $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f})$. Indeed, assume that $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ and let $(\mathfrak{a}, \mathfrak{f}) = \mathfrak{c}$ and $(\mathfrak{b}, \mathfrak{f}) = \mathfrak{d}$. By assumption there exists $\lambda \in 1 + \mathfrak{a}^{-1}\mathfrak{f}$ be such that $\lambda\mathfrak{a} = \mathfrak{b}$. We have $(1, \mathfrak{a}^{-1}\mathfrak{f}) = \mathfrak{c}\mathfrak{a}^{-1}$ and $(\lambda, \mathfrak{a}^{-1}\mathfrak{f}) = \mathfrak{d}\mathfrak{a}^{-1}$. Since $(\lambda, \mathfrak{a}^{-1}\mathfrak{f}) = (1, \mathfrak{a}^{-1}\mathfrak{f})$ we deduce that $\mathfrak{d}\mathfrak{a}^{-1} = \mathfrak{c}\mathfrak{a}^{-1}$ which implies that $\mathfrak{d} = \mathfrak{c}$.

For every $\mathcal{O}$-integral ideal $\mathfrak{d} \supseteq \mathfrak{f}$ we let

$$I_{\mathcal{O}}[\mathfrak{d}, \mathfrak{f}] := \{\mathfrak{b} \subseteq \mathcal{O} : \mathfrak{b} \text{ is an invertible integral } \mathcal{O}\text{-ideal such that } (\mathfrak{f}, \mathfrak{b}) = \mathfrak{d}\}.$$

It follows that $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ admits the following stratification:

$$I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}} = \bigcup_{\mathfrak{d}|\mathfrak{f}} I_{\mathcal{O}}[\mathfrak{d}, \mathfrak{f}]/\sim_{\mathfrak{f}}.$$

Using (2.9), we see that the monoid structure on $I_{\mathcal{O}}(1)$ descends to a (finite) monoid structure on $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$. For an element $\mathfrak{a} \in I_{\mathcal{O}}(1)$ let us denote by $[\mathfrak{a}]$ the class of $\mathfrak{a}$ modulo $\sim_{\mathfrak{f}}$. We claim that the set of invertible elements of $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is exactly $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. Indeed, assume that $[\mathfrak{a}] \in I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is such that there exists $[\mathfrak{b}] \in I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ such that $[\mathfrak{a}][\mathfrak{b}] = [\mathcal{O}]$. Then $(\mathfrak{a}\mathfrak{b}, \mathfrak{f}) = \mathcal{O}$ which implies that $(\mathfrak{a}, \mathfrak{f}) = \mathcal{O}$ and therefore $[\mathfrak{a}] \in I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. Conversely, if $[\mathfrak{a}] \in I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ then for all $n \in I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ one has that $[\mathfrak{a}^n] \in I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. Since $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$ is finite, there exist integers $1 \leq n < m$ such that $[\mathfrak{a}^n] = [\mathfrak{a}^m]$. Thus there exists $\lambda \in \mathfrak{a}^{-m}\mathfrak{f} + 1$ such that $\lambda\mathfrak{a}^m = \mathfrak{a}^n$. From this we deduce that $\lambda - 1 \in \mathfrak{a}^{-m}\mathfrak{f}$ and that $\lambda - 1 \in \mathfrak{a}^{n-m}$. Using the fact that $\mathfrak{a}$ and $\mathfrak{f}$ are comaximal we find that $\lambda - 1 \in \mathfrak{a}^{-m}\mathfrak{f} \cap \mathfrak{a}^{n-m} = \mathfrak{a}^{n-m}\mathfrak{f}$. Thus we have $[\mathfrak{a}^{m-n}] = [\mathcal{O}]$ wich implies that $[\mathfrak{a}]$ is invertible.

Definition 2.2 We set $C_{\mathcal{O}}(\mathfrak{f}) := I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. We call $C_{\mathcal{O}}(\mathfrak{f})$ the *narrow extended ideal class group of K of conductor $[\mathcal{O}, \mathfrak{f}]$*. In the special case where $\mathfrak{f}$ is $\mathcal{O}$-invertible we simply say that $C_{\mathcal{O}}(\mathfrak{f})$ is the *narrow extended ideal class group of K of conductor $\mathfrak{f}$*. Note that this makes sense since the order $\mathcal{O}$ is already encoded in $\mathfrak{f}$ ($\mathrm{End}_K(\mathfrak{f}) = \mathcal{O}$). By class field theory, the ideal class group $C_{\mathcal{O}}(\mathfrak{f})$ corresponds to an abelian extension of K which we denote by $K([\mathcal{O}, \mathfrak{f}]\infty)$ where $\infty = \infty_1\infty_2$ stands for the product of the two distinct real places of K. We call $K([\mathcal{O}, \mathfrak{f}]\infty)$ the *narrow extended class field of conductor $[\mathcal{O}, \mathfrak{f}]$*. In the case where $\mathfrak{f}$ is $\mathcal{O}$-invertible we simply write $K(\mathfrak{f}\infty)$ rather than the more cumbersome notation $K([\mathcal{O}, \mathfrak{f}]\infty)$.

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We note that $K([\mathcal{O}, \mathfrak{f}]\infty)$ is an extension of K unramified outside the finite places of K not dividing $[\mathcal{O} : \mathfrak{f}] \cdot \mathrm{cond}(\mathcal{O})$. On the other hand, the ramification above ∞_1 and ∞_2 is a more delicate question and depends on the sign and congruences of a fundamental unit of K.

We define

(1) $P_{\mathcal{O},1}(\mathfrak{f}) = \left\{ \frac{\alpha}{\beta}\mathcal{O} : \alpha, \beta \in \mathcal{O}, (\alpha, \mathfrak{f}) = (\beta, \mathfrak{f}) = \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}} \right\}$,

(2) $P_{\mathcal{O},1}(\mathfrak{f}\infty) = \left\{ \frac{\alpha}{\beta}\mathcal{O} : \alpha, \beta \in \mathcal{O}, (\alpha, \mathfrak{f}) = (\beta, \mathfrak{f}) = \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}}, \frac{\alpha}{\beta} \gg 0 \right\}$.

(3) $Q_{\mathcal{O},1}(\mathfrak{f}) := \left\{ \frac{\alpha}{\beta} \in K : \alpha, \beta \in \mathcal{O}, (\alpha, \mathfrak{f}) = (\beta, \mathfrak{f}) = \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}} \right\}$.

(4) $Q_{\mathcal{O},1}(\mathfrak{f}\infty) := \left\{ \frac{\alpha}{\beta} \in K : \alpha, \beta \in \mathcal{O}, (\alpha, \mathfrak{f}) = (\beta, \mathfrak{f}) = \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}}, \frac{\alpha}{\beta} \gg 0 \right\}$.

We have a natural map $Q_{\mathcal{O},1}(\mathfrak{f}) \to P_{\mathcal{O},1}(\mathfrak{f})$ (resp. $Q_{\mathcal{O},1}(\mathfrak{f}\infty) \to P_{\mathcal{O},1}(\mathfrak{f}\infty)$) which is given by $\lambda \mapsto \lambda\mathcal{O}$. One may prove that for $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}}(\mathfrak{f})$, $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists a $\lambda\mathcal{O} \in P_{\mathcal{O},1}(\mathfrak{f}\infty)$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Keeping in mind that $P_{\mathcal{O},1}(\mathfrak{f}\infty) \not\subseteq I_{\mathcal{O}}(\mathfrak{f})$ we may nevertheless identify $C_{\mathcal{O}}(\mathfrak{f})$ with the "quotient" $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$ in the sense just explained above.

We will need the following elementary proposition:

Proposition 2.1 *Let $n, f \in \mathbb{Z}_{\geq 1}$ and let $\mathcal{O}$ and $\mathcal{O}'$ be orders such that $\mathrm{cond}(\mathcal{O}) = m$ and $\mathrm{cond}(\mathcal{O}') = nm$ (in particular $\mathcal{O}' \subseteq \mathcal{O}$). Set $\mathfrak{f} = f\mathcal{O}$ and $\mathfrak{f}' = f\mathcal{O}'$.*

(i) Let $\mathfrak{a} \in I_{\mathcal{O}}(n\mathfrak{f})$ then $\mathfrak{a} \cap \mathcal{O}'$ is $\mathcal{O}'$-invertible. Moreover, the natural map

$$(2.10) \qquad\qquad \Theta : I_{\mathcal{O}}(n\mathfrak{f}) \to I_{\mathcal{O}'}(\mathfrak{f}')$$
$$\mathfrak{a} \mapsto \Theta(\mathfrak{a}) = \mathfrak{a} \cap \mathcal{O}'$$

is multiplicative, i.e., for all $\mathfrak{a}, \mathfrak{a}' \in I_{\mathcal{O}}(n\mathfrak{f})$ one has $\Theta(\mathfrak{a}\mathfrak{a}') = \Theta(\mathfrak{a})\Theta(\mathfrak{a}')$.

(ii) If $\mathfrak{a}, \mathfrak{a}' \in I_{\mathcal{O}}(n\mathfrak{f})$ are such that $\mathfrak{a} \sim_{nf} \mathfrak{a}'$ then one has $\Theta(\mathfrak{a}) \sim_f \Theta(\mathfrak{a}')$ and therefore Θ induces a natural onto map

$$(2.11) \qquad\qquad \widetilde{\Theta} : C_{\mathcal{O}}(n\mathfrak{f}) \to C_{\mathcal{O}'}(\mathfrak{f}').$$

Proof See Appendix C.

We note that the map $\widetilde{\Theta}$ in the above proposition corresponds to the restriction map

$$\mathrm{Gal}(K(\mathfrak{f}n\infty)/K) \overset{res}{\to} \mathrm{Gal}(K(\mathfrak{f}'\infty)/K).$$

3 An equivalence relation on pair of lattices with cyclic quotient

For every $\tau \in K \backslash \mathbb{Q}$ we define

$$Q_\tau(x, y) := A(x - \tau y)(x - \tau^\sigma y) = Ax^2 + Bxy + Cy^2,$$

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with $A > 0$, $A, B, C \in \mathbb{Z}$ and $(A, B, C) = 1$ to be the unique *primitive quadratic form* associated to τ. Usually, we will denote the coefficient of x^2 of $Q_\tau(x, y)$ by $A_\tau = A$. We note here that if a is positive integer such that $a\Lambda_\tau \subseteq \mathcal{O}_\tau$ then $A_\tau | a$. Recall that for every $\mathbb{Z}$-lattice $L \subseteq K$ we define $\mathrm{End}_K(L) = \{\lambda \in K : \lambda L \subseteq L\}$ and when $\tau \in K \backslash \mathbb{Q}$ we denote $\mathrm{End}_K(\Lambda_\tau)$ simply by $\mathcal{O}_\tau$. For an element $\tau \in K \backslash \mathbb{Q}$ one can check that $\mathcal{O}_\tau = \mathbb{Z} + A_\tau \tau \mathbb{Z}$. For every $\mathbb{Z}$-lattice $L \subseteq K$ we define $\mathbf{N}(L)$ to be the absolute value of the determinant of a matrix with rational coefficients that takes a $\mathbb{Z}$-basis of $\mathcal{O}_K$ to a $\mathbb{Z}$-basis of L. Note that for an element $a \in K^\times$ and an arbitrary order $\mathcal{O}$ one has the formula $\mathbf{N}(a\mathcal{O}) = |\mathbf{N}_{K/\mathbb{Q}}(a)|\mathbf{N}(\mathcal{O})$. One may also check that

$$(3.1) \qquad \mathbf{N}(\Lambda_\tau) = \frac{\mathrm{cond}(\mathcal{O}_\tau)}{A_\tau} \quad \text{and} \quad \Lambda_\tau \Lambda_{\tau^\sigma} = \frac{1}{A_\tau} \mathcal{O}_\tau.$$

If $\mathfrak{a}$ is an invertible integral $\mathcal{O}$-ideal we obtain from the first equality of (3.1) that $\mathbf{N}(\mathfrak{a}) = \mathrm{cond}(\mathcal{O})[\mathcal{O} : \mathfrak{a}]$. In particular, if $\mathfrak{a} = A_\tau \Lambda_\tau$ then we have that $A_\tau = [\mathcal{O}_\tau : \mathfrak{a}]$.

Let $\mathcal{L}$ be the set of all lattices of K. It is convenient to define a map that takes an element of $K \backslash \mathbb{Q}$ to an *integral lattice* of K.

Definition 3.1 *We set*

$$I_{\text{-}} : K \backslash \mathbb{Q} \to \mathcal{L}$$
$$\tau \mapsto I_\tau := A_\tau \Lambda_\tau,$$

where A_τ is the x^2-coefficient of $Q_\tau(x, y)$.

We record the following two useful formulas

$$(3.2) \qquad \text{(i)} \quad I_\tau^{-1} = \Lambda_{\tau^\sigma} \quad \text{and} \quad \text{(ii)} \quad \tau^\sigma I_\tau = I_{\frac{1}{\tau}}.$$

From now on we take the following convention:

Definition 3.2 *Let $\mathcal{O} \subseteq \mathcal{O}'$ be two orders and let $\mathfrak{a} \in \mathcal{L}^{\mathcal{O}'}$ and $L \in \mathcal{L}^{\mathcal{O}}$. Assume furthermore that $\mathfrak{a}$ is $\mathcal{O}'$-integral. Then the product $\mathfrak{a}L$ is taken to mean $(\mathfrak{a} \cap \mathcal{O})L$ where the last product is the product as two $\mathcal{O}$-ideals.*

Let $N = \prod_{i=1}^{r} l_i^{e_i}$ where l_i's are distinct prime numbers which split in K and where the e_i's are arbitrary positive integers. Choose a splitting of $\mathfrak{N}$, i.e., an integral $\mathcal{O}_K$-ideal such that $\mathfrak{N}\mathfrak{N}^\sigma = N\mathcal{O}_K$. Note that for any lattice $L \in \mathcal{L}^{\mathcal{O}}$ one has that $\mathfrak{N}L$ is again $\mathcal{O}$-invertible and that $L/\mathfrak{N}L \simeq \mathbb{Z}/N\mathbb{Z}$.

We set $H(\mathfrak{N}) = \{\tau \in K : \mathfrak{N}\Lambda_\tau = \Lambda_{N\tau}, \tau - \tau^\sigma > 0\}$, and $H^{\mathcal{O}}(\mathfrak{N}) = \{\tau \in H(\mathfrak{N}) : \mathcal{O}_\tau = \mathcal{O}\}$. We note that

$$(3.3) \qquad \mathfrak{N}\Lambda_\tau = \Lambda_{N\tau} \iff N\Lambda_\tau = \mathfrak{N}^\sigma \Lambda_{N\tau}.$$

We have a natural stratification

$$(3.4) \qquad H(\mathfrak{N}) = \bigcup_{\mathcal{O}} H^{\mathcal{O}}(\mathfrak{N}),$$

where the union is disjoint and taken over all orders of K. We note that if $\tau \in H^{\mathcal{O}}(\mathfrak{N})$ then necessarily $N|A_\tau$. In particular, one may deduce from this that

$$(3.5) \qquad\qquad \tau \in H^{\mathcal{O}}(\mathfrak{N}) \iff \frac{-1}{N\tau} \in H^{\mathcal{O}}(\mathfrak{N}).$$

We also define $\mathcal{L}(\mathfrak{N}) = \{[L, \mathfrak{N}L] \in \mathcal{L}^2\}$ and $\mathcal{L}^{\mathcal{O}}(\mathfrak{N}) = \{[L, \mathfrak{N}L] \in \mathcal{L}(\mathfrak{N}) : \operatorname{End}_K(L) = \mathcal{O}\}$, where $\mathcal{O}$ is an arbitrary order of K. Similarly to (3.4) we have a natural stratification

$$(3.6) \qquad\qquad \mathcal{L}(\mathfrak{N}) = \bigcup_{\mathcal{O}} \mathcal{L}^{\mathcal{O}}(\mathfrak{N}).$$

Definition 3.3 *Let $\mathcal{O}$ be a fixed order and let $\mathfrak{g} \subseteq \mathcal{O}$ be a fixed $\mathcal{O}$-ideal. Let $[\mathfrak{a}, \mathfrak{b}], [\mathfrak{a}', \mathfrak{b}'] \in \mathcal{L}^{\mathcal{O}}(\mathfrak{N})$. We say that $[\mathfrak{a}, \mathfrak{b}] \sim_{\mathfrak{g}} [\mathfrak{a}', \mathfrak{b}']$ if and only if there exists $\lambda \in (\mathfrak{a}^{-1}\mathfrak{g} + 1)$, $\lambda \gg 0$, such that $[\lambda\mathfrak{a}, \lambda\mathfrak{b}] = [\mathfrak{a}', \mathfrak{b}']$.*

One may check that $\sim_{\mathfrak{g}}$ gives rise to an equivalence relation on the set $\mathcal{L}^{\mathcal{O}}(\mathfrak{N})$.

Now let $f \in \mathbb{Z}_{\geq 1}$ be an integer coprime to Nd_K. We now focus our attention on the two sets $\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N})$ and $\mathcal{L}(\mathfrak{N})$. On each of these two sets we will define an equivalence relation and it will be shown that their respective quotients are in natural bijection. Consider the map

$$\psi : \mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N}) \to \mathcal{L}(\mathfrak{N})$$
$$(r, \tau) \mapsto [\widetilde{r}I_\tau, \widetilde{r}\mathfrak{N}I_\tau]$$

where $\sim : \mathbb{Z}/f\mathbb{Z} \to \{1, \ldots, f\}$ is the unique map such that for any $r \in \mathbb{Z}/f\mathbb{Z}$ one has $\widetilde{r} \equiv r$ (mod f).

Definition 3.4 *Let $(r, \tau), (r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N})$. We say that $(r, \tau) \sim_f (r', \tau')$ if and only if there exists a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN)$ such that $d^{-1}r \equiv r'$ (mod f) and $\tau' = \gamma\tau$. Let $[L, M], [L', M'] \in \mathcal{L}(\mathfrak{N})$. We say that $[L, M] \sim_f [L', M']$ if and only if there exists $\lambda \in (L^{-1}f + 1)$, $\lambda \gg 0$, such that $[\lambda L, \lambda M] = [L', M']$.*

We denote the class (r, τ) modulo $\sim_f$ by $[(r, \tau)]$ and the class $[L, M]$ modulo $\sim_f$ by $[[L, M]]$.

Proposition 3.1 *The map*

$$\widetilde{\psi} : (\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N}))/\sim_f \to \mathcal{L}(\mathfrak{N})/\sim_f$$
$$[(r, \tau)] \mapsto [\psi(r, \tau)],$$

is well defined and induces a bijection of sets.

Proof See Appendix C.

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3.1 An adelic action on $\mathcal{L}(\mathfrak{N})$

In this subsection we first define an adélic action on the set $\mathcal{L}^{\mathcal{O}}(\mathfrak{N})/\sim_{\mathfrak{g}}$. This adélic action is the key ingredient that allows to define the conjectural Shimura reciprocity law for the p-adic invariant u_D (see Conjecture 9.2). Second of all, using the identification between $\mathcal{L}(\mathfrak{N})/\sim_f$ and $\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N})/\sim_f$ (see Proposition 3.1) we transport this adélic action to the set $\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N})/\sim_f$. This adélic action is used to define the conjectural Shimura reciprocity law for the p-adic invariant u_C.

Let $C_K = \mathbf{A}_K^\times/K^\times$ be the idèle class group of K. By class field theory we have a short exact sequence

$$0 \to C_K^\circ \to C_K \overset{rec}{\to} G_{K^{ab}/K} \to 0,$$

where C_K° is the connected component of C_K which contains the identity. For every integral $\mathcal{O}_K$-ideal $\mathfrak{m}$ class field theory gives us a natural onto map

$$\pi_{\mathfrak{m}} : C_K \to C_{\mathcal{O}_K}(\mathfrak{m})$$

where $C_{\mathcal{O}_K}(\mathfrak{m}) = I_{\mathcal{O}_K}(\mathfrak{m})/P_{\mathcal{O}_K,1}(\mathfrak{m}\infty)$ stands for the usual narrow ray class group of conductor $\mathfrak{m}$ of K.

Definition 3.5 *Let $\mathcal{O}$ be a fixed order of K and let $\mathfrak{g} \subseteq \mathcal{O}$ be an $\mathcal{O}$-ideal. Let $n = \operatorname{cond} \mathcal{O}$, $g = [\mathcal{O} : \mathfrak{g}]$ and let*

$$(3.7) \qquad\qquad \pi_{gn} : C_K \to C_{\mathcal{O}_K}(gn),$$

be the surjective map given by class field theory. For every $c \in C_K$ and $[[L, M]] \in \mathcal{L}^{\mathcal{O}}(\mathfrak{N})/\sim_{\mathfrak{g}}$ (see Definition 3.3 for the meaning of $\sim_{\mathfrak{g}}$) we define

$$(3.8) \qquad\qquad c \star [[L, M]] = [[(\mathfrak{a} \cap \mathcal{O})L, (\mathfrak{a} \cap \mathcal{O})M]],$$

where $\pi_{gn}(c) = [\mathfrak{a}]$.

We note that the group action (3.8) is well defined. Indeed if $\mathfrak{a}, \mathfrak{a}' \in I_{\mathcal{O}_K}(gn)$ and $\mathfrak{a} \sim_{gn} \mathfrak{a}'$ then from Proposition 2.1 one has that $(\mathfrak{a} \cap \mathcal{O}) \sim_g (\mathfrak{a}' \cap \mathcal{O})$ which in turns implies that $(\mathfrak{a} \cap \mathcal{O}) \sim_{\mathfrak{g}} (\mathfrak{a}' \cap \mathcal{O})$. Note also that since $(\mathfrak{a}, n\mathcal{O}_K) = \mathcal{O}_K$ one has that $\mathfrak{a} \cap \mathcal{O}$ is $\mathcal{O}$-invertible. Thus $(\mathfrak{a} \cap \mathcal{O})L$ and $(\mathfrak{a} \cap \mathcal{O})M$ are again $\mathcal{O}$-invertible ideals.

Using Proposition 2.1 we have a natural onto projection map

$$\widetilde{\Theta} : C_{\mathcal{O}_K}(gn) \to C_{\mathcal{O}}(g)$$
$$[\mathfrak{a}] \mapsto [\mathfrak{a} \cap \mathcal{O}].$$

Moreover, the natural map $p : C_{\mathcal{O}}(g) \to C_{\mathcal{O}}(\mathfrak{g})$ is also onto. It thus follows that the map $p \circ \widetilde{\Theta}$ is onto. If one restricts the action of C_K to a stratum $\mathcal{L}^{\mathcal{O}}(\mathfrak{N})/\sim_{\mathfrak{g}}$ then one may check that the action of C_K factors through the generalized ideal class group $C_{\mathcal{O}}(\mathfrak{g})$ in the sense that for all $[[L, M]] \in \mathcal{L}^{\mathcal{O}}(\mathfrak{N})/\sim_{\mathfrak{g}}$ one has that

$$c \star [[L, M]] = [[\mathfrak{c}L, \mathfrak{c}M]],$$

where $[\mathfrak{c}] = p \circ \widetilde{\Theta} \circ \pi_{gn}(c)$ and π_{gn} is the map which appears in (3.7).

Thanks to Proposition 3.1, we may now define an action of C_K on the set $(\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N}))/ \sim_f$.

Definition 3.6 *Let* $[(r,\tau)] \in (\mathbb{Z}/f\mathbb{Z} \times H(\mathfrak{N}))/ \sim_f$ *and let* $\widetilde{\psi}[(r,\tau)] = [[L,M]] \in \mathcal{L}^{\mathcal{O}_\tau}(\mathfrak{N})/ \sim_f$. *For* $c \in C_K$ *we define*

$$c \star [(r,\tau)] := \widetilde{\psi}^{-1}(c \star [[L,M]]).$$

4 Zeta functions attached to real quadratic number fields

In this section we introduce certain zeta functions attached to K that play a key role in the construction of the two p-adic invariants u_C and u_D. There are two sets of notation available if one wants to define these zeta functions. The first set of notation uses the language of quadratic forms. The second set of notation uses the language of ideals. Each of these two ways has its own advantages and inconveniences. The language of quadratic forms was privileged in [DD06] and [Cha09c] while the language of ideals was used in [Das08]. One should note that for a general number field, the language of quadratic form has no equivalent. The reader will find in this section the precise definitions of the zeta functions that appear in the construction of u_C and u_D and the relations between them. We also collected some key facts about special values of partial zeta functions at negative integers which are used implicitly in the proof of Theorem 1.1

Let $\mathcal{O}$ be an arbitrary order of K and let $\mathfrak{f}$ be an $\mathcal{O}$-integral ideal. We denote the group of units (resp. totally positive units) of $\mathcal{O}^\times$ which are congruent to 1 modulo $\mathfrak{f}$ by $\mathcal{O}(\mathfrak{f})^\times$ (resp. $\mathcal{O}(\mathfrak{f}\infty)^\times$). In the case where $\mathfrak{f}$ is a principal ideal generated by an integer f we will write $\mathcal{O}(f\infty)^\times$ rather than the ugly notation $\mathcal{O}(f\mathcal{O}\infty)^\times$.

Definition 4.1 *Let* $\tau \in K\backslash\mathbb{Q}$. *We define* $\eta_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ *to be the unique generator of* $Stab_{\Gamma_1(f)}(\tau)$ *such that* $c\tau + d > 1$. *We also set* $\epsilon(\eta_\tau) := c\tau + d$.

Definition 4.2 *For every pair* $(r,\tau) \in \mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}$ *we define*

(1) $\zeta((r,\tau),s) = - \displaystyle\sum_{\{(m,n)\in\mathbb{Z}^2\backslash(0,0)\}/\langle\eta_\tau\rangle} \frac{\text{sign}(Q_\tau(m,n))}{|Q_\tau(m,n)|^s} e^{\frac{-2\pi i r n}{f}}, \quad \Re(s) > 1,$

(2) $\widehat{\zeta}((r,\tau),s) = f^{2s} \displaystyle\sum_{\{0\neq(m,n)\equiv(r,0) \pmod{f}\}/\langle\eta_\tau\rangle} \frac{\text{sign}(Q_\tau(m,n))}{|Q_\tau(m,n)|^s}, \quad \Re(s) > 1.$

The first summation is taken over a complete set of representatives of $\{(m,n) \in \mathbb{Z}^2\backslash(0,0)\}$ modulo a right action of η_τ which we define below. Similarly the second summation is taken over a complete set of representatives of

(4.1) $$\left\{0 \neq (m,n) \in \mathbb{Z}^2 : (m,n) \equiv (r,0) \pmod{f}\right\}$$

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modulo the right action of η_τ.

The right action of a matrix $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z})$ on a pair (x, y) is given by

(4.2)
$$\begin{pmatrix} x & y \end{pmatrix} \sharp \begin{pmatrix} a & b \\ c & d \end{pmatrix} := (-by + dx, ay - cx).$$

We have used the symbol $\sharp$ in order to distinguish it from the standard right multiplication of a row vector by a 2 by 2 matrix. We note that

$$\begin{pmatrix} x & y \end{pmatrix} \sharp \gamma = \left(\gamma^{-1} \begin{pmatrix} x \\ y \end{pmatrix} \right)^t.$$

However, we have preferred to avoid the use of the transpose operation on vectors and matrices so that the notation which appears in (4.2) will be privileged. This right action is forced upon us for the following reason: Let $\mathcal{O}(f\infty)^\times = \langle \epsilon \rangle$ with $\epsilon > 1$. Then the action of the unit ϵ on the column vector $\begin{pmatrix} \tau & 1 \end{pmatrix}^t$ is given by the standard left multiplication by the matrix η_τ, i.e.,

$$\eta_\tau \begin{pmatrix} \tau \\ 1 \end{pmatrix} = \epsilon \begin{pmatrix} \tau \\ 1 \end{pmatrix}.$$

Now say that $\eta_\tau = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ then since

$$\begin{pmatrix} -y & x \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = x - \tau y$$

we find that

(4.3)
$$\begin{pmatrix} -y & x \end{pmatrix} \begin{pmatrix} \epsilon\tau \\ \epsilon \end{pmatrix} = \begin{pmatrix} -y & x \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix},$$

which forces the right action defined in (4.2). We note that (2) of Definition 4.2 makes sense since for $\gamma \in \Gamma_1(f)$ one has

$$\begin{pmatrix} r & 0 \end{pmatrix} \sharp \gamma \equiv \begin{pmatrix} r & 0 \end{pmatrix} \pmod{f}.$$

We also note that the summations in (1) and (2) don't depend on the choice of the representatives.

Remark 4.1 This subtle point concerning the right action of η_τ on the set (4.1) was overlooked in [Cha09c]. It was wrongly stated on p. 24 of [Cha09c] that the matrix η_τ was acting on the left by the rule $(x, y) \mapsto (ax + by, cx + dy)$. This mistake had no implications for the results stated in [Cha09c] but it should nevertheless be corrected. In fact the right action defined in (4.2) will play a key role later on (see for example (7.10) and (7.11)).

Remark 4.2 The zeta functions appearing in (1) and (2) of Definition 4.2 satisfy the identities $\zeta((r,\tau),s) = \zeta((-r,\tau),s)$ and $\widehat{\zeta}((r,\tau),s) = \widehat{\zeta}((-r,\tau),s)$. In the special case where $f = 1$ one has $\zeta((0,\tau),s) = -\widehat{\zeta}((0,\tau),s)$. One may also check that $\widehat{\zeta}((0,\tau),s)$ coincides with the zeta function $\zeta_{Q_\tau}(s)$ which appears in equation (55) of [DD06].

Recall that there is an action of $\Gamma_0(f)$ on the set $\mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}$ given by

$$\gamma \star (r,\tau) = \left(d^{-1}r, \frac{a\tau + b}{c\tau + d}\right) \ \text{ for } \ \gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f) \ \text{ and } \ (r,\tau) \in \mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}.$$

Let $(r,\tau),(r',\tau') \in \mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}$. A direct calculation shows that if (r,τ) is equivalent to (r',τ') modulo $\Gamma_0(f)$ then $\zeta((r,\tau),s) = \zeta((r',\tau'),s)$ and $\widehat{\zeta}((r,\tau),s) = \widehat{\zeta}((r',\tau'),s)$.

The zeta functions (1) and (2) are related by a functional equation.

Theorem 4.1 *The function $\zeta((r,\tau),s)$ admits a meromorphic continuation to all of $\mathbb{C}$. Moreover, let $F_{w_1}(s) = \mathrm{disc}(Q_\tau)^{s/2}\pi^{-s}\Gamma\left(\frac{s+1}{2}\right)^2$. Then*

(4.4) $$-F_{w_1}(s)\zeta((r,\tau),s) = F_{w_1}(1-s)\widehat{\zeta}((r,\tau),1-s).$$

Proof This result can be deduced from the computations carried by Siegel in [Sie68]. For a detailed proof which builds on ideas of [Sie68], see the proof of Theorem 8.2 of [Cha07]. For a different proof which uses the functional equation of a theta function see Theorem 1.1 of [Cha09b]. $\square$

Remark 4.3 In particular, Theorem 4.1 allows us to give a meaning to the special values of $\widehat{\zeta}((r,\tau),s)$ at negative integers. The zeta function appearing in (1) will not be used in this paper. The author included it only in order to state the functional equation (4.4).

Let $\mathcal{O}$ be a fixed order of K and let $\mathfrak{f}$ be an $\mathcal{O}$-integral ideal (not necessarily invertible) which is distinct from $\mathcal{O}$.

Definition 4.3 *We say that an invertible $\mathcal{O}$-ideal $\mathfrak{c}$ is $\mathfrak{f}$-int if $\mathfrak{c}$ can be written as $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ where $\mathfrak{a}$ and $\mathfrak{b}$ are invertible integral $\mathcal{O}$-ideals such that $(\mathfrak{b},\mathfrak{f}) = \mathcal{O}$.*

Remark 4.4 We have used the notation $\mathfrak{f}$-int in order to avoid any confusion with the notion of $\mathcal{O}$-integrality that was defined in Section 2. Note also that when $\mathfrak{c}$ is an integral invertible $\mathcal{O}$-ideal then it is automatically $\mathfrak{f}$-int.

Definition 4.4 *Let $\mathfrak{c}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int and choose a writing $\mathfrak{c} = \mathfrak{a}\mathfrak{b}^{-1}$ such that $(\mathfrak{b},\mathfrak{f}) = \mathcal{O}$. We define*

$$\Gamma_{\mathfrak{c}}(\mathfrak{f}) := \mathcal{O}(\infty)^\times \cap (\mathfrak{f}\mathfrak{c}^{-1} + n_{\mathfrak{c}}),$$

where $n_{\mathfrak{c}} \in \mathbb{Z}$ is an arbitrarily chosen integer contained in $\mathfrak{b} \subseteq \mathfrak{c}^{-1}$ such that $n_{\mathfrak{c}} \equiv 1 \pmod{\mathfrak{f}}$. Note that the existence of such an integer $n_{\mathfrak{c}}$ is guaranteed precisely because $\mathfrak{c}$ is assumed to be $\mathfrak{f}$-int.

Remark 4.5 Let $\mathfrak{a}$ and $\mathfrak{b}$ be $\mathfrak{f}$-int invertible $\mathcal{O}$-ideals. Then if there exists a $\lambda \in K^\times$ such that $\mathfrak{a} = \lambda\mathfrak{b}$ and that $(\mathfrak{a},\mathfrak{f}) = (\mathfrak{b},\mathfrak{f})$ then one has that $\Gamma_\mathfrak{a}(\mathfrak{f}) = \Gamma_\mathfrak{b}(\mathfrak{f})$. In particular, if one has that $\mathfrak{a} \sim_\mathfrak{f} \mathfrak{b}$ then $\Gamma_\mathfrak{b}(\mathfrak{f}) = \Gamma_\mathfrak{a}(\mathfrak{f})$.

We would like now to introduce partial zeta functions twisted by a *sign character* $w : K^\times \to \{\pm 1\}$.

Definition 4.5 *Let $\mathfrak{c}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int. We define*

$$\zeta(\mathfrak{c},\mathfrak{f},w,s) = \mathbf{N}(\mathfrak{c})^{-s} \sum_{\Gamma_\mathfrak{c}(\mathfrak{f})\backslash\{0\neq\mu\in(\mathfrak{c}^{-1}\mathfrak{f}+n_\mathfrak{c})\}} \frac{w(\mu)}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s}, \qquad \Re(s) > 1,$$

where $n_\mathfrak{c} \in \mathbb{Z}$ is an arbitrarily chosen integer as in Definition 4.4.

The summation is taken over a complete set of representatives of $\{0 \neq \mu \in (\mathfrak{f}\mathfrak{c}^{-1} + n_\mathfrak{c})\}$ modulo $\Gamma_\mathfrak{c}(\mathfrak{f})$. One may check that $\zeta(\mathfrak{c},\mathfrak{f},w,s)$ is independent of the choice of $n_\mathfrak{c}$ and that the first entry of $\zeta(\mathfrak{c},\mathfrak{f},w,s)$ depends on $\mathfrak{c}$ only modulo $\sim_\mathfrak{f}$. Let $\mathfrak{b}$ be an invertible integral $\mathcal{O}$-ideal. Then if $\mathfrak{c}$ is $\mathfrak{f}$-int the $\mathcal{O}$-ideal $\mathfrak{c}\mathfrak{b}$ is automatically $\mathfrak{f}\mathfrak{b}$-integral and one has

$$(4.5) \qquad\qquad \zeta(\mathfrak{c},\mathfrak{f},w,s) = \zeta(\mathfrak{c}\mathfrak{b},\mathfrak{b}\mathfrak{f},w,s).$$

Remark 4.6 Similarly, one can define such zeta functions for an arbitrary number field. A functional equation similar to (4.4) holds, see [Cha09b]. In [Cha10], the author worked out some of their arithmetic properties.

We also define *classical partial zeta functions* attached to an arbitrary order $\mathcal{O}$.

Definition 4.6 *For every invertible $\mathcal{O}$-ideal $\mathfrak{c}$ we define*

$$\zeta(\mathfrak{c},\mathfrak{f}\infty,s) = \mathbf{N}(\mathfrak{c})^{-s} \sum_{\Gamma_\mathfrak{c}(\mathfrak{f})\backslash\{0\neq\mu\in(\mathfrak{f}\mathfrak{c}^{-1}+n_\mathfrak{c}),\mu\gg0\}} \frac{1}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s} \qquad \Re(s) > 1.$$

This function admits a meromorphic continuation to all of $\mathbb{C}$ with a single pole (of order one) at $s = 1$. If $[\mathfrak{a},\mathfrak{f}]$ is primitive of conductor $[\mathcal{O},\mathfrak{g}]$ we say that $\zeta(\mathfrak{a},\mathfrak{f},w,s)$ (resp. $\zeta(\mathfrak{a},\mathfrak{f}\infty,s)$) is primitive of conductor $[\mathcal{O},\mathfrak{g}]$.

Remark 4.7 If the ideal $\mathfrak{a}$ is an integral $\mathcal{O}_K$-ideal (necessarily invertible) coprime to $\mathfrak{f}$ one readily sees that

$$(4.6) \qquad\qquad \zeta(\mathfrak{a},\mathfrak{f}\infty,s) = \zeta_R(K(\mathfrak{f}\infty)/K,\sigma_\mathfrak{a},s),$$

where $\sigma_\mathfrak{a} \in G_\mathfrak{f} := \mathrm{Gal}(K(\mathfrak{f}\infty)/K)$ is the Frobenius at $\mathfrak{a}$ and R is the set of finite places of K which divide $\mathfrak{f}$. The zeta function $\zeta_R(K(\mathfrak{f}\infty)/K,\sigma_\mathfrak{a},s)$ is the one that appears on the left hand side of (1.3).

The next two identities below give us a way to write a partial zeta function weighted by a sign character w as a linear combination of classical partial zeta functions and vice-versa. Let $\{\lambda_i\}_{i=1}^4$ be a complete set of representatives of $Q_{\mathcal{O},1}(\mathfrak{f})/Q_{\mathcal{O},1}(\mathfrak{f}\infty)$ (for the definitions of

$Q_{\mathcal{O},1}(\mathfrak{f})$ and $Q_{\mathcal{O},1}(\mathfrak{f}\infty)$ see Section 2). Then for any invertible $\mathcal{O}$-ideal $\mathfrak{b}$ which is $\mathfrak{f}$-int and any sign character w, a direct computation shows that

$$(4.7) \qquad \zeta(\mathfrak{b},\mathfrak{f},w,s) = \sum_{i=1}^{4} |\mathbf{N}(\lambda_i)|^{-s} w(\lambda_i) \zeta(\mathfrak{b}\lambda_i, \mathfrak{f}\infty, s),$$

and that

$$(4.8) \qquad 4\zeta(\mathfrak{b},\mathfrak{f}\infty,s) = \sum_{w \text{ is a sign character}} \zeta(\mathfrak{b},\mathfrak{f},w,s).$$

There are two sign characters that will play a special role in our context, namely $w_0 := 1$ and $w_1 := \text{sign} \circ \mathbf{N}_{K/\mathbb{Q}}$. We have the following lemma which relates special values at negative integers of $\zeta(\mathfrak{a},\mathfrak{f},w_i,s)$ (for $i \in \{0,1\}$) with the ones of $\zeta(\mathfrak{a},\mathfrak{f}\infty,s)$.

Lemma 4.1 *Let $k \in \mathbb{Z}_{\geq 1}$ and $k \equiv i \pmod 2$. Then*

$$4\zeta(\mathfrak{a},\mathfrak{f}\infty,1-k) = \zeta(\mathfrak{a},\mathfrak{f},w_i,1-k).$$

Proof This lemma follows from (4.8) and the key observation that $\zeta(\mathfrak{a},\mathfrak{f},w,1-k) = 0$ for $w = \pm\text{sign}$ and $k \in \mathbb{Z}_{\geq 1}$. For a proof and an explanation of the latter fact see p. 812 of [Cha10]. $\square$

Remark 4.8 One can show that if there exists an $\epsilon \in \mathcal{O}(\mathfrak{f})^{\times}$ such that $w_1(\epsilon) = \text{sign}(\mathbf{N}_{K/\mathbb{Q}}(\epsilon)) = -1$ then $\zeta(\mathfrak{a},\mathfrak{f},w_1,s)$ is identically equal to 0.

Definition 4.7 *Let G_1 and G_2 be subgroups of $\mathcal{O}_K(\infty)^{\times}$ where $G_1 = \langle \epsilon_1 \rangle$, $G_2 = \langle \epsilon_2 \rangle$ and $\epsilon_1, \epsilon_2 > 1$. We define*

$$[G_1 : G_2] \in \mathbb{Q}_{>0}$$

to be the unique positive rational number r such that $\epsilon_1^r = \epsilon_2$.

The next lemma says the zeta function associated to a quadratic form which appears in (2) of Definition 4.2 is the same (up to a simple fudge factor) to a partial zeta function twisted by the sign character w_1.

Lemma 4.2 *Let $(r,\tau) \in \mathbb{Z}/f\mathbb{Z} \times K \backslash \mathbb{Q}$ and let $\mathfrak{f} = f\mathcal{O}_\tau$. Then*

$$(4.9) \qquad \widehat{\zeta}((r,\tau),s) = \mu(r,\tau)\mathbf{N}(\mathfrak{f})^s \zeta(\widetilde{r}I_\tau, \mathfrak{f}, w_1, s),$$

where

$$(4.10) \qquad \mu(r,\tau) := [\Gamma_{\widetilde{r}I_\tau}(\mathfrak{f}) : \langle \epsilon(\eta_\tau) \rangle] \in \mathbb{Q}_{>0}.$$

Proof This is straightforward computation. $\square$

Remark 4.9 We note that in the special case where $\tau \in H^{\mathcal{O}_K}(\mathfrak{N})$ is such that $(A_\tau, f) = 1$, it may be shown that $\epsilon(\eta_\tau)$ is a generator of $\mathcal{O}_K(\mathfrak{f}\infty)^{\times}$ and therefore if $r \in (\mathbb{Z}/f\mathbb{Z})^{\times}$ one has that $\mu(r,\tau) = 1$.

5 Two groups of divisors

In this section we introduce two groups of divisors $\mathrm{Div}_f(N)$ and $\widetilde{\mathrm{Div}}(\mathfrak{N})$ which are free $\mathbb{Z}$-modules of finite type endowed with an additional structure of a $\mathbb{Z}/f\mathbb{Z}$-module. The first group of divisors captures in essence the choice of the family of modular units $\{\beta_r(z)\}_{r\in\mathbb{Z}/f\mathbb{Z}}$ which appeared in the introduction. The second group of divisors may be viewed as a convenient way of "twisting" Shintani zeta functions in order to regularize their special values at negative integers.

For the rest of the section we fix a triple (p, f, N) as in the introduction where now N is no longer assumed to be square-free. We let $N = \prod_{i=1}^r l_i^{e_i}$ and $\mathfrak{N} = \prod_{i=1}^r \eta_i^{e_i}$ where η_i is a choice of a prime of $\mathcal{O}_K$ above l_i. Thus one has $\mathcal{O}_K/\mathfrak{N}\mathcal{O}_K \simeq \mathbb{Z}/N\mathbb{Z}$.

5.1 The group $\mathrm{Div}_f(N)$

Let $\mathrm{Div}(N)$ be the free abelian group generated by the symbols $\{[d] : d|N, d > 0\}$ and let

$$\deg : \mathrm{Div}(N) \to \mathbb{Z}$$

$$\sum_{d|N} n(d)[d] \mapsto \sum_{d|N} n(d)d,$$

be the degree map. We denote by $\mathrm{Div}^0(N)$ the kernel of deg. It is also convenient to define an involution $* : \mathrm{Div}(N) \to \mathrm{Div}(N)$ given by the rule $[d]^* = \left[\frac{N}{d}\right]$. Thus if $\delta = \sum_{d|N} n(d)[d]$ one has $\delta^* = \sum_{d|N} n(d)\left[\frac{N}{d}\right]$. We define

$$\mathrm{Div}_f(N) := \mathrm{Div}(N) \times \mathbb{Z}/f\mathbb{Z}.$$

A typical element of $\delta \in \mathrm{Div}_f(N)$ will be denoted by

$$\delta = \sum_{d|N, r\in\mathbb{Z}/f\mathbb{Z}} n(d,r)[d,r] \quad \text{where} \quad n(d,r) \in \mathbb{Z},$$

where $[d, n]$ is a shorthand notation for the element $([d], r) \in \mathrm{Div}(N) \times \mathbb{Z}/f\mathbb{Z}$.

We endow the abelian group $\mathrm{Div}_f(N)$ with a left $\mathbb{Z}/f\mathbb{Z}$-module structure which is given on a generator $[d, j]$ by the rule $r \star [d, j] = [d, rj]$ for $r \in \mathbb{Z}/f\mathbb{Z}$. Then we extend $\star$ to all of $\mathrm{Div}_f(N)$ by $\mathbb{Z}$-linearity. For a divisor $\delta \in \mathrm{Div}_f(N)$ we will also use the short hand notation δ_r which is taken to mean $r \star \delta$. We extend the involution $*$ to $\mathrm{Div}_f(N)$ by the rule $[d, r]^* = [\frac{N}{d}, -r]$. The $\mathbb{Z}$-module $\mathrm{Div}_f(N)$ admits a direct sum decomposition given by

$$\mathrm{Div}_f(N) = \bigoplus_{t\in\mathbb{Z}/f\mathbb{Z}} \mathrm{Div}_f(N)_t,$$

where $\mathrm{Div}_f(N)_t = \{\delta \in \mathrm{Div}_f(N) : \delta = \sum_{d|N} n(d)[d,t]\}$. For every $t \in \mathbb{Z}/f\mathbb{Z}$, we have a natural projection maps $\pi_t : \mathrm{Div}_f(N) \to \mathrm{Div}_f(N)_t$ given by $\sum_{d|N, r\in\mathbb{Z}/f\mathbb{Z}} n(d,r)[d,r] \mapsto \sum_{d|N} n(d,t)[d]$. We also have a degree map on $\mathrm{Div}_f(N)_t$ which is given by $\sum_{d|N} n(d,t)[d] \mapsto \sum_{d|N} n(d,t)d$.

Let (p, f, N) be a triple as in the introduction. The next definition is crucial for the construction of the p-adic invariants u_{DD} and u_C.

Definition 5.1 *We say that a divisor $\delta \in \mathrm{Div}_f(N)$ is a good divisor (with respect to the triple p, f, N) if the following two conditions are satisfied:*

(1) *For all $t \in \mathbb{Z}/\mathbb{Z}$ one have $\deg(\pi_t(\delta)) = 0$,*

(2) *$p \star \delta = \delta$.*

Remark 5.1 In Appendix B, it is explained how to associate a modular unit $\beta_\delta(z)$ to a divisor $\delta \in \mathrm{Div}_f(N)$. The condition (1) implies that the modular unit $\beta_\delta(z)$ has no zeros nor poles on the set $\Gamma_0(fN)\{\infty\}$ and (2) implies that $\frac{\beta_\delta(z)}{\beta_\delta(pz)}$ is U_p-invariant (see Section 4 of [Cha] for further details).

5.2 The group $\widetilde{\mathrm{Div}}_f(\mathfrak{N})$

In this subsection we fix an order $\mathcal{O}$, an $\mathcal{O}$-integral ideal $\mathfrak{f} = f\mathcal{O}$ and an element $\tau \in H^{\mathcal{O}}(\mathfrak{N})$. We let S be the set of places of K that consists exactly of all the infinite ones, $\wp = p\mathcal{O}_K$ and the ones that ramify in $K(\mathfrak{f}\infty)$. We note that $K(\mathfrak{f}\infty)$ is an unramified extension of K outside $f \cdot \mathrm{cond}(\mathcal{O})$. We let $R = S \backslash \{\wp\}$.

Define $\widetilde{\mathrm{Div}}(\mathfrak{N})$ be the free $\mathbb{Z}$-module generated by the symbols $[\mathfrak{d}]$ where $\mathfrak{d}|\mathfrak{N}$. An element $\widetilde{\delta} \in \widetilde{\mathrm{Div}}(\mathfrak{N})$ will be denoted by

$$\sum_{\mathfrak{d}|\mathfrak{N}} n(\mathfrak{d})[\mathfrak{d}],$$

where $n(\mathfrak{d}) \in \mathbb{Z}$. We also define an involution $*$ on $\widetilde{\mathrm{Div}}(\mathfrak{N})$ which is given on generators by $[\mathfrak{d}]^* = \left[\frac{\mathfrak{N}}{\mathfrak{d}}\right]$.

We now fix an identification between $\widetilde{\mathrm{Div}}(\mathfrak{N})$ and $\mathrm{Div}(N)$.

Definition 5.2 *Let*
$$\Phi : \widetilde{\mathrm{Div}}(\mathfrak{N}) \to \mathrm{Div}(N)$$
be the map which is given on generators by $[\mathfrak{d}] \mapsto \left[\frac{\mathbf{N}(\mathfrak{N})}{\mathbf{N}(\mathfrak{d})}\right]$. Then we extend Φ to all of $\widetilde{\mathrm{Div}}(\mathfrak{N})$ by $\mathbb{Z}$-linearity.

We note that $\Phi(\widetilde{\delta}^*) = \Phi(\widetilde{\delta})^*$

Because of our choice of Φ we define the degree map on $\widetilde{\mathrm{Div}}(\mathfrak{N})$ in the following way:

$$\deg : \widetilde{\mathrm{Div}}(\mathfrak{N}) \to \mathbb{Z}$$

$$\sum_{\mathfrak{d}\mathfrak{N}} n(\mathfrak{d})[\mathfrak{d}] \mapsto \sum_{\mathfrak{d}|\mathfrak{N}} n(\mathfrak{d})\mathbf{N}\left(\frac{\mathfrak{N}}{\mathfrak{d}}\right).$$

In this way, elements of degree 0 in $\widetilde{\mathrm{Div}}(\mathfrak{N})$ map to elements of degree 0 in $\mathrm{Div}(N)$ under Φ. We denote the kernel of deg by $\widetilde{\mathrm{Div}}^0(\mathfrak{N})$.

Definition 5.3 *Let $\mathfrak{d}'$ and $\mathfrak{d}$ be two divisors of $\mathfrak{N}$. We say that $\mathfrak{d}'$ is consecutive to $\mathfrak{d}$ if $\frac{\mathfrak{d}'}{\mathfrak{d}} = \eta_i$ for some i.*

The next lemma will play a key role later on

Lemma 5.1 *The group $\widetilde{\mathrm{Div}}^0(\mathfrak{N})$ is generated by elements of the form*

$$d'[\mathfrak{d}'] - d[\mathfrak{d}],$$

where $\mathfrak{d}'$ is consecutive to $\mathfrak{d}$, $d' = \mathbf{N}(\mathfrak{d}')$ and $d = \mathbf{N}(\mathfrak{d})$.

Proof This is an easy induction argument. $\quad\square$

We define $\widetilde{\mathrm{Div}}_f(\mathfrak{N}) := \widetilde{\mathrm{Div}}(\mathfrak{N}) \times \mathbb{Z}/f\mathbb{Z}$ and endow it with its natural structure of $\mathbb{Z}/f\mathbb{Z}$-module. An element $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$ will be denoted as $\sum_{\mathfrak{d}|\mathfrak{N}, r\in\mathbb{Z}/f\mathbb{Z}} n(\mathfrak{d},r)[\mathfrak{d},r]$. There is a natural direct sum decomposition of $\widetilde{\mathrm{Div}}(\mathfrak{N})$ which is given by

$$\widetilde{\mathrm{Div}}_f(\mathfrak{N}) = \bigoplus_{t\in\mathbb{Z}/f\mathbb{Z}} \widetilde{\mathrm{Div}}_f(\mathfrak{N})_t,$$

where $\widetilde{\mathrm{Div}}_f(\mathfrak{N})_t = \{\widetilde{\delta} \in \widetilde{\mathrm{Div}}(\mathfrak{N}) : \widetilde{\delta} = \sum_{\mathfrak{d}|\mathfrak{N}} n(\mathfrak{d})[\mathfrak{d},t]\}$. The isomorphism Φ induces in a natural way an isomorphism of $\mathbb{Z}/f\mathbb{Z}$-module between $\widetilde{\mathrm{Div}}_f(\mathfrak{N})$ and $\mathrm{Div}_f(N)$ which we again denote by Φ. For every $t \in \mathbb{Z}/f\mathbb{Z}$, we let $\pi_t : \widetilde{\mathrm{Div}}_f(\mathfrak{N}) \to \widetilde{\mathrm{Div}}_f(\mathfrak{N})_t$ denote the natural projection map. Note that $\pi_t \circ \Phi = \Phi \circ \pi_t$.

Finally, we say that a divisor $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$ is a good divisor (with respect to the triple (p, f, N)) if and only if $\Phi(\widetilde{\delta}) \in \mathrm{Div}_f(N)$ is a good divisor.

6 Zeta functions attached to divisors

Let (p, f, N) be a triple as in the introduction. Now we would like to associate to any divisor $\delta \in \mathrm{Div}_f(N)$ and every pair $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}$ two zeta functions. One can think of these zeta functions as being "twisted" by the divisor δ.

Definition 6.1 *Let $\delta = \sum_{d|N, r\in\mathbb{Z}/f\mathbb{Z}} n(d,r)[d,r] \in \mathrm{Div}_f(N)$ be a good divisor and let $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times K\backslash\mathbb{Q}$. We define*

(1) $\displaystyle \zeta(\delta, (r,\tau), s) = \sum_{d|N, j\in\mathbb{Z}/f\mathbb{Z}} n(d,j)\, d^{-s} \cdot \widehat{\zeta}((rj, d\tau), s),$

(2) $\displaystyle \zeta^*(\delta, (r,\tau), s) = \sum_{d|N, j\in\mathbb{Z}/f\mathbb{Z}} n\left(\frac{N}{d}, j\right)\left(\frac{N}{d}\right)^s \cdot \widehat{\zeta}((-rj, d\tau^*), s),$

where $\tau^* = \frac{-1}{fN\tau}$ and $\widehat{\zeta}$ is the zeta function which appears in (2) of Definition 4.2.

Remark 6.1 We would like to warn the reader about a typo in [Cha09c]. In (2) of Definition 5.4 of [Cha09c] one should replace the term d_0^s by $(\frac{N_0}{d_0})^s$. Also the definition of (1) above varies slightly from the definition which appears in Definition 5.4 of [Cha09c] and Definition 9.2 of [Cha]. This should not result in any inconsistency since all the statements that we will make in this paper concerning $\zeta(\delta, (r, \tau), s)$ will be proved without using any reference to the papers [Cha09c] and [Cha]. We note however that the special value at $s = 0$ is independent of the chosen definition.

Lemma 6.1 *We have the following relations between the three types of zeta functions:*

$$(6.1) \qquad\qquad \zeta^*(\delta, (r, \tau), s) = N^s \zeta(\delta^*, (r, \tau^*), s).$$

Proof This is a straightforward computation. $\square$

In a similar way, one may associate a zeta function to a divisor $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$.

Definition 6.2 *Let $\mathfrak{c}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int. For every divisor $\widetilde{\delta} = \sum_{\mathfrak{d}|\mathfrak{N}} \widetilde{n}(\mathfrak{d}, r)[\mathfrak{d}, r] \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$ we define*

$$\zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}\infty, s) := \sum_{\mathfrak{d}|\mathfrak{N}, r \in \mathbb{Z}/f\mathbb{Z}} \widetilde{n}(\mathfrak{d}, r) \mathbf{N}(\mathfrak{d})^{-s} \zeta\left(\widetilde{r}\mathfrak{c}(\mathfrak{d}^\sigma)^{-1}, \mathfrak{f}\infty, s\right).$$

Remark 6.2 Assume that $\mathcal{O} = \mathcal{O}_K$ and that N is square-free. Set $T = \{\eta : \eta|\mathfrak{N}\}$ and $\widetilde{\delta} = \prod_{i=1}^r ([1] - l_i[\eta_i]) \in \widetilde{\mathrm{Div}}(\mathfrak{N})$. Then in this case one has

$$(6.2) \qquad\qquad \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}\infty, s) = \zeta_{R, T^\sigma}(K(\mathfrak{f}\infty)/K, \sigma_\mathfrak{a}, s),$$

where $\sigma_\mathfrak{a} \in G_\mathfrak{f}$ is the Frobenius at $\mathfrak{a}$ and $\zeta_{R, T^\sigma}(K(\mathfrak{f}\infty/K)\sigma_\mathfrak{a}^{-1}, s)$ is the zeta function which appears in (1.5) and $T^\sigma = \{\eta^\sigma : \eta \in T\}$.

The next lemma relates special values at even negative integers of the zeta function which appears in (1) of Definition 6.1 with the one which appears in Definition 6.2.

Lemma 6.2 *Assume that $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$ and let $\mathfrak{f} = f\mathcal{O}_\tau$. Let $k \in \mathbb{Z}_{\geq 1}$ with $k \equiv 1 \pmod 2$. Then we have*

$$(6.3) \qquad\qquad 4\mu(r, \tau)\zeta_{\widetilde{\delta}}(\widetilde{r}I_\tau, \mathfrak{f}\infty, 1-k) = \mathbf{N}(\mathfrak{f})^{-(1-k)}\zeta(\delta^*, (r, \tau), 1-k),$$

where $\Phi(\widetilde{\delta}) = \delta$ and $\mu(r, \tau)$ is the positive rational number which appears in (4.10) and I_τ is the ideal attached to τ which appears in Definition 3.1.

Proof Let $\widetilde{\delta} = \sum_{d|N, j \in \mathbb{Z}/f\mathbb{Z}} \widetilde{n}(\mathfrak{d}, j)[\mathfrak{d}] \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$. By definition of the application Φ (see Definition 5.2) we have

$$\Phi(\widetilde{\delta}) = \sum_{d|N, j \in \mathbb{Z}/f\mathbb{Z}} n(d, j)\left[\frac{N}{d}\right] \quad \text{and therefore} \quad \Phi(\widetilde{\delta}^*) = \sum_{d|N, j \in \mathbb{Z}/f\mathbb{Z}} n(d, j)[d],$$

where for all $j \in \mathbb{Z}/f\mathbb{Z}$ and divisor $\mathfrak{d}|\mathfrak{N}$ one has that $n(\mathbf{N}(\mathfrak{d}), j) := \widetilde{n}(\mathfrak{d}, j)$. Set $\mathfrak{a} = \widetilde{r}I_\tau$. By definition of $\zeta_{\widetilde{\delta}}$, we have

$$
(6.4) \qquad \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}\infty, s) = \sum_{\mathfrak{d}|\mathfrak{N}, j \in \mathbb{Z}/f\mathbb{Z}} \widetilde{n}(\mathfrak{d}, j) \mathbf{N}(\mathfrak{d})^{-s} \zeta\left(\frac{\mathfrak{a}\mathfrak{d}}{\mathbf{N}(\mathfrak{d})}, \mathfrak{f}\infty, s \right).
$$

For every divisor $\mathfrak{d}|\mathfrak{N}$ one has that

$$
(6.5) \qquad \zeta\left(\frac{\mathfrak{a}\mathfrak{d}}{\mathbf{N}(\mathfrak{d})}, \mathfrak{f}, w_1, s \right) = \zeta\left(\frac{\widetilde{r}A_\tau}{d} \Lambda_\tau \mathfrak{d}, \mathfrak{f}, w_1, s \right) = \frac{1}{\mu(r, \tau)} \mathbf{N}(\mathfrak{f})^{-s} \widehat{\zeta}((r, d\tau), s),
$$

where the first equality follows from the fact that $\frac{A_\tau}{d} = A_{d\tau}$, $\mathfrak{d}\Lambda_\tau = \Lambda_{d\tau}$ and the second equality follows from Lemma 4.2.

Combining (6.5) and Lemma 4.1 one finds that for every odd integer $k \in \mathbb{Z}_{\geq 1}$

$$
(6.6) \qquad 4\zeta\left(\frac{\widetilde{r}A_\tau}{d} \Lambda_\tau \mathfrak{d}, \mathfrak{f}\infty, 1 - k \right) = \mathbf{N}(\mathfrak{f})^{-(1-k)} \widehat{\zeta}((r, d\tau), 1 - k).
$$

Finally, combining (6.4) and (6.6) we find that

$$
4\zeta_{\widetilde{\delta}}(\widetilde{r}I_\tau, \mathfrak{f}\infty, 1 - k) = \mu(r, \tau) \mathbf{N}(\mathfrak{f})^{-(1-k)} \zeta(\delta^*, (r, \tau), 1 - k),
$$

where $\Phi(\widetilde{\delta}) = \delta$. $\qquad\square$

Remark 6.3 Suppose that $\widetilde{\delta} \in \widetilde{\mathrm{Div}}(\mathfrak{N})$ is chosen so that $p \star \widetilde{\delta} = \widetilde{\delta}$. Then a direct computation shows that

$$
(6.7) \qquad \zeta_{\widetilde{\delta}, \{p\}}(\mathfrak{a}, \mathfrak{f}\infty, s) = \left(1 - \frac{1}{p^{2s}} \right) \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}\infty, s).
$$

The subscript $\{p\}$ means that we only sum over elements which are coprime to p. Assume that every prime divisor $\mathfrak{q}|\mathfrak{f}$ is invertible (for example this is automatically satisfied when $\mathcal{O}$ is the maximal order). Then under this assumption, for all residue class $r \in \mathbb{Z}/f\mathbb{Z}$ one has that $(\mathfrak{f}, \widetilde{r}\mathfrak{a})$ is again $\mathcal{O}$-invertible. Then, in light of the identity (4.5) we see that the zeta function $\zeta_{\widetilde{\delta}, \{p\}}(\mathfrak{a}, \mathfrak{f}\infty, s)$ may be written in the following form

$$
(6.8) \qquad \zeta_{\widetilde{\delta}, \{p\}}(\mathfrak{a}, \mathfrak{f}\infty, s) = \sum_{\mathfrak{d}|\mathfrak{N}, r \in \mathbb{Z}/f\mathbb{Z}} n(\mathfrak{d}, r) \mathbf{N}(\mathfrak{d})^{-s} \zeta_{\{p\}}\left(\frac{\widetilde{r}\mathfrak{a}\mathfrak{d}}{(\mathfrak{f}, \widetilde{r}\mathfrak{a})}, \frac{\mathfrak{f}}{(\mathfrak{f}, \widetilde{r}\mathfrak{a})}\infty, s \right),
$$

which is a linear combination of *primitive zeta functions* of conductor $\mathfrak{f}'$ for various divisors $\mathfrak{f}'|\mathfrak{f}$. In particular, it follows from the work of Deligne and Ribet (see [DR80]) that the special values of $\zeta_{\widetilde{\delta}, \{p\}}(\mathfrak{a}, \mathfrak{f}\infty, s)$ at negative even integers which are congruent to 0 modulo $2(p-1)$ can be p-adically interpolated. We denote the corresponding p-adic zeta function by $\zeta_{\widetilde{\delta}, p}(\mathfrak{a}, \mathfrak{f}\infty, s)$.

7 Shintani zeta functions

In this section we first recall the notion of a cone decomposition in the setting of a real quadratic number field. Then we introduce the so-called Shintani zeta function associated to the choice of such a cone decomposition. Recall that K comes equipped with a fixed embedding $K \subseteq \mathbb{R}$ and that $\sigma : K \to K$ denotes the non-trivial automorphism of K. Let $\iota : K \to \mathbb{R}^2$ be the embedding given by $x \mapsto (x, x^\sigma)$. Under ι, every invertible $\mathcal{O}$-ideal $\mathfrak{a} \subseteq K$ may be viewed as a lattice in $\mathbb{R}^2$. For the rest of this section, we will view elements of K as elements of $\mathbb{R}^2$ via ι. Note that the group $K^\times$ acts naturally on $\mathbb{R}^2$ and that its subgroup $K_+^\times = \{x \in K^\times : x \gg 0\}$ acts naturally on the positive quadrant $Q := \mathbb{R}_{>0}^2$.

Definition 7.1 *For $\mathbb{R}$-linearly independent vectors $v_1, \ldots, v_r \in Q \cap K$ (in our setting $r \leq 2$) we let*

$$C = C(v_1 \ldots, v_r) = \left\{ \sum_{i=1}^r c_i v_i \in Q : c_i \in \mathbb{R}_{>0} \right\}.$$

We call C a Shintani cone of dimension r. We say that $\mathcal{D} \subseteq Q$ is a Shintani set if it can be written as a finite disjoint union of Shintani cones.

Let W be a finite union of Shintani cones. One can show that W can always be written as a finite disjoint union of possibly smaller Shintani cones. Therefore W is a Shintani set.

Let $\mathcal{O}$ be an arbitrary order of K and let $\mathfrak{f} = f\mathcal{O}$. Let $\mathcal{O}(\mathfrak{f}\infty)^\times = \langle \epsilon \rangle$ where ϵ is the unique generator such that $\epsilon > 1$. The group $\mathcal{O}(\mathfrak{f}\infty)^\times$ acts discretely on Q and admits a fundamental domain (the existence of a fundamental in the setting of real quadratic field is obvious). It is convenient to define a privileged choice of a fundamental domain for the action of $\mathcal{O}(\mathfrak{f}\infty)^\times$ on Q, namely

$$(7.1) \qquad\qquad \mathcal{D}_{\mathfrak{f}}^{can} := C(1, \epsilon) \cup C(1).$$

Now we would like to extend slightly a key definition that was introduced in [Das08].

Definition 7.2 *A prime $\mathcal{O}_K$-ideal η is called $\mathcal{O}$-**good** for a Shintani cone C if*

(1) $\mathbf{N}(\eta)$ *is a rational prime ℓ;*

(2) *the cone C may be written $C = C(v_1, \ldots, v_r)$ such that for all i, $v_i \in \mathcal{O}$ and $v_i \notin (\eta \cap \mathcal{O})$.*

In general we say that η is $\mathcal{O}$-good for a subset $W \subseteq Q$ if W may be written as a finite disjoint union of Shintani cones $W = \bigcup_i C_i$ such that η is $\mathcal{O}$-good for each C_i.

Remark 7.1 Let $\mathfrak{f} = f\mathcal{O}_K$. Then any prime $\mathcal{O}_K$-ideal η of degree 1 is $\mathcal{O}_K$-good for the canonical fundamental domain $\mathcal{D}_{\mathfrak{f}}^{can}$. Suppose that η is $\mathcal{O}_K$-good for a Shintani cone $C = C(v_1, v_2)$ where $v_1, v_2 \in \mathcal{O}_K$ and $v_1, v_2 \notin \eta$. Let $l = \mathbf{N}(\eta)$. Then if n is a positive integer coprime to l we readily see that $nv_1, nv_2 \in \mathcal{O}_n$ and $nv_1, nv_2 \notin \eta_n = (\mathcal{O}_n \cap \eta)$. In particular, η is $\mathcal{O}_n$-good for the Shintani cone C.

Definition 7.3 *A finite set of places T of K is said to be $\mathcal{O}$-good for a Shintani set $\mathcal{D}$ if $\mathcal{D}$ can be written as a finite disjoint union of Shintani cones $\mathcal{D} = \bigcup C_i$ such that for each cone C_i there are at least two primes in T of different residue characteristic which are $\mathcal{O}$-good for C_i or one prime $\eta \in T$ which is $\mathcal{O}$-good for C_i and has absolute ramification $\leq l - 2$ where $l = \mathbf{N}(\eta)$.*

For the rest of the section we fix an order $\mathcal{O}$ of K and an $\mathcal{O}$-integral ideal $\mathfrak{f}$.

Definition 7.4 *Let $W \subseteq Q$ be an arbitrary subset, $\mathfrak{c}$ be an invertible $\mathcal{O}$-ideal and let $x \in K$. For complex numbers s such that $\Re(s) > 1$ we define*

$$(7.2) \qquad \zeta(\mathfrak{c}, \mathfrak{f}, x, W, s) := \mathbf{N}(\mathfrak{c})^{-s} \sum_{0 \neq \mu \in (\mathfrak{c}^{-1}\mathfrak{f} + x) \cap W} \frac{1}{|\mathbf{N}(\mu)|^s}.$$

For $\lambda \in K_+^\times$ and $\Re(s) > 1$ one has the formula

$$(7.3) \qquad \mathbf{N}(\lambda)^{-s} \zeta(\mathfrak{c}, \mathfrak{f}, x, W, s) = \zeta(\lambda^{-1}\mathfrak{c}, \mathfrak{f}, \lambda x, \lambda W, s).$$

In general, for an arbitrary subset $W \subseteq Q$ the function $\zeta(\mathfrak{c}, \mathfrak{f}, x, W, s)$ will not admit a meromorphic continuation to all of $\mathbb{C}$. However, there is an important special case where it does namely in the case where W is a Shintani cone.

Proposition 7.1 *(Shintani) Let C be a Shintani cone. Then the function $\zeta(\mathfrak{c}, \mathfrak{f}, x, C, s)$ admits a meromorphic to all of $\mathbb{C}$. Moreover, if $k \in \mathbb{Z}_{\geq 1}$, the special value $\zeta(\mathfrak{c}, \mathfrak{f}, x, C, 1 - k)$ is a rational number.*

Proof This follows from Proposition 1 of [Shi76]. $\square$

Let $W = \bigcup_i C_i$ (disjoint union) be a Shintani domain. Since

$$\zeta(\mathfrak{c}, \mathfrak{f}, x, W, s) = \sum_i \zeta(\mathfrak{c}, \mathfrak{f}, x, C_i, s),$$

we see that Proposition 7.1 continues to hold for such W. In particular, Proposition 7.1 holds true with $W = \mathcal{D}$ where $\mathcal{D}$ is any fundamental domain for the action of $\mathcal{O}(\mathfrak{f}\infty)^\times$ on Q.

Definition 7.5 *We say that a meromorphic function $f(s)$ on the complex plane is a Shintani zeta function (for the real quadratic field K) if there exist quantities $\mathfrak{c}, \mathfrak{f} \subseteq K$, $x \in K$ and a Shintani set W such that $f(s) = \zeta(\mathfrak{c}, \mathfrak{f}, x, W, s)$.*

7.1 $\mathbb{Q}$-valued distributions on $\mathcal{O}_{K_p}$

In this subsection, we want to define $\mathbb{Q}$-valued distributions on $\mathcal{O}_K$ using special values at $s = 0$ of Shintani zeta functions. Let (p, f, N) be a triple as in the introduction and let

$\mathfrak{N} = \prod_{i=1}^{r} \eta_i^{e_i}$ where $\mathfrak{N}\mathfrak{N}^\sigma = N\mathcal{O}_K$. For the rest of the section we fix an order $\mathcal{O}$ and we let $\mathfrak{f} = f\mathcal{O}$.

From now on, for an arbitrary compact-open set $U \subseteq \mathcal{O}_{K_p}$ and an arbitrary subset $W \subseteq Q$ the notation $W \cap U$ is taken to mean $W \cap (U \cap K)$. The next proposition is an easy corollary of Proposition 7.1.

Proposition 7.2 *Let W be a finite union of Shintani cones and let $U \subseteq \mathcal{O}_{K_p}$ be a compact open set. Then the function*

$$\zeta(\mathfrak{c}, \mathfrak{f}, x, W \cap U, s),$$

admits a meromorphic continuation to all of $\mathbb{C}$ and its special values at negative integers are rational numbers.

Proof Note that $(\mathfrak{c}^{-1}\mathfrak{f} + x) \cap U$ may be written as

$$(\mathfrak{c}^{-1}\mathfrak{f} + x) \cap U = \bigcup_i (\mathfrak{b}^{-1}\mathfrak{f} + y_i),$$

where the union is finite and disjoint, $\mathfrak{b}$ is a suitable invertible $\mathcal{O}$-ideal and $y_i \in K$. It thus follows that

$$\zeta(\mathfrak{c}, \mathfrak{f}, x, W \cap U, s) = \sum_i \zeta(\mathfrak{b}, \mathfrak{f}, y_i, W, s).$$

This concludes the proof. $\square$

Definition 7.6 *Let $W \subseteq Q$ be an arbitrary subset and let*

$$\widetilde{\delta} = \sum_{\mathfrak{d}|\mathfrak{N}, r\in\mathbb{Z}/f\mathbb{Z}} n(\mathfrak{d}, r)[\mathfrak{d}, r] \in \widetilde{\mathrm{Div}}_f(\mathfrak{N}).$$

For an invertible $\mathcal{O}$-ideal $\mathfrak{c}$ and $\lambda \in K$ we define

$$\zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}, x, W, s) := \sum_{\mathfrak{d}|\mathfrak{N}, r\in\mathbb{Z}/f\mathbb{Z}} n(\mathfrak{d}, r)\zeta(r\mathfrak{c}(\mathfrak{d}^\sigma)^{-1}, \mathfrak{f}, x, W, s).$$

Definition 7.7 *Let $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$ and let W be a finite union of Shintani cones. We define a $\mathbb{Q}$-valued distribution on $\mathcal{O}_{K_p}$ by the rule*

(7.4) $$U \mapsto \zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}, x, W \cap U, 0),$$

where U is an arbitrary compact open set of $\mathcal{O}_{K_p}$.

Remark 7.2 In the case where $\mathfrak{c}$ is an integral $\mathcal{O}_K$ ideal coprime to $f\mathcal{O}_K$, $\widetilde{\delta} = \prod_{i=1}^r (1 - l_i[\eta_i])$, $x = 1$ and $\mathcal{D}$ is a Shintani set one has

$$\zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}, x, \mathcal{D} \cap U, 0) = \nu(\mathfrak{c}, \mathcal{D}, U),$$

where $\nu(\mathfrak{c}, \mathcal{D}, \_)$ is the measure defined on the line (20) of [Das08].

Proposition 7.3 *Let $\lambda \in K_+^\times$, $x \in K$ and let $\mathfrak{a}$ and $\mathfrak{b}$ be two invertible $\mathcal{O}$-ideals. Let W be a Shintani domain. Then the following two formulas hold:*

(1) $\zeta_{\widetilde{\delta}}(\mathfrak{a}\mathfrak{b}, \mathfrak{f}\mathfrak{b}, x, W, 0) = \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, x, W),$

(2) $\zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, x, W \cap U, 0) = \zeta_{\widetilde{\delta}}(\lambda^{-1}\mathfrak{a}, \mathfrak{f}, \lambda x, \lambda(W \cap U), 0).$

Proof This a straightforward computation. $\square$

Proposition 7.4 *Let $\mathcal{D}$ be a fundamental domain for the action of $\mathcal{O}_K(\mathfrak{f}\infty)^\times$ on the positive quadrant Q and let $\mathfrak{c}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int. Then we have*

$$(7.5) \qquad \zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}, n_{\mathfrak{c}}, \mathcal{D} \cap \mathcal{O}_{K_p}, 0) = \zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}\infty, 0),$$

where $\zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}\infty, s)$ is the zeta function which appears in Definition 6.2 and $n_{\mathfrak{c}}$ is an integer chosen as in Definition 4.4. Moreover, if $p \star \widetilde{\delta} = \widetilde{\delta}$ then

$$(7.6) \qquad \zeta_{\widetilde{\delta}}(\mathfrak{c}, \mathfrak{f}, n_{\mathfrak{c}}, \mathcal{D} \cap \mathcal{O}_{K_p}^\times, 0) = \zeta_{\widetilde{\delta}, \{p\}}(\mathfrak{c}, \mathfrak{f}\infty, 0) = 0.$$

The subscript $\{p\}$ of $\zeta_{\widetilde{\delta}, \{p\}}$ means that one restricts the sum over elements coprime to p.

Proof The proof of (7.5) is straightforward and (7.6) follows from (6.7). $\square$

Let T be the set of all places of K which divide $\mathfrak{N}$. For technical reasons we will assume until the end of the section that $(\mathrm{cond}(\mathcal{O}), N) = 1$. Therefore if $\eta_0 \in T = \{\eta | \mathfrak{N}\}$ it gives rise to a discrete valuation on $\mathcal{O}$ which we denote by v_{η_0}. We can now state a variant of a key proposition that was proved in [Das08]:

Proposition 7.5 *Let C be a Shintani cone of dimension m and assume that there exists $\eta \in T$ which is $\mathcal{O}$-good for C. Let $\mathfrak{a}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int and such that $v_\eta(\mathfrak{a}) \leq 0$. Let $l = [\mathcal{O} : \eta]$ and let $y \in K$. Then*

$$\zeta(\mathfrak{a}, \mathfrak{f}, y, C, 0) - l\zeta(\mathfrak{a}\eta^{-1}, \mathfrak{f}, y, C, 0) \in \mathbb{Z}[\tfrac{1}{l}],$$

where the denominator is at most $l^{m/(l-1)}$.

Proof We follow closely a part of the proof of Proposition 6.1 of [Das08]. Let $\eta' = \eta \cap \mathcal{O}$. Since η is $\mathcal{O}$-good for C we may write $C = C(v_1, \ldots, v_r)$ with $v_i \in \mathcal{O}$ and $v_i \notin \eta'$. We claim that we can always find a positive integer $n_i \geq 1$ such that $n_i v_i \in \mathfrak{a}^{-1}\mathfrak{f}$ but $n_i v_i \notin \mathfrak{a}^{-1}\mathfrak{f}\eta'$. Let us prove this. Since $v_\eta(\mathfrak{a}) \leq 0$, we may write $\mathfrak{a} = \frac{\mathfrak{m}}{\mathfrak{n}}$ where $\mathfrak{m}$ and $\mathfrak{n}$ are invertible integral $\mathcal{O}$-ideals and $v_\eta(\mathfrak{m}) = 0$. Note that $\mathfrak{a}^{-1} = \mathfrak{n}\mathfrak{m}^{-1}\mathfrak{f} \supseteq \mathfrak{n}\mathfrak{f}$. Let n_i be a positive integer such that $n_i v_i \in \mathfrak{n}\mathfrak{f}$ and such that $v_\eta(n_i) = v_\eta(\mathfrak{n})$ (such an integer n_i exists since $v_i \notin \mathcal{O}$). Since $v_\eta(v_i) = v_\eta(\mathfrak{m}) = v_\eta(\mathfrak{f}) = 0$ we see that $n_i v_i \notin \mathfrak{a}^{-1}\mathfrak{f}\eta'$. So this proves our claim. Therefore, without loss of generality, we may assume that $v_i \in \mathfrak{a}^{-1}\mathfrak{f}$ and $v_i \notin \mathfrak{a}^{-1}\mathfrak{f}\eta'$.

Any element $\alpha \in C$ may be written uniquely as $\alpha = \sum_{i=1}^r (x_i + z_i)v_i$, for real numbers $0 < x_i \leq 1$ and non-negative integers z_i. Since $v_i \in \mathfrak{a}^{-1}\mathfrak{f}$, the element α lie in $\mathfrak{a}^{-1}\mathfrak{f} + y$ if and only if $\sum x_i v_i$ does. Thus if we let

$$\Omega(\mathfrak{c}, y, v) = \{x \in \mathfrak{c} + y : x = \sum x_i v_i \text{ with } 0 < x_i \leq 1\},$$

then

$$\zeta(\mathfrak{a}, \mathfrak{f}, y, C, s) = \sum_{x \in \Omega(\mathfrak{a}^{-1}\mathfrak{f}, y, v)} \sum_{z_1, \ldots, z_r = 0} \mathbf{N}\left(\sum (x_i + z_i) v_i\right)^{-s}.$$

One has $\zeta(\mathfrak{a}, \mathfrak{f}, y, C, s) = Z(\mathfrak{a}^{-1}\mathfrak{f}, y, C, s)$ where

$$(7.7) \qquad\qquad Z(\mathfrak{b}, y, C, s) = \sum_{\alpha \in (\mathfrak{b}+y) \cap C} \mathbf{N}(\alpha)^{-s},$$

is the zeta function which appears on line (69) of [Das08]. From this point the rest of the argument is identical to the end of the proof of Proposition 6.1 of [Das08], so we skip it.
$\square$

7.2 $\mathbb{Q}$-valued distributions on $\mathbb{Z}_p^2$

In this subsection we want define $\mathbb{Q}$-valued distributions on $\mathbb{Z}_p^2$ by using the distributions constructed in the previous section on $\mathcal{O}_{K_p}$. For every pair of integers (u, v) and for every non-negative integer n we let

$$(7.8) \qquad\qquad U_{u,v,n} = \{(x, y) \in \mathbb{Z}_p^2 : (x, y) \equiv (u, v) \pmod{p^n}\},$$

be the ball of radius $\frac{1}{p^n}$ centered at (u, v). We denote the set of all balls of $\mathbb{Z}_p^2$ by $\mathcal{B}$. We note that a distribution on $\mathbb{Z}_p^2$ is completely determined by its values on elements of $\mathcal{B}$ since the set $\mathcal{B}$ is a basis for the topology of $\mathbb{Z}_p^2$. There is a right action of $GL_2(\mathbb{Z}_p)$ on $\mathbb{Z}_p^2$ which is given by the following rule

$$\begin{pmatrix} x & y \end{pmatrix} \sharp \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} dx - by & -cx + ay \end{pmatrix},$$

where $\begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ and $\begin{pmatrix} x & y \end{pmatrix} \in \mathbb{Z}_p^2$. We have chosen this action in agreement with (4.3). We thus obtain a right action of $GL_2(\mathbb{Z}_p)$ on $\mathcal{B}$ which is given on balls by the rule

$$(7.9) \qquad\qquad (U_{u,v,n}) \sharp \gamma = U_{du-bv, -cu+av, n},$$

where $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)$ and $U_{u,v,n} \in \mathcal{B}$. We note that the right action in (7.9) may be written in terms of the usual left action of $GL_2(\mathbb{Z}_p)$ on $\mathbb{Z}_p^2$ namely

$$(U_{u,v,n}) \sharp \gamma = \gamma^{-1} U_{u,v,n},$$

where for a matrix $\eta = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{Z}_p)$, $\eta U_{u,v,n} := U_{au+bv, cu+dv}$.

The space of $\mathbb{Q}$-valued distributions on $\mathbb{Z}_p^2$, which we denote by $Dist(\mathbb{Z}_p^2, \mathbb{Q})$, has an induced right action by $GL_2(\mathbb{Z}_p)$. For a distribution $\mu \in Dist(\mathbb{Z}_p^2, \mathbb{Q})$ and an element $\gamma \in GL_2(\mathbb{Z}_p)$ we define the *right γ-twist* of μ by the rule

$$(7.10) \qquad \mu^\gamma(U) := \mu(U \sharp \gamma^{-1}) = \mu(\gamma U).$$

We also endow the space of rational binary quadratic forms with a left action of $GL_2(\mathbb{Q})$ by the rule

$$(7.11) \qquad \gamma Q(x, y) := Q((\begin{array}{cc} x & y \end{array}) \sharp \gamma) \quad \text{for} \quad \gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{Q}).$$

We want to define a $\mathbb{Q}$-valued distribution on $\mathbb{Z}_p^2$ using the distribution which appears in Definition 7.7. In order to do so we need to choose an identification of $\mathbb{Z}_p^2$ with $\mathcal{O}_{K_p}$.

Definition 7.8 *For every $\tau \in K \backslash \mathbb{Q}$, we define an injective map*

$$\phi_\tau : \mathbb{Z}_p^2 \to K_p$$
$$(x, y) \mapsto x - y\tau^\sigma.$$

Note that in general, the image ϕ_τ is not necessarily equal to $\mathcal{O}_{K_p}$. We say that an element $\tau \in \mathcal{H}_p \cap K$ is *reduced* if for $j = 0, \ldots, p-1$, $|\tau - j|_p \geq 1$ and $\left|\frac{1}{\tau}\right|_p \geq 1$. When τ is reduced, it is easy to see that the map ϕ_τ gives an isomorphism between $\mathbb{Z}_p^2$ and $\mathcal{O}_{K_p}$.

We note that

$$(7.12) \qquad \phi_\tau(\gamma U) = (-c\tau^\sigma + a)\phi_{\gamma^{-1}\tau}(U).$$

We are now ready to define $\mathbb{Q}$-valued distributions on $\mathbb{X}$.

Definition 7.9 *Let $\tau \in \mathcal{H}_p \cap K$ where τ is reduced and let W be a Shintani set. Let $x \in K$ and let $\mathfrak{a}$ be a fractional ideal. For every divisor $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$ we define the following $\mathbb{Q}$-valued distribution on $\mathbb{Z}_p^2$:*

$$U \mapsto \nu_{\widetilde{\delta}}(\mathfrak{a}, \tau, x, W)(U) := \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, x, W \cap \phi_\tau(U), 0),$$

for U an arbitrary compact-open set of $\mathbb{Z}_p^2$.

The next lemma will play a key role later on

Lemma 7.1 *Let $\tau \in K \backslash \mathbb{Q}$ and $x \in K^\times$. Suppose that $\gamma = \left(\begin{array}{cc} a & b \\ c & d \end{array} \right) \in GL_2(\mathbb{Z})$ and that $-c\tau^\sigma + a \gg 0$. Then we have*

$$\nu_{\widetilde{\delta}}^\gamma(I_\tau, \tau, x, W) = \nu_{\widetilde{\delta}}(I_{\gamma\tau}, \gamma\tau, (-c\tau^\sigma + a)^{-1}x, (-c\tau^\sigma + a)^{-1}W).$$

Proof This is a straightforward computation. $\square$

8 $\mathbb{Z}$-valued measures on $\mathbb{X}$

In [Cha09c], a family of $\mathbb{Z}$-valued measures $\widetilde{\mu}_\delta$ (see Definition 8.1 below) on $\mathbb{X}$ was constructed using periods of Eisenstein series. Here $\mathbb{X} = \mathbb{Z}_p \times \mathbb{Z}_p \backslash (p\mathbb{Z}_p \times p\mathbb{Z}_p)$ denotes the set of primitive vectors of $\mathbb{Z}_p^2$. The author defined the p-invariant u_C as a certain multiplicative integral on the space $\mathbb{X}$ which involves the measure $\widetilde{\mu}_\delta$. In a similar way, the invariant u_D is defined as p-adic multiplicative integral on the space $\mathcal{O}_{K_p}$ which involves the measure $\nu_{\widetilde{\delta}}$ of Definition 7.9. In Section 9, precise definitions of u_C and u_D are given. The key ingredient that allows us to relate the invariant u_C to the invariant u_D are explicit formulas of the measures $\widetilde{\mu}_\delta$ and $\nu_{\widetilde{\delta}}$ on balls of $\mathbb{X}$.

Definition 8.1 *Assume that $\mathfrak{d}'$ is consecutive to $\mathfrak{d}$ and let $\eta = \frac{\mathfrak{d}'}{\mathfrak{d}}$, $l = \mathbf{N}(\eta)$, $d = \mathbf{N}(\mathfrak{d})$ and $d' = ld$. Let us fix an integer $1 \leq j \leq f$ and let*

$$\delta = \frac{N}{d}[d, j] - \frac{N}{d'}[d', j] \in \mathrm{Div}_f(N).$$

Let $\infty = \frac{1}{0}$ be the standard cusp at infinity and $\frac{a}{c} \in \Gamma_0(fN)\{\infty\}$. For every ball $U_{u,v,s} \subseteq \mathbb{X}$, we define

$$\widetilde{\mu}_\delta \left\{ \infty \to \frac{a}{c} \right\} (U_{u,v,s})$$

$$(8.1) \quad := -12 \left(\frac{N}{d} \right) \sum_{1 \leq h \leq \frac{c}{fd}} \widetilde{B}_1 \left(\frac{a}{\frac{c}{fd}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) - \frac{dfu}{p^s} \right) \widetilde{B}_1 \left(\frac{1}{\frac{c}{fd}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) \right)$$

$$+ 12 \left(\frac{N}{d'} \right) \sum_{1 \leq h' \leq \frac{c}{fd'}} \widetilde{B}_1 \left(\frac{a}{\frac{c}{fd'}} \left(h' + \frac{v}{p^s} + \frac{j}{f} \right) - \frac{d'fu}{p^s} \right) \widetilde{B}_1 \left(\frac{1}{\frac{c}{fd'}} \left(h' + \frac{v}{p^s} + \frac{j}{f} \right) \right),$$

where $\widetilde{B}_1(x) = \{x\} - \frac{1}{2} + \frac{\mathbb{1}_{\mathbb{Z}}(x)}{2}$, $0 \leq \{x\} < 1$ denotes the fractional part of a real number x and $\mathbb{1}_{\mathbb{Z}}(x)$ stands for the characteristic function of $\mathbb{Z}$.

Remark 8.1 The expression which appears on the right hand side of (8.1) is a special case of Dedekinds sums that have been considered in [Hal85].

Using Lemma 5.1, we first extend by linearity, the definition of $\widetilde{\mu}_\delta \left\{ \infty \to \frac{a}{c} \right\}$ to any divisor $\delta \in \mathrm{Div}^0(N)_j$. In a second step, we extend the definition of $\widetilde{\mu}_\delta \left\{ \infty \to \frac{a}{c} \right\}$ to all $\delta \in \mathrm{Div}_f(N)$ which satisfies (1) of Definition 5.1.

The next proposition justifies the previous definition.

Proposition 8.1 *Let $\delta \in \mathrm{Div}_f(N)$ be a good divisor for the triple (p, f, N). Then the assignment $U_{u,v,s} \mapsto \widetilde{\mu}_\delta \left\{ \infty \to \frac{a}{c} \right\} (U_{u,v,s})$ gives rise to a $\mathbb{Z}$-valued measure on $\mathbb{X}$ with total measure zero, i.e., $\widetilde{\mu}_{\delta_r} \left\{ \infty \to \frac{a}{c} \right\} (\mathbb{X}) = 0$.*

Proof See Proposition 14.1 in [Cha07]. $\square$

In the next subsection we will give a different proof of Proposition 8.1 by relating the measure $\widetilde{\mu}_\delta$ to the measure $\nu_{\widetilde{\delta}}$.

It will be convenient to rewrite (8.1) in a different way. Every element $1 \leq h \leq \frac{c}{fd}$ may be written uniquely as $h = \frac{c}{fd'}r + h'$ for some $0 \leq r \leq l-1$ and $1 \leq h' \leq \frac{c}{fd'}$. We have

$$\sum_{1 \leq h' \leq \frac{c}{fd'}} \widetilde{B}_1\left(\frac{a}{\frac{c}{fd'}}\left(h' + \frac{v}{p^s} + \frac{j}{f}\right) - \frac{d'fu}{p^s}\right) \widetilde{B}_1\left(\frac{1}{\frac{c}{fd'}}\left(h' + \frac{v}{p^s} + \frac{j}{f}\right)\right)$$

$$= \sum_{1 \leq h' \leq \frac{c}{fd'}} \sum_{0 \leq r \leq l-1} \widetilde{B}_1\left(\frac{a}{\frac{c}{fd}}\left(h' + r\frac{c}{fd'} + \frac{v}{p^s} + \frac{j}{f}\right) - \frac{dfu}{p^s}\right) \widetilde{B}_1\left(\frac{1}{\frac{c}{fd'}}\left(h' + r\frac{c}{fd'} + \frac{v}{p^s} + \frac{j}{f}\right)\right)$$

$$= \sum_{1 \leq h \leq \frac{c}{fd}} \widetilde{B}_1\left(\frac{a}{\frac{c}{fd}}\left(h + \frac{v}{p^s} + \frac{j}{f}\right) - \frac{dfu}{p^s}\right) \widetilde{B}_1\left(\frac{1}{\frac{c}{fd'}}\left(h + \frac{v}{p^s} + \frac{j}{f}\right)\right)$$

where the first equality follows from the distribution relation $\sum_{j=0}^{l-1} \widetilde{B}_1(x + \frac{j}{l}) = \widetilde{B}_1(lx)$ and that $\widetilde{B}_1(x+1) = \widetilde{B}_1(x)$. The second equality follows from the definition of h. We may thus rewrite (8.1) as

$$\widetilde{\mu}_\delta\left\{\infty \to \frac{a}{c}\right\}(U_{u,v,s}) = -12 \sum_{1 \leq h \leq \frac{c}{fd}} \widetilde{B}_1\left(\frac{a}{\frac{c}{fd}}\left(h + \frac{j}{f} + \frac{v}{p^s}\right) - \frac{fdu}{p^s}\right)$$

(8.2)
$$\cdot \left[\left(\frac{N}{d}\right)\widetilde{B}_1\left(\frac{1}{\frac{c}{fd}}\left(h + \frac{j}{f} + \frac{v}{p^s}\right)\right) - \left(\frac{N}{d'}\right)\widetilde{B}_1\left(\frac{1}{\frac{c}{fd'}}\left(h + \frac{j}{f} + \frac{v}{p^s}\right)\right)\right].$$

Remark 8.2 Let $f = 1$ and let $\delta \in Div_f(N) = Div_f(N)$ be a fixed good divisor for the triple (p, f, N). One may verify that the right hand side of (8.2) coincides with the measure which appears in Proposition 3.2 of [Das07]. In particular, since in this special case the p-adic measures which are used to define u_{DD} and u_C agree one gets that $u_{DD}(\alpha_\delta, \tau) = u_{C,\delta}(1, \tau)$. We would like to point out here that the factor 12 which appears in the right hand side of 8.1 is not optimal. It will follow from Proposition 8.4 that under the additional assumption that $l \geq 5$ one has that $\frac{1}{6}\widetilde{\mu}_\delta\left\{\infty \to \frac{a}{c}\right\}(U_{u,v,s})$ is an integer. This fact was used by Dasgupta in [Das08] in the following case: let $N = l$ be a prime and set $\delta = [l] - [1]$. Looking at the right hand side of (8.2) one may check that $\frac{1}{6}\widetilde{\mu}_\delta\left\{\infty \to \frac{a}{c}\right\}(U_{u,v,s})$ coincides with the measure which appears on line (82) of [Das08].

8.1 Explicit formulas of $\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)$ on balls of $\mathbb{X}$

In this subsection we give explicit formulas of the measure $\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)$ (see Definition 7.9) when evaluated on balls of $\mathbb{X}$. Let (p, f, N) be a triple as in the introduction, $\mathfrak{f} = f\mathcal{O}$ and let $\epsilon > 1$ be a generator of $\mathcal{O}(\mathfrak{f}\infty)^\times$.

Proposition 8.2 *Assume that $\mathfrak{d}'$ is consecutive to $\mathfrak{d}$ and let $\eta = \frac{\mathfrak{d}'}{\mathfrak{d}}$, $l = \mathbf{N}(\eta)$, $d = \mathbf{N}(\mathfrak{d})$ and $d' = ld$. Set*

$$\widetilde{\delta} = d[\mathfrak{d}] - d'[\mathfrak{d}'].$$

Note that $\deg(\widetilde{\delta}) = 0$. Let $U_{u,v,n}$ be a ball contained in $\mathbb{X}$ and assume that $u \equiv v \equiv 0$ (mod f). Let $x = x_1 + x_2\tau$ with $x_i \in \mathbb{Z}$ and let $C = C(1, \epsilon)$ where $\epsilon > 1$ is a generator of $\mathcal{O}(\mathfrak{f}\infty)^\times$. Let be $\tau \in K\backslash\mathbb{Q}$ and suppose $\frac{\tau}{d'} \in H^{\mathcal{O}}(\mathfrak{N})$ and set $\mathfrak{b} = I_\tau$. Then we have the following formula:

(8.3) $$\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)(U_{u,v,n}) = \sum_{h \pmod{d'c}} B_1^*(y_1(h))[dB_1^*(ly_2(h)) - d'B_1^*(y_2(h))],$$

where

$$y_1(h) = \frac{a}{d'c}\left(h - \frac{v}{fp^n} + \frac{x_2}{f}\right) + \frac{u}{fd'p^n} + \frac{x_1}{fd'}$$

$$y_2(h) = \frac{-1}{d'c}\left(h - \frac{v}{fp^n} + \frac{x_2}{f}\right).$$

Here $B_1^*(y) = y' - \frac{1}{2}$ where y' is the unique real number $0 < y' \leq 1$ such that $y - y' \in \mathbb{Z}$.

Proof We will prove Proposition 8.2 under the simplifying assumption that $p \equiv 1 \pmod{f}$ since it simplifies the presentation. It is easy to adapt the proof to the general case since it only amounts to taking a "trace" over the subgroup $\langle p \pmod{f}\rangle$ of $(\mathbb{Z}/f\mathbb{Z})^\times$. Note that

$$\phi_\tau(U_{u,v,n}) = u - v\tau + p^N \mathcal{O}_{K_p}.$$

Using the assumptions that $p \equiv 1 \pmod{f}$, $(\mathfrak{b}, p) = 1$ and that $u \equiv v \equiv 0 \pmod{f}$ we readily see that

$$(\mathfrak{b}^{-1}\mathfrak{d}^\sigma f + x) \cap \phi_\tau(U_{u,v,n}) = \mathfrak{b}^{-1}\mathfrak{d}^\sigma fp^n + u - v\tau + p^n(x_1 + x_2\tau).$$

We have $\mathfrak{b}^{-1} = \Lambda_\tau$. Since $\frac{\tau}{d'} \in H^{\mathcal{O}}(\mathfrak{N})$ we may deduce from (3.3) that

(8.4) $$\mathfrak{b}^{-1}\mathfrak{d}^\sigma = d\mathbb{Z} + \tau\mathbb{Z} \quad \text{and} \quad \mathfrak{b}^{-1}(\mathfrak{d}')^\sigma = d'\mathbb{Z} + \tau\mathbb{Z}.$$

Thus

$$\mathfrak{b}^{-1}\mathfrak{d}^\sigma p^n f = p^n fd\mathbb{Z} + p^n fN\tau\mathbb{Z}.$$

Let $\eta_\tau = \begin{pmatrix} a & b \\ c & \widetilde{d} \end{pmatrix} \in \Gamma_1(f) \cap \Gamma_0(N)$. Make the crucial observation that $(a, fN) = 1$. Note that $1 < \epsilon = c\tau + \widetilde{d}$ and thus $\epsilon^{-1} = -c\tau + a$ where $0 < \epsilon^{-1} < 1$.

Let $m = \text{cond}(\mathcal{O}) = 1$ where $(m, N) = 1$. Set

$$w_1 = d'fp^n \quad \text{and} \quad w_2 = d'fp^n\epsilon^{-1}.$$

For $i \in \{1, 2\}$ we have

$$\frac{w_i}{l} \in \mathfrak{b}^{-1}\mathfrak{d}^\sigma p^n f \quad \text{but} \quad \frac{w_i}{l} \notin \mathfrak{b}^{-1}(\mathfrak{d}')^\sigma p^n f,$$

as required. Unfolding the definition of $\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)\,(U_{u,v,n})$ we find that

$$\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)\,(U_{u,v,n})$$

(8.5)
$$= dZ(\mathfrak{b}^{-1}\mathfrak{d}fp^n, u - v\tau + p^n x, w, 0) - d'Z(\mathfrak{b}^{-1}(\mathfrak{d}')^\sigma fp^n, u - v\tau + p^n x, w, 0),$$

where $Z(\mathfrak{a}, y, w, s)$ is the function which appears in (7.7). Applying the formula which appears in (77) of [Das08] to the right hand side of (8.5) we obtain

$$\nu_{\widetilde{\delta}}(\mathfrak{b}, \tau, x, C)\,(U_{u,v,n})$$

(8.6)
$$= d \sum_{y \in \Omega(\mathfrak{b}^{-1}\mathfrak{d}^\sigma pf^n, u-v\tau+p^n x, w, 0)} B_1(y_1)B_1(y_2) - d' \sum_{y' \in \Omega(\mathfrak{b}^{-1}(\mathfrak{d}')^\sigma pf^n, u-v\tau+p^n x, w, 0)} B_1(y_1')B_1(y_2').$$

Using (8.4) we find that $\mathfrak{b}^{-1}\mathfrak{d}^\sigma fp^n$ is the set of elements of the form

$$hfp^n\tau + jfp^n d \quad \text{for} \quad h, j \in \mathbb{Z}.$$

Now consider the system of linear equations (in the y_i's)

$$hp^n fN\tau + jfp^n d + u - v\tau + p^n(x_1 + x_2\tau) = y_1(d'fp^n) + y_2 d'fp^n \epsilon^{-1},$$

Solving the system we find

(8.7)
$$y_2 = \frac{-1}{d'c}\left(h - \frac{v}{fp^n} + \frac{x_2}{f}\right)$$

$$y_1 = \frac{j}{l} + \frac{u}{fd'p^n} + \frac{x_1}{fd'} + \frac{a}{d'c}\left(h - \frac{v}{fp^n} + \frac{x_2}{f}\right).$$

For each residue class modulo $d'c$, there exists a unique integer h in that class such that $0 < y_2 \leq 1$. For this fixed h, and each possible residue class modulo l, there exists a unique integer j in that class such that $0 < y_1 \leq 1$. Thus the first summation in (8.6) equals

(8.8)
$$\sum_{j \pmod l} \sum_{h \pmod {d'c}} B_1^*\left(\frac{a}{d'c}\left(h - \frac{v}{fp^n} + \frac{x_2}{f}\right) + \frac{j}{l} + \frac{u}{fd'p^n} + \frac{x_1}{fd'}\right) B_1^*\left(\frac{1}{d'c}\left(h - \frac{v}{fp^n} - \frac{x_2}{f}\right)\right).$$

Now every element $0 \leq h \leq \frac{d'c}{N} - 1$ can be written uniquely as $h = \frac{dc}{N}r + s$ with $0 \leq r \leq l - 1$ and $0 \leq s \leq \frac{dc}{N} - 1$. Thus (8.8) may be written as

$$\sum_{r,s} B_1^*\left(\frac{r}{l} + \frac{s}{d'c} + \frac{1}{d'c}\left(-\frac{v}{fp^n} + \frac{x_2}{f}\right)\right) \sum_{j \pmod l} B_1^*\left(\frac{ra}{l} + \frac{as}{d'c} - \frac{a}{d'c}\left(-\frac{v}{fp^n} + \frac{x_2}{f}\right) + \frac{j}{l} + \frac{u}{fd'p^n} + \frac{x_1}{fd'}\right).$$

Using the distribution relation $\sum_{j=0}^{m-1} B_1^*(x + \frac{j}{m}) = B_1^*(mx)$ for $m = l$ and the fact that $B^*(x+1) = B^*(x)$ we may rewrite the expression above as

$$(8.9) \qquad \sum_{h \pmod{d'c}} \widetilde{B}_1^* \left(\frac{-1}{dc} \left(h - \frac{v}{fp^n} + \frac{x_2}{f} \right) \right) \widetilde{B}_1^* \left(\frac{a}{d'c} \left(h - \frac{v}{fp^n} + \frac{x_2}{f} \right) + \frac{u}{fd'p^n} + \frac{x_1}{fd'} \right).$$

So this computes the first summation of (8.6). A similar calculation to what we did shows that the second summation in (8.6) is equal to

$$(8.10) \qquad \sum_{h \pmod{d'c}} B_1^* \left(\frac{-1}{d'c} \left(h - \frac{v}{fp^n} + \frac{x_2}{f} \right) \right) B_1^* \left(\frac{a}{d'c} \left(h - \frac{v}{fp^n} + \frac{x_2}{f} \right) + \frac{u}{fd'p^n} + \frac{x_1}{fd'} \right).$$

This shows (8.3) and therefore it concludes the proof. $\square$

Proposition 8.3 *We use the same notation as in the previous proposition. Let*

$$\widetilde{\delta} = d[\mathfrak{d}, j] - d'[\mathfrak{d}', j] \quad \text{and} \quad \delta = \Phi(\widetilde{\delta}) = d \left[\frac{N}{d}, j \right] - d' \left[\frac{N}{d'}, j \right].$$

Let $\tau \in H^{\mathcal{O}}(\mathfrak{N})$ and set $\mathfrak{b} = I_{fN\tau}$. Note that $\frac{Nf\tau}{d'} \in H^{\mathcal{O}}(\mathfrak{N})$. Let $\gamma = \begin{pmatrix} -f^2N^2 & 0 \\ 0 & -fN \end{pmatrix}$ and define

$$(8.11) \qquad \xi(U_{u,v,n}) := 12\nu_{\widetilde{\delta}}^{\gamma}(\mathfrak{b}, fN\tau, fN\tau, C)(U_{u,v,n}) - \widetilde{\mu}_{\delta} \left\{ \infty \to \frac{a}{c} \right\} (U_{u,v,n}).$$

Then for every ball $U_{u,v,n} \subseteq \mathbb{X}$ one has $\frac{1}{3}\xi(U_{u,v,n}) \in \mathbb{Z}$ and

$$(8.12) \qquad \xi(U_{u,v,n}) + \xi(U_{-u,-v,n}) = 0.$$

Proof Using (8.3) we find that

$$12\nu_{\widetilde{\delta}}^{\gamma}(\mathfrak{b}, fN\tau, fN\tau, C)(U_{u,v,n})$$

$$= -12d' \sum_{1 \le h \le \frac{cd'}{fN}} B_1^* \left(\frac{a}{\frac{cd'}{fN}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) - \frac{\frac{N}{d'}fu}{p^s} \right) B_1^* \left(\frac{1}{\frac{cd'}{fN}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) \right)$$

$$- 12d \sum_{1 \le h \le \frac{cd}{fN}} B_1^* \left(\frac{a}{\frac{cd}{fN}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) - \frac{\frac{N}{d}fu}{p^s} \right) B_1^* \left(\frac{1}{\frac{cd}{fN}} \left(h + \frac{v}{p^s} + \frac{j}{f} \right) \right).$$

Using (8.1) one may check that the right hand side of the equality above gives the same expression as $\widetilde{\mu}_{\delta} \left\{ \infty \to \frac{a}{c} \right\} (U_{u,v,n})$, except that one evaluates $\widetilde{B}_1$ rather than B_1^*. Once this observation is made, the proof of (8.12) is elementary. The complete proof may be found in Appendix C. $\square$

We can now give a new proof of Proposition 8.1

Proposition 8.4 *Let*

$$\widetilde{\delta} = d[\mathfrak{d}, j] - d'[\mathfrak{d}', j] \quad and \quad \delta = \Phi(\widetilde{\delta}) = d\left[\frac{N}{d}, j\right] - d'\left[\frac{N}{d'}, j\right].$$

Then $m_l \cdot \widetilde{\mu}_\delta \left\{ \infty \to \frac{a}{c} \right\}$ is a $\mathbb{Z}$-valued measure where

$$m_l = \begin{cases} 4 & if \quad l = 2 \\ 3 & if \quad l = 3 \\ 1 & if \quad l \geq 5. \end{cases}$$

Proof From Proposition 8.3 we see that it is enough to show that for every ball $U_{u,v,n} \subseteq \mathbb{X}$ one has

$$m_l \cdot \nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C)(U_{u,v,n}) \in \mathbb{Z}.$$

We may write $(\mathfrak{b}^{-1}\mathfrak{f} + z)$ in the following way:

$$(8.13) \qquad (\mathfrak{b}^{-1}\mathfrak{f} + z) \cap U_{u,v,n} = \bigcup_{i=1}^{d}(\mathfrak{a}^{-1} + y_i),$$

where $\mathfrak{a}^{-1} = \mathfrak{b}^{-1}\mathfrak{f}\mathfrak{p}^n$, $\mathfrak{p} = p\mathcal{O}_K$ and $y_i \in K$. Since $(\mathfrak{f}\mathfrak{p}, \eta) = \mathcal{O}$ and $v_\eta(\mathfrak{b}) \leq 0$ we again have $v_\eta(\mathfrak{a}) \leq 0$. From (8.13) we get

$$(8.14) \qquad \zeta(\mathfrak{b}, \mathfrak{f}, z, C \cap U_{u,v,n}) = \sum_{i=1}^{d} \zeta(\mathfrak{a}, \mathfrak{f}, y_i, C, 0).$$

From this one gets that

$$\zeta_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}, z, C \cap U_{u,v,n}, 0) = \sum_{i=1}^{d} \zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, y_i, C, 0).$$

From Proposition 7.5 we find that $\zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, y_i, C, 0) \in \mathbb{Z}[\frac{1}{l}]$ with a denominator at most $l^{m/(l-1)}$. In particular, if $l > 3$ we find that $\zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, y_i, C, 0) \in \mathbb{Z}$. If $l = 3$ we have $\zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, y_i, C, 0) \in \frac{1}{3}\mathbb{Z}$ and if $l = 2$ we have $\zeta_{\widetilde{\delta}}(\mathfrak{a}, \mathfrak{f}, y_i, C, 0) \in \frac{1}{4}\mathbb{Z}$. This concludes the proof. $\square$

9 The p-adic invariants u_C and u_D

In this section we give the precise definitions of the p-adic invariants u_C and u_D. We also recall two conjectures which predict that u_C and u_D are strong p-units in appropriate abelian extensions of K. Finally, we state conjectural Shimura reciprocity laws which give an analytic description of $G_{K^{ab}/K}$ on u_C and u_D. For the rest of this section we fix a triple (p, f, N) and an order $\mathcal{O}$ of conductor coprime to N. We let $\mathfrak{N}\mathfrak{N}^\sigma = N\mathcal{O}_K$ be a splitting of N and we let $\mathfrak{f} = f\mathcal{O}$. Finally, we fix an element $\tau \in H^{\mathcal{O}}(\mathfrak{N})$.

9.1 Definition of u_C

Let $\mathbb{X} = (\mathbb{Z}_p \times \mathbb{Z}_p)\backslash(p\mathbb{Z}_p \times p\mathbb{Z}_p)$ be the set of primitive vectors of $\mathbb{Z}_p^2$. Assume furthermore that τ is reduced. Let $\delta \in \mathrm{Div}_f(N)$ be a good divisor and let $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$.

Definition 9.1 *Then the formula for the p-adic invariant $u_{C,\delta}(r, \tau)$ is given by*

$$(9.1) \qquad u_{C,\delta}(r, \tau) := p^{3\zeta^*(\delta,(r,\tau),0)} \fint_{\mathbb{X}} (x - \tau y) d\widetilde{\mu}_{\delta_r}\{\infty \to \eta_\tau\infty\}(x, y),$$

where the measure $\widetilde{\mu}_{\delta_r}\{\infty \to \eta_\tau\infty\}$ is the one which appears in Definition 8.1, the matrix η_τ is the one which appears in Definition 4.1 and $\zeta^*(\delta, (r, \tau), s)$ is the zeta function which appears in Definition 6.1.

The multiplicative integral in (9.1) indicates that one considers a limit of "Riemann products" rather than usual Riemann sums where the right hand side of (9.1) is taken to mean

$$(9.2) \qquad \fint_{\mathbb{X}} (x - \tau y) d\widetilde{\mu}_{\delta_r}\{\infty \to \gamma_\tau\infty\}(x, y) := \lim_{||\mathcal{U}|| \to 0} \prod_{U \in \mathcal{U}} (x_U - \tau y_U)^{\widetilde{\mu}_{\delta_r}\{\infty \to \gamma_\tau\infty\}(U)} \in K_p^\times,$$

where $\mathcal{U}$ is a cover of $\mathbb{X}$ by disjoint compact open sets, (x_U, y_U) is an arbitrary point of $U \in \mathcal{U}$, and the p-adic limit is taken over increasingly fine covers $\mathcal{U}$. The product in (9.2) makes sense since the measures $\widetilde{\mu}_{\delta_r}\{\infty \to \gamma\infty\}$ is $\mathbb{Z}$-valued and not only $\mathbb{Z}_p$-valued, thanks to Proposition 8.4. The appellation of p-adic invariant for $u_C(r, \tau)$ is appropriate in the light of the following proposition:

Proposition 9.1 *Assume that $(r, \tau) \sim_f (r', \tau')$ then*

$$(9.3) \qquad u_{C,\delta}(r, \tau) = u_{C,\delta}(r', \tau') \pmod{\mu_{p^2-1}}.$$

Proof See Theorem 1.2 of [Cha09c]. $\square$

Remark 9.1 As a consequence of relating the p-adic invariant u_C to the p-adic invariant u_D we will show that (9.3) continues to hold without the modulo μ_{p^2-1}.

Using the fact that τ is reduced and taking ord_p on both sides of (9.1) one finds that

$$(9.4) \qquad \mathrm{ord}_p(u_{C,\delta}(r, \tau)) = 3\zeta_p^*(\delta, (r, \tau), 0).$$

We let $*$ be the involution on $H(\mathfrak{N})$ given by $\tau \mapsto \tau^* = \frac{-1}{fN\tau}$. We note that $*$ preserves the orientation of τ in the sense that if $\tau > \tau^\sigma$ then $\tau^* > (\tau^*)^\sigma$. However, the involution $*$ does not preserve the stratification (3.6) since in general the order $\mathcal{O}_\tau$ will be different from the order $\mathcal{O}_{\tau^*}$.

Now we would like to define an adélic action of C_K on u_C be before doing so we need to make a few remarks. Let $\mathcal{O}^*$ be the unique order of K such that $f \cdot \mathrm{cond}\,\mathcal{O} = \mathrm{cond}\,\mathcal{O}^*$. Proposition 2.1 gives us the maps

$$(9.5) \qquad \Theta : I_{\mathcal{O}}(f^2) = I_{\mathcal{O}}(f) \to I_{\mathcal{O}^*}(f),$$

and

$$\widetilde{\Theta} : C_{\mathcal{O}}(f^2) \to C_{\mathcal{O}^*}(f)$$

where Θ is multiplicative and $\widetilde{\Theta}$ is an onto group homomorphism. From Class field theory, the map $\widetilde{\Theta}$ corresponds to the restriction map $res : Gal(K(\mathfrak{f}^2\infty)/K) \to Gal(K(\mathfrak{f}^*\infty)/K)$ where $\mathfrak{f} = f\mathcal{O}$ and $\mathfrak{f}^* = f\mathcal{O}^*$. We may now define the adélic action of C_K on u_C.

Definition 9.2 Let $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N}, f)$ and set $\mathfrak{b} = \widetilde{r} I_\tau$. Note that $[\mathfrak{b}, \mathfrak{b}\mathfrak{N}] \in \mathcal{L}^{\mathcal{O}}(\mathfrak{N})$. Let us denote the class of $[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]$ in $\mathcal{L}^{\mathcal{O}}(\mathfrak{N})/ \sim_f$ by $[[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]]$. Let $c \in C_K$ and let $c \star [[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]] = [[\mathfrak{a}, \mathfrak{a}\mathfrak{N}]]$ where $\star$ is the action which appears in Definition 3.5. For $c \in C_K$ we define

$$c \star u_C(r, \tau) := u_C(r', \tau'),$$

where $(r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$ is chosen so that

(9.6) $$\Theta(\mathfrak{a})\widetilde{r} I_{\tau^*} \sim_f \widetilde{r'} I_{\tau'^*},$$

where $\tau^* = \frac{-1}{fN\tau}$.

A few remarks here are in order. The map $\Theta : I_{\mathcal{O}}(f) \to I_{\mathcal{O}^*}(f)$ is the map which appears in (9.5). Also note that because $\tau \in H^{\mathcal{O}}(\mathfrak{N}, f)$ $((A_\tau, f) = 1)$ one has that $\mathcal{O}_{\tau^*} = \mathcal{O}^*$ so that $I_{\tau^*}, I_{\tau'^*} \in I_{\mathcal{O}^*}(1)$. We note that in general the two ideals $I_{\tau^*}, I_{\tau'^*}$ belong to the stratum

$$I_{\mathcal{O}^*}[f\mathcal{O}, \mathfrak{f}^*] := \{\mathfrak{b} \subseteq \mathcal{O}^* : \mathfrak{b} \text{ is an invertible integral } \mathcal{O}^*\text{-ideal such that } (\mathfrak{f}^*, \mathfrak{b}) = f\mathcal{O}\},$$

Finally, the existence of a pair $(r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N}, f)$ such that (9.6) holds is guaranteed by Corollary 9.1.

Conjecture 9.1 Let $[(r, \tau)] \in (\mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N}, f))/ \sim_f$ and assume that τ is reduced. Let $\mathcal{O} = \mathcal{O}_\tau$, $\mathfrak{f} = f\mathcal{O}$, $\wp = p\mathcal{O}_K$ and $L = K(\mathfrak{f}\infty)^{\langle \sigma_\wp \rangle}$. Then $u_C(r, \tau)$ is a strong p-unit in L. Let $rec : C_K \to G_{K^{ab}/K}$ be the reciprocity map given by class field theory. Let $c \in C_K$ be an idèle of K and let $rec(c) = \sigma \in \text{Gal}(K^{ab}/K)$. Then

$$u_{C,\delta}(r, \tau)^{\sigma^{-1}} = c \star u_{C,\delta}(r', \tau'),$$

where $\star$ is the action which appears in Definition 9.2.

Remark 9.2 We would like to warn the reader that the statement of the Shimura reciprocity law for u_C as it appears in Conjecture 4.1 of [Cha09c] is wrong since it did not take into account the twist by the involution $*$. The relation (9.6) plays a key role in the proof of the compatibility of the Shimura reciprocity law between the two p-adic invariants u_C and u_D. In fact, this adelic action on u_C is forced upon us. The author discovered this mistake in the process of working out the fine details of this compatibility. We would also like to point out that even if the Shimura reciprocity formula which appears in Conjecture 4.1 of [Cha09a] was wrong, the calculations performed in [Cha09a] (which was based on it) remains valid, thanks to Proposition 9.5 which implies that: if $\{[(r_i, \tau_i)]\}_{i \in I}$ is a full orbit of $[(r, \tau)]$ under the action of $C_{\mathcal{O}}(f)$ then $\{[(r_i, \tau_i^*)]\}_{i \in I}$ is a full orbit of $[(r, \tau^*)]$ under the action of $C_{\mathcal{O}^*}(f)$. From that we deduce that $\{u_C(r_i, \tau_i)\}_{i \in I}$ is the full orbit of $u_C(r, \tau)$ under the action of $G_{K^{ab}/K}$.

9.2 Definition of u_D

We now want to describe Dasgupta's invariant u_D. Let S be the set of places that contains the infinite places of K, $\wp = p\mathcal{O}_K$ and the ones that ramify in $K(\mathfrak{f}\infty)$. We let $R = S\backslash\{\wp\}$ and $T = \{\eta|\mathfrak{N}\}$. Let $\widetilde{\delta} \in \widetilde{\mathrm{Div}}(\mathfrak{N})$ be a good divisor for the triple (p, f, N). Let us fix an order $\mathcal{O}$ of K such that $(\mathrm{cond}(\mathcal{O}), N) = 1$. Let $\mathfrak{f} \subseteq \mathcal{O}$ be an arbitrary $\mathcal{O}$-ideal (not necessarily invertible) and let $\mathfrak{b} \subseteq K$ be an arbitrary invertible $\mathcal{O}$-ideals which is $\mathfrak{f}$-int. We also let $\mathcal{D}$ be a Shintani domain for the action of $\Gamma_\mathfrak{b}(\mathfrak{f})$ (see Definition 4.4) on the positive quadrant $Q = \mathbb{R}^2_{>0}$ and make the crucial assumption that T is $\mathcal{O}$-good for $\mathcal{D}$.

Definition 9.3 *Following [Das08], we define*

$$(9.7) \qquad u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathcal{D}) := p^{\zeta_{\widetilde{\delta}}(\mathfrak{b},\mathfrak{f}\infty,0)} \fint_{\mathcal{O}_{K_p}^\times} z \cdot d\nu(\mathfrak{b}, \mathcal{D})(z),$$

where $\nu(\mathfrak{b}, \mathcal{D})(U) = \zeta_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}, n_\mathfrak{b}, \mathcal{D} \cap U, 0)$ and $\zeta_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}\infty, s)$ is the zeta function which appears in Definition 6.2. For the definition of the multiplicative integral which appears in (9.7) see Section 3 of [Das08].

Remark 9.3 We note that (9.7) was defined in [Das08] when $\mathcal{O} = \mathcal{O}_K$, $\mathfrak{b}$ is an $\mathcal{O}_K$-ideal coprime to $\mathfrak{f} = f\mathcal{O}_K$ and

$$\widetilde{\delta} = \sum_{\mathfrak{d}|\mathfrak{N}, r\in\mathbb{Z}/f\mathbb{Z}} (-1)^{\widetilde{\tau}(\mathfrak{d})} \mathbf{N}(\mathfrak{d})[\mathfrak{d}, r] \in \widetilde{\mathrm{Div}}_f(\mathfrak{N}),$$

where $\mathfrak{N}$ square-free and $\widetilde{\tau}(\mathfrak{d})$ is the number of divisors of $\mathfrak{d}$. We note that in this special case one has $\Gamma_\mathfrak{b}(\mathfrak{f}) = \mathcal{O}_K(\mathfrak{f}\infty)^\times$.

The following key proposition was proved in [Das08].

Proposition 9.2 *Let $\mathcal{D}$ and $\mathcal{D}'$ be two Shintani domains for the action of $\mathcal{O}(\mathfrak{f}\infty)^\times$ on Q. Assume that there exists a prime $\eta|\mathfrak{N}$ which is $\mathcal{O}$-good for $\mathcal{D}$ and $\mathcal{D}'$. Suppose also that $\sigma_\mathfrak{b} = \sigma_{\mathfrak{b}'}$. Then*

$$u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathcal{D}) = u_{D,\widetilde{\delta}}(\mathfrak{b}', \mathcal{D}').$$

Proof All the proofs in Section 5 of [Das08] carry over mutatis mutandis to our more general setting. □

Definition 9.4 *Let $\mathcal{D}$ be a fundamental domain for the action of $\mathcal{O}(\mathfrak{f}\infty)^\times$ on Q. Assume that a prime $\eta|\mathfrak{N}$ which is $\mathcal{O}$-good for $\mathcal{D}$. We set*

$$u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}) := u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathcal{D}).$$

We say that $u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f})$ is primitive of conductor $[\mathcal{O}, \mathfrak{g}]$ if the pair $[\mathfrak{b}, \mathfrak{f}]$ is primitive of conductor $[\mathcal{O}, \mathfrak{g}]$ (resp. non-primitive) if the pair $[\mathfrak{b}, \mathfrak{f}]$ is primitive of conductor $[\mathcal{O}, \mathfrak{g}]$ (resp. non-primitive).

Now let us define an adélic action of C_K on u_D.

Definition 9.5 Let $\mathfrak{b}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int and let $c \in C_K$. Note that $[\mathfrak{b}, \mathfrak{b}\mathfrak{N}] \in \mathcal{L}^{\mathcal{O}}(\mathfrak{N})$. Let us denote the class of $[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]$ in $\mathcal{L}^{\mathcal{O}}(\mathfrak{N})/\sim_{\mathfrak{f}}$ by $[[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]]$ and let $c \star [[\mathfrak{b}, \mathfrak{b}\mathfrak{N}]] = [[\mathfrak{a}, \mathfrak{a}\mathfrak{N}]]$ where $\star$ is the action which appears in Definition 3.5. We define

$$c \star u_D(\mathfrak{b}, \mathfrak{f}) := u_D(\mathfrak{a}, \mathfrak{f}).$$

We may now state the main algebraicity conjecture on the p-adic invariant u_D and the its corresponding Shimura reciprocity formula.

Conjecture 9.2 Let $\mathfrak{b}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int. Let $\mathfrak{n}$ be an invertible $\mathcal{O}$-ideal such that

$$\Gamma_{\mathfrak{b}}(\mathfrak{f}) \supseteq \mathcal{O}(\mathfrak{n}\infty)^{\times}.$$

Then $u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f})$ is a strong p-unit in $K(\mathfrak{n}\infty)^{\langle \sigma_\wp \rangle}$. Let $rec : C_K \to G_{K^{ab}/K}$ be the reciprocity map given by class field theory. Let $c \in C_K$ be an idèle of K and let $rec(c) = \sigma \in \mathrm{Gal}(K^{ab}/K)$. Then

$$u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f})^{\sigma^{-1}} = c \star u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}).$$

Remark 9.4 If $\mathfrak{b}$ is an integral invertible $\mathcal{O}$-ideal then one has that $\Gamma_{\mathfrak{b}}(\mathfrak{f}) \supseteq \mathcal{O}(\mathfrak{f}\infty)^{\times}$ and therefore, Conjecture 9.2 predicts that $u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}) \in K(\mathfrak{f}\infty)^{\langle \sigma_\wp \rangle}$. Now let us assume that the pair $[\mathfrak{b}, \mathfrak{f}]$ is primitive of conductor $[\mathcal{O}, \mathfrak{g}]$. In this case, there exists a $\lambda \in K^{\times}$ such that $(\lambda \mathfrak{b}, \lambda \mathfrak{f}) = \mathcal{O}$ and $\lambda \mathfrak{f} = \mathfrak{g}$ so that

$$\Gamma_{\mathfrak{b}}(\mathfrak{f}) = \Gamma_{\lambda \mathfrak{b}}(\mathfrak{g}) \supseteq \mathcal{O}(\mathfrak{g}\infty)^{\times}.$$

Therefore, Conjecture 9.2 predicts that

$$u_{D,\widetilde{\delta}}(\mathfrak{b}, \mathfrak{f}) \in K(\mathfrak{g}\infty)^{\langle \sigma_\wp \rangle}.$$

Remark 9.5 In general, the author thinks that the following strengthening on the field of definition of $u_D(\mathfrak{b}, \mathfrak{f})$ should hold true. So let $\mathfrak{b}$ be an invertible $\mathcal{O}$-ideal which is $\mathfrak{f}$-int and suppose that

$$\Gamma_{\mathfrak{b}}(\mathfrak{f}) \supseteq \mathcal{O}'(\mathfrak{n}'\infty)^{\times},$$

where $\mathcal{O}'$ is some order and $\mathfrak{n}'$ is an invertible $\mathcal{O}'$-ideal. Then the author expects that $u_D(\mathfrak{b}, \mathfrak{f}) \in K(\mathfrak{n}'\infty)$.

9.3 Relation between u_C and u_D

The next proposition gives a relationship between the invariant u_C to the invariant u_D.

Proposition 9.3 Let $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$ and let $\delta = \Phi(\widetilde{\delta})$ be a good divisor for the triple (p, f, N). Then we have

$$(9.8) \qquad u_{C,\delta}(r, \tau)^{\nu(r,\tau)} = u_{D,\widetilde{\delta}}(\widetilde{r}I_{\tau^*}, \mathfrak{f}^*)^{12},$$

where $\mathfrak{f}^* = f\mathcal{O}_{\tau^*}$ and

$$(9.9) \qquad \nu(r, \tau) := \mu(r, \tau^*)^{-1},$$

where $\mu(r, \tau^*)$ is the rational number which appears in (4.10). Moreover, the Shimura reciprocity law for u_C agrees with the one for u_D.

Proof From Lemma 5.1 it is enough to prove (9.8) for

$$\widetilde{\delta} = (d[\mathfrak{d}] - d'[\mathfrak{d}']) \quad \text{where} \quad d = \mathbf{N}(\mathfrak{d}) \quad \text{and} \quad d' = \mathbf{N}(\mathfrak{d}'),$$

where $\mathfrak{d}'$ consecutive to $\mathfrak{d}$. We have

$$\zeta^*(\delta, (r, \tau), 0) = \zeta(\delta^*, (r, \tau^*), 0) = 4\mu(r, \tau^*)\zeta_{\widetilde{\delta}}(\widetilde{r}A_{\tau^*}\Lambda_{\tau^*}, \mathfrak{f}^*\infty, 0),$$

where the first equality follows from Lemma 6.1 and the second equality follows from Lemma 6.2. Using (9.4) with the equality above we find that

$$\nu(r, \tau) \cdot v_p(u_{C,\delta}(r, \tau)) = 12 \cdot v_p(u_{D,\widetilde{\delta}}(\widetilde{r}A_{\tau^*}\Lambda_{\tau^*}, \mathcal{D})).$$

This proves that the p-adic valuation on both sides of (9.8) are equal. Therefore it remains to show that the multiplicative integrals on both sides of (9.8) agree.

Let $\mathfrak{b} = \widetilde{r}I_\tau$ and $\gamma = \begin{pmatrix} -f^2N & 0 \\ 0 & -fN \end{pmatrix}$. For $C' = C(1)$ or $C' = C(\epsilon)$, we have

$$(9.10) \qquad \nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C')(-U) = -\nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C')(U).$$

From (9.10), we get that

$$\fint_{\mathbb{X}} (x - fN\tau y)d\nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C(1)) = \fint_{\mathbb{X}} (x - \tau y)d\nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon)) = 1.$$

Thus one has

$$(9.11) \quad \fint_{\mathbb{X}} (x - fN\tau y)d\nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon, 1)) = \fint_{\mathbb{X}} (x - fN\tau y)d\nu_{\widetilde{\delta}}^\gamma(\mathfrak{b}, fN\tau, fN\tau, \mathcal{D}).$$

Let ξ (which depends of $\widetilde{\delta}$) be the measure which appears in Proposition 8.3. From (8.12) we deduce that

$$(9.12) \qquad \fint_{\mathbb{X}} \xi(x, y) = 1.$$

Moreover, because $\nu_{\widetilde{\delta}}^{\gamma}(\mathfrak{b}, fN\tau, fN\tau, C)$ has total measure 0 we have

(9.13)
$$\oint_{\mathbb{X}}(x - fN\tau y)d\nu_{\widetilde{\delta}}(\mathfrak{b}, fN\tau, fN\tau, C(1,\epsilon))(fx, y) = \oint_{\mathbb{X}}(x - fN\tau y)d\nu_{\widetilde{\delta}}^{\gamma}(\mathfrak{b}, fN\tau, fN\tau, C(1,\epsilon))(x, y).$$

Combining (9.12) and (9.13) and (8.11) we find that

$$\left(\oint_{\mathbb{X}}(x - fN\tau y)d\nu_{\widetilde{\delta}}(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon, 1))(x, y)\right)^{12} = \left(\oint_{\mathbb{X}}(x - \tau y)d\widetilde{\mu}_r\{\infty \to \eta_\tau\infty\}(x, y)\right)^{\nu(r,\tau)}.$$

Thus in order to show the multiplicative parts on both sides of (9.8) agree, it is enough to show that

(9.14)
$$\oint_{\mathbb{X}}(x - fN\tau y)d\nu_{\widetilde{\delta}}(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon, 1))(x, y) = \oint_{\mathbb{X}}(x - \tau^* y)d\nu_{\widetilde{\delta}}(\mathfrak{b}^*, \tau^*, 1, C(\epsilon, 1))(x, y),$$

where $\mathfrak{b}^* = I_{\tau^*}$ and $\tau^* = \frac{-1}{fN\tau}$.

Applying Lemma 7.1 with the matrix $\gamma = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ we find that

$$\nu_{\widetilde{\delta}}^{\gamma}\left(\mathfrak{b}^*, \tau^*, 1, \frac{1}{fN_0\tau}C(\epsilon, 1)\right) = \nu_{\widetilde{\delta}}(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon, 1)).$$

It thus follows that

$$\oint_{\mathbb{X}}(x - fN\tau y)d\nu_{\widetilde{\delta}}(\mathfrak{b}, fN\tau, fN\tau, C(\epsilon, 1))(x, y) = \oint_{\mathbb{X}}(x - \tau^* y)d\nu_{\widetilde{\delta}}\left(\mathfrak{b}^*, \tau^*, 1, \frac{1}{fN_0}C(\epsilon, 1)\right)(y, fNx)$$
$$= \oint_{\mathbb{X}}(x - \tau^* y)d\nu_{\widetilde{\delta}}(\mathfrak{b}^*, \tau^*, 1, C(\epsilon, 1))(x, y),$$

where the last equality follows from the independence on the choice of the fundamental domain and the fact that $\nu_{\widetilde{\delta}}\left(\mathfrak{b}^*, \tau^*, 1, \frac{1}{fN_0}C(\epsilon, 1)\right)(y, fNx)$ has total measure zero. This shows (9.14).

Finally, the fact that the two Shimura reciprocity laws agree follows directly from their definitions. This concludes the proof. $\square$

9.4 From $K(\mathfrak{f}^*\infty)$ to $K(\mathfrak{f}\infty)$

In this subsection we would like to show the equivalence between Conjecture 9.1 and Conjecture 9.2 under special assumptions on f and $n = cond(\mathcal{O})$. Let $u_C(r, \tau)$ be the p-adic invariant associated to a pair $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N}, f)$. (here we only assume that $(A_\tau, f) = 1$ in order to simplify the presentation). It follows from Proposition 9.3 that one has

(9.15)
$$u_C(r, \tau)^{\nu(r,\tau)} = u_D(\widetilde{r}I_{\tau^*}, \mathfrak{f}^*)^{12},$$

and that the Shimura reciprocity law of u_C is compatible with the Shimura reciprocity law of u_D. Therefore the equivalence between the two conjectures will follow if we can show that the field of definition for $u_C(r, \tau)$ which is predicted by Conjecture 9.1 agrees with the one predicted by Conjecture 9.2.

Let $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$. Then Conjecture 9.2 predicts that $u_D(\widetilde{r}I_{\tau^*}, \mathfrak{f}^*)$ is a strong p-unit in $K(\mathfrak{f}^*\infty)^{\langle \sigma_\wp \rangle}$ where $\mathfrak{f}^* = f\mathcal{O}_{\tau^*}$. Now assume that $\tau \in H^{\mathcal{O}}(\mathfrak{N}, f)$ so that $\mathrm{cond}(\mathcal{O}_{\tau^*}) = f \cdot \mathrm{cond}(\mathcal{O})$ and that $r \in (\mathbb{Z}/f\mathbb{Z})^\times$. Under these assumptions we have that $\nu(r, \tau) = 1$ and that $K(\mathfrak{f}\infty) \subseteq K(\mathfrak{f}^*\infty) \subseteq K(\mathfrak{f}^2\infty)$. Conjecture 9.1, when applied to the left hand side of (9.15), predicts that

$$(9.16) \qquad u_C(r, \tau) \in K(\mathfrak{f}\infty)^{\langle \sigma_\wp \rangle}.$$

On the other hand, Conjecture 9.2, when applied to the right hand side of (9.15), only predicts, a priori, that

$$(9.17) \qquad u_C(r, \tau) = u_D(\widetilde{r}I_{\tau^*}, \mathfrak{f}^*)^{12} \in K(\mathfrak{f}^*\infty)^{\langle \sigma_\wp \rangle}.$$

In general $K(\mathfrak{f}\infty) \subsetneqq K(\mathfrak{f}^*\infty)$. Therefore it seems that Conjecture 9.1 gives a finer statement than Conjecture 9.2. In the remainder of the section we prove the following:

Proposition 9.4 *Conjecture 9.2 $\Longrightarrow$ (9.16).*

Let us explain our strategy of proof. The inclusions $K(\mathfrak{f}\infty) \subseteq K(\mathfrak{f}^*\infty) \subseteq K(\mathfrak{f}^2\infty)$ give rise to the two restriction maps

$$(9.18) \qquad \begin{aligned} res_1 &: \mathrm{Gal}(K(\mathfrak{f}^2\infty)/K) \to \mathrm{Gal}(K(\mathfrak{f}^*\infty)/K), \\ res_2 &: \mathrm{Gal}(K(\mathfrak{f}^2\infty)/K) \to \mathrm{Gal}(K(\mathfrak{f}\infty)/K), \end{aligned}$$

which correspond to the two projections

$$(9.19) \qquad \begin{aligned} \widetilde{\Theta}_1 &: I_{\mathcal{O}}(f)/P_{\mathcal{O},1}(f^2\infty) \to I_{\mathcal{O}^*}(f)/P_{\mathcal{O}^*,1}(f\infty), \\ \widetilde{\Theta}_2 &: I_{\mathcal{O}}(f)/P_{\mathcal{O},1}(f^2\infty) \to I_{\mathcal{O}}(f)/P_{\mathcal{O},1}(f\infty). \end{aligned}$$

When $\tau \in H^{\mathcal{O}}(\mathfrak{N}, f)$, we will show that $(P_{\mathcal{O},1}(f\infty) \cap I_{\mathcal{O}}(f) \pmod{\sim_f}) = ker_2(\widetilde{\Theta}_2)$ acts trivially on the class $[\widetilde{r}I_{\tau^*}]$ and therefore, from Conjecture 9.2, this will mean that the action of $\mathrm{Gal}(K(\mathfrak{f}^2\infty)/K)$ on $u_D(\widetilde{r}I_{\tau^*}, \mathfrak{f}^*)$ factors through $\mathrm{Gal}(K(\mathfrak{f}\infty)/K)$. Having this additional information, we thus see a postiori that Conjecture 9.2 implies the finer statement (9.16).

We want to prove the following:

Proposition 9.5 *Let $(r, \tau) \in (\mathbb{Z}/f\mathbb{Z})^\times \times H^{\mathcal{O}}(\mathfrak{N}, f)$. Denote the class of (r, τ) modulo $\sim_f$ by $[(r, \tau)]$. Assume that $\{(r_i, \tau_i)\}_{i \in I}$ is a complete set of representatives for the action of $C_{\mathcal{O}}(f)$ on $[(r, \tau)]$. Let $\mathcal{O}^* = \mathcal{O}_{\tau^*}$. Then $\{(r_i, \tau_i^*)\}$ is a complete set of representatives for the action of $C_{\mathcal{O}^*}(f)$ on $[(r, \tau^*)]$. Moreover, the action of $C_{\mathcal{O}^*}(f)$ on $[(r, \tau^*)]$ factors through $C_{\mathcal{O}}(f)$.*

In order to prove the previous proposition we need to prove a lemma.

Lemma 9.1 *Let* $(r, \tau), (r', \tau') \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$. *Then we have that* $(r, \tau) \sim_f (r', \tau')$ *if and only if* $(r', \tau^*) \sim_f (r, \tau'^*)$.

Proof First note that since $(\widetilde{r}, f) = (\widetilde{r}', f)$ and that $1 \leq \widetilde{r}, \widetilde{r}' \leq f$ one has that $\frac{\widetilde{r}}{\widetilde{r}'}$ is a unit modulo f. From the Proposition 3.1 we know that there exists a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN)$ such that $\gamma\tau = \tau'$ and $d^{-1}r \equiv r' \pmod{f}$. Therefore we have

$$(9.20) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau + d) \begin{pmatrix} \tau' \\ 1 \end{pmatrix}.$$

We can rewrite (9.20) in the following way

$$(9.21) \qquad \begin{pmatrix} a & -bfN \\ -\frac{c}{fN} & d \end{pmatrix} \begin{pmatrix} 1 \\ \tau^* \end{pmatrix} = (c\tau + d)\frac{\tau'}{\tau} \begin{pmatrix} 1 \\ \tau'^* \end{pmatrix}.$$

It thus follows that $a + bf\tau^* = (c\tau + d)\frac{\tau'}{\tau}$ and we can thus rewrite (9.21) as

$$(9.22) \qquad \begin{pmatrix} d & -\frac{c}{fN} \\ -bfN & a \end{pmatrix} \begin{pmatrix} \tau^* \\ 1 \end{pmatrix} = (-bfN\tau^* + a) \begin{pmatrix} \tau'^* \\ 1 \end{pmatrix}.$$

From (9.22) we may deduce that

$$\frac{\widetilde{r}}{\widetilde{r}'}(bfN(\tau^*)^\sigma + a)\widetilde{r}' A_{\tau^*}\Lambda_{\tau^*} = \widetilde{r} A_{\tau'^*}\Lambda_{\tau'^*}.$$

Since $\begin{pmatrix} d & -\frac{c}{fN} \\ -bfN & a \end{pmatrix} \in \Gamma_0(fN)$ and $\frac{dr'}{r} \equiv 1 \pmod{f}$ we have

$$\frac{\widetilde{r}}{\widetilde{r}'}(-bfN(\tau^*)^\sigma + a) \in (\widetilde{r}' A_{\tau^*}\Lambda_{\tau^*})^{-1}f + 1.$$

It thus follows that $\widetilde{r}' A_{\tau^*}\Lambda_{\tau^*} \sim_f \widetilde{r} A_{\tau'^*}\Lambda_{\tau'^*}$ and therefore from Proposition 3.1 we find that $(r', \tau^*) \sim_f (r, \tau'^*)$. $\square$

We may now deduce the following key corollary

Corollary 9.1 *Let* $[(r, \tau)] \in (\mathbb{Z}/f\mathbb{Z})^\times \times H^{\mathcal{O}}(\mathfrak{N})/ \sim_f$ *and let*

$$C_{\mathcal{O}}(f) \cdot [(r, \tau)] = \{[(r_i, \tau_i)]\}_{i \in I},$$

be the full orbit of $[(r, \tau)]$ *under the action of* $C_{\mathcal{O}}(f)$. *Then*

$$C_{\mathcal{O}^*}(f) \cdot [(r, \tau^*)] = \{[(r_i, \tau_i^*)]\}_{i \in I}.$$

Proof Without lost of generality, we may assume that the orbit $C_{\mathcal{O}}(f) \cdot [(r, \tau)]$ is written as

$$\{[(r_{ij}, \tau_i)]\}_{i \in I, j \in J},$$

so that if $i, i' \in I$ are such that $(*, \tau_i) \sim_f (*, \tau_{i'})$ then $i = i'$. Assume that for distinct elements $i, i' \in I$ there exists elements $j, j' \in J$ such that

(9.23) $$(r_{ij}, \tau_i^*) \sim_f (r_{i'j'}, \tau_{i'}^*).$$

First note that $\frac{r_{ij}}{r_{i'j'}} := s$ is a unit modulo f. Applying Lemma 9.1 to (9.23), we find that $(r_{i'j'}, \tau_i) \sim_f (r_{ij}, \tau_{i'})$. Multiplying the first coordinate of the previous relation by s we find that $(r_{ij}, \tau_i) \sim_f (sr_{ij}, \tau_{i'})$. In particular, this implies that $i = i'$ (so $\tau_i = \tau_{i'}$) and that there exists $\epsilon \in \mathcal{O}(\infty)^{\times}$ such that $\epsilon \equiv s \pmod{f}$. Now note that

$$(r_{ij}, \tau_i) \sim_f (sr_{ij}, \tau_{i'}) \sim_f (s^2 r_{i'j'}, \tau_{i'}) \sim_f (r_{i'j'}, \tau_{i'}).$$

Therefore we must have $r_{ij} = r_{i'j'}$. We thus conclude that the elements in the set $\{(r_{ij}, \tau_i^*)\}_{i \in I, j \in J}$ are inequivalent modulo $\sim_f$. Reversing this argument we see that this set gives a complete list of representatives modulo $\sim_f$. This concludes the proof. $\square$

Proof of Proposition 9.5 Let $n = \mathrm{cond}(\mathcal{O})$. Note that if $\tau \in H^{\mathcal{O}}(\mathfrak{N}, f)$ then $Q_{\tau}(x, y) = Ax^2 + Bxy + Cy^2$ with $(A, f) = 1$ and $B^2 - 4AC = n^2 d_K$. Let $[(r, \tau)]$ be the equivalence class of (r, τ) modulo $\sim_f$. Since $(A, f) = 1$ and $N | A$ we have $Q_{\tau^*}(x, y) = \mathrm{sign}(C)(Cf^2 Nx^2 - Bfxy + \frac{A}{N}y^2)$ so that $\mathrm{cond}(\mathcal{O}_{\tau^*}) = fn$. Let $\mathcal{O}^* = \mathcal{O}_{\tau^*}$ so that $\tau^* \in H^{\mathcal{O}^*}(\mathfrak{N})$. We let Θ_1, Θ_2 be as in (9.18) and $\widetilde{\Theta}_1, \widetilde{\Theta}_2$ be as in (9.19).

We note that for $\lambda \mathcal{O} \in P_{\mathcal{O},1}(f\infty)$ one has that $\Theta_1(\lambda \mathcal{O}) = \lambda \mathcal{O}^*$ (we leave the proof of this fact as an exercise). We claim that $\widetilde{\Theta}_1(ker_2(\widetilde{\Theta}_2))$ "acts trivially" on the class $[(r, \tau^*)]$. By acting trivially we mean that if $\lambda \mathcal{O} \in P_{\mathcal{O},1}(f\infty)$ then $[\lambda \widetilde{r} I_{\tau^*}] = [\widetilde{r} I_{\tau^*}]$ where the brackets mean the class modulo $\sim_f$. Let $\lambda \mathcal{O} \in P_{\mathcal{O},1}(f\infty)$ and assume that $\lambda \in (\widetilde{r} I_{\tau^*})^{-1} f + 1$. In particular one has that $\lambda \in Q_{\mathcal{O},1}(f\infty) \cap (\widetilde{r} I_{\tau^*})^{-1}$. We have that

(9.24) $$(\widetilde{r} I_{\tau^*})^{-1} f + 1 = \frac{1}{\widetilde{r}} \Lambda_{(\tau^*)^{\sigma}} f + 1 = \frac{1}{\widetilde{r}}\left(f\mathbb{Z} + \left(\frac{-B - n\sqrt{D}}{NC}\right)\mathbb{Z}\right) + 1,$$

and a direct computation shows that

(9.25) $$Q_{\mathcal{O},1}(f\infty) \cap (\widetilde{r} I_{\tau^*})^{-1} \subseteq \frac{1}{\widetilde{r}}\left(f\mathbb{Z} + \left(\frac{-B - n\sqrt{D}}{NC}\right)\mathbb{Z}\right) + 1.$$

Combining (9.24) and (9.25) we find that $\lambda \in Q_{\mathcal{O},1}(f\infty) \cap (\widetilde{r} I_{\tau^*})^{-1} \subseteq (\widetilde{r} I_{\tau^*})^{-1} f + 1$. Therefore $\widetilde{\Theta}_1(ker(\widetilde{\Theta}_2))$ acts trivially on $[(r, \tau^*)]$ as claimed.

Let $\{\mathfrak{a}_i\}_{i \in I}$ be a complete set of representatives of $I_{\mathcal{O}}(f)$ modulo $P_{\mathcal{O},1}(f\infty)$ and let

$$[(r_i, \tau_i)] := \psi^{-1}([\mathfrak{a}_i \mathfrak{a}, \mathfrak{a}_i \mathfrak{a} \mathfrak{N}]),$$

where $\mathfrak{a} = \widetilde{r} I_{\tau}$. Note that for all i one has that $(r_i, \tau_i) \in (\mathbb{Z}/f\mathbb{Z})^{\times} \times H^{\mathcal{O}}(\mathfrak{N}, f)$. From Corollary 9.1 one may deduce that $\{[(r_i, \tau_i^*)]\}_{i \in I}$ gives a complete set of representatives of $[(r, \tau^*)]$ under the action $C_{\mathcal{O}^*}(f)$. Moreover, since $\widetilde{\Theta}_1(ker(\widetilde{\Theta}_2))$ acts trivially on each class $[(r_i, \tau_i^*)]$ we see that the action of $C_{\mathcal{O}^*}(f)$ factors through $C_{\mathcal{O}}(f) \simeq C_{\mathcal{O}^*}(f)/\widetilde{\Theta}_1(ker(\widetilde{\Theta}_2))$. This concludes the proof. $\square$

10 Norm formulas for u_D

In this section we first prove a product formula for the p-adic invariant u_D which involve different orders of K. Second, we prove a non-trivial relationship between the p-adic invariants u_D and u_D^* where u_D^* is the "p-adic invariant u_D twisted by the the involution $*$" where $*$ is the involution given by $\tau \mapsto \frac{-1}{fN\tau}$. Thirdly, we prove the main result of this paper (Theorem 10.1) which may be viewed as a precise version of Theorem 1.1. Finally, we end this section by proving Proposition 10.3 which corrects a mistake that appeared in [Das08].

For the rest of this subsection we fix two orders $\mathcal{O}$ and $\mathcal{O}'$ of conductor coprime to N, a splitting of N namely a factorization of $N\mathcal{O}_K$ as $\mathfrak{N}\mathfrak{N}^\sigma$, a good divisor $\widetilde{\delta} \in \widetilde{\mathrm{Div}}_f(\mathfrak{N})$. For the whole section, we suppress the divisor $\widetilde{\delta}$ from the notation.

Proposition 10.1 *Let $\mathfrak{b} \subseteq \mathcal{O}$ (resp. $\mathfrak{b}' \subseteq \mathcal{O}'$) be an invertible $\mathcal{O}$-ideal (resp. an invertible $\mathcal{O}'$-ideal). Let $\mathfrak{g} \subseteq \mathcal{O}$ (resp. $\mathfrak{g} \subseteq \mathcal{O}'$) be an $\mathcal{O}$-ideal (resp. an $\mathcal{O}'$ ideal) which is not necessarily $\mathcal{O}$-invertible (resp. $\mathcal{O}'$-invertible). Assume that $\mathfrak{b}^{-1}\mathfrak{g} \supseteq \mathfrak{b}'^{-1}\mathfrak{g}'$, $([\mathfrak{b}^{-1}\mathfrak{g} : \mathfrak{b}'^{-1}\mathfrak{g}'], pN) = 1$, that $\Gamma_\mathfrak{b}(\mathfrak{g}) \supseteq \Gamma_{\mathfrak{b}'}(\mathfrak{g}')$ and set $e = [\Gamma_\mathfrak{b}(\mathfrak{g}) : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')]$. Assume furthermore that the following condition is satisfied:*

(†) *For all $x \in \mathfrak{b}^{-1}\mathfrak{g}$ $(x \neq -1)$, $(x+1)\mathfrak{b}'$ is $\mathfrak{g}'$-int and $\Gamma_{(x+1)\mathfrak{b}'}(\mathfrak{g}') \supseteq \Gamma_{\mathfrak{b}'}(\mathfrak{g}')$.*

Let $\{\lambda\}$ be a complete set of representatives of $\mathfrak{b}^{-1}\mathfrak{g}/\mathfrak{b}'^{-1}\mathfrak{g}'$ where each λ is chosen to be totally positive and let $\mu_\lambda = \lambda + 1$. Then

$$(10.1) \qquad u_D(\mathfrak{b}, \mathfrak{g})^e = \prod_\lambda u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')^{h_\lambda},$$

where the h_λ's are integers such that

$$h_\lambda = [\Gamma_{\mu_\lambda \mathfrak{b}'}(\mathfrak{g}') : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')].$$

Moreover, (10.1) is compatible with the adélic action of C_K on u_D which appears in Definition 9.5.

We note that by assumption $\mathfrak{b}$ is $\mathfrak{g}$-int so that the left hand side of (10.1) makes sense. Moreover, using (†) we find that $\mu_\lambda \mathfrak{b}'$ is $\mathfrak{g}'$-int so that the p-adic invariant appearing in the product of the right hand side of (10.1) makes sense. We would like to emphasize here that the left hand side of (10.1) is a p-adic invariant relative to the order $\mathcal{O}$ and that the right hand side of (10.1) is a product of p-adic invariants relative to the order $\mathcal{O}'$. So when $\mathcal{O} \neq \mathcal{O}'$, (10.1) gives a non-trivial relationship between p-adic invariants associated to different orders.

Remark 10.1 We note that since by assumption $\Gamma_\mathfrak{b}(\mathfrak{g}) \supseteq \Gamma_{\mathfrak{b}'}(\mathfrak{g}')$ it is expected (see Remark 9.5) that the abelian extension generated by the $u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')$'s over K contains the abelian extension generated by $u_D(\mathfrak{b}, \mathfrak{g})$ over K.

Proof Using Lemma 5.1 it is enough to show the proposition for

$$\widetilde{\delta} = \sum_{r \in \langle p \pmod{f} \rangle} (d'[\mathfrak{d}', r] - d[\mathfrak{d}, r]),$$

where $\mathfrak{d}'$ is consecutive to $\mathfrak{d}$, $\eta = \frac{\mathfrak{d}'}{\mathfrak{d}}$, $d = \mathbf{N}(\mathfrak{d})$ and $d' = \mathbf{N}(\mathfrak{d}')$. First note that since $\mathfrak{b}$ (resp. $\mathfrak{b}'$) is an $\mathcal{O}$-integral ideal (resp. $\mathcal{O}'$-integral) we have

(10.2) $$\Gamma_{\mathfrak{b}}(\mathfrak{g}) = (\mathfrak{b}^{-1}\mathfrak{g} + 1) \cap \mathcal{O}(\infty)^\times \quad \text{and} \quad \Gamma_{\mathfrak{b}'}(\mathfrak{g}') = (\mathfrak{b}'^{-1}\mathfrak{g}' + 1) \cap \mathcal{O}'(\infty)^\times.$$

Let $\widetilde{\mathcal{D}} = \mathcal{D}_{\mathcal{O}}^{can}$ be the canonical fundamental domain for the action of $\mathcal{O}(\infty)^\times$ on the positive quadrant Q. Using the assumption that $(\mathrm{cond}(\mathcal{O}), N) = 1$, one may check that if $\eta | \mathfrak{N}$ then η is $\mathcal{O}$-good for $\widetilde{\mathcal{D}}$. Let

$$\mathcal{D} = \bigcup_{\epsilon} \epsilon \widetilde{\mathcal{D}},$$

where the set $\{\epsilon\}$ is a complete set of representatives of $\mathcal{O}(\infty)^\times / \Gamma_{\mathfrak{b}}(\mathfrak{f})$. Note that $\mathcal{D}$ is a fundamental domain for the action of $\Gamma_{\mathfrak{b}}(\mathfrak{f})$ on the positive quadrant Q. One may also check that for every prime $\eta | \mathfrak{N}$, η is $\mathcal{O}$-good for $\mathcal{D}$. Finally, set

(10.3) $$\mathcal{D}' = \bigcup_{\epsilon'} \epsilon' \mathcal{D},$$

where $\{\epsilon'\}$ is a complete set of representatives of $\Gamma_{\mathfrak{b}}(\mathfrak{g}) / \Gamma_{\mathfrak{b}'}(\mathfrak{g}')$. In a similar way, $\mathcal{D}'$ is a fundamental domain for the action of $\Gamma_{\mathfrak{b}'}(\mathfrak{g}')$ on Q and for every prime $\eta | \mathfrak{N}$, η is $\mathcal{O}'$-good for $\mathcal{D}'$. By assumption we have

$$\mathfrak{b}^{-1}\mathfrak{g} + 1 = \bigcup_{\lambda} (\mathfrak{b}'^{-1}\mathfrak{g}' + \lambda + 1),$$

where $\{\lambda\}$ is a complete set of totally positive representatives of $\mathfrak{b}^{-1}\mathfrak{g} / \mathfrak{b}'^{-1}\mathfrak{g}'$.

For every ϵ' we have

$$\epsilon'\mathcal{D} \cap (\mathfrak{b}^{-1}\mathfrak{g} + 1) = \bigcup_{\lambda} \epsilon'\mathcal{D} \cap (\mathfrak{b}'^{-1}\mathfrak{g}' + \lambda + 1)$$

(10.4) $$= \bigcup_{\lambda} \mu_\lambda \left(\mu_\lambda^{-1} \epsilon'\mathcal{D} \cap ((\mu_\lambda \mathfrak{b}')^{-1}\mathfrak{g}' + 1) \right),$$

where $\mu_\lambda = 1 + \lambda$ (note that $\mu_\lambda \neq 0$ since $\lambda \gg 0$). Let $U \subseteq \mathcal{O}_{K_p}^\times$ be an arbitrary compact-open set. Using (10.4), (10.3) and the homogeneous property of the measure $\nu_{\widetilde{\delta}}$, we may deduce that

(10.5) $$\sum_{\epsilon'} \nu_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{g}, 1, \epsilon'\mathcal{D})(U) = \sum_{\lambda} \nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1}\mathcal{D}')(\mu_\lambda^{-1}U)$$

(10.6) $$= \sum_{\lambda} \nu_{\widetilde{\delta}} h_\lambda(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1}\mathcal{D}'_\lambda)(\mu_\lambda^{-1}U),$$

where $\mathcal{D}'_\lambda$ is a fundamental domain for the action of $\Gamma_{\mu_\lambda \mathfrak{b}'}(\mathfrak{g}')$ on Q chosen so that

$$\mathcal{D}' = \bigcup_{\{\epsilon_i(\lambda)\}_i} \epsilon_i(\lambda)\mathcal{D}'_\lambda,$$

where $\{\epsilon_i(\lambda)\}_i$ is a complete set of representatives of $\Gamma_{\mu_\lambda \mathfrak{b}'}(\mathfrak{g}')/\Gamma_{\mathfrak{b}'}(\mathfrak{g}')$. Moreover, we note that η is $\mathcal{O}$-good for the domain $\mathcal{D}'_\lambda$. Here the h_λ's are positive integers such that

$$(10.7) \qquad\qquad h_\lambda = [\Gamma_{\mu_\lambda \mathfrak{b}'}(\mathfrak{g}') : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')].$$

If we take $U = \mathcal{O}_{K_p}$ in (10.5) and we unfold the definition of $\nu_{\widetilde{\delta}}$ then we find

$$(10.8) \qquad\qquad e\zeta_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{g}\infty, 0) = \sum_\lambda h_\lambda \zeta_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}'\infty, 0),$$

where $e = [\Gamma_{\mathfrak{b}}(\mathfrak{g}) : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')]$. Using (10.8) we find that

$$ev_p(u_D(\mathfrak{b}, \mathfrak{g})) = \sum_\lambda h_\lambda v_p(u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')).$$

We have an isomorphism $K_p^\times \simeq p^{\mathbb{Z}} \times \mathcal{O}_{K_p}^\times$. Let us denote the projection on the i-th coordinate by π_i. From (10.8) we get that $\pi_1(u_D(\mathfrak{b}, \mathfrak{g})^e) = \pi_1(\prod_\lambda u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')^{h_\lambda})$. Thus in order to show (10.1) it remains to show that $\pi_2(u_D(\mathfrak{b}, \mathfrak{g})^e)$ coincides with $\pi_2(\prod_\lambda u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')^{h_\lambda})$. Using the fact that $p \star \widetilde{\delta} = \widetilde{\delta}$ one finds that

$$\nu_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{g}, 1, \mathcal{D})(\mathcal{O}_{K_p}^\times) = \zeta_{\{p\}}(\mathfrak{b}, \mathfrak{g}\infty, 0) = 0,$$

and

$$(10.9) \qquad\qquad \nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1}\mathcal{D}')(\mathcal{O}_{K_p}^\times) = \zeta_{\{p\}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}'\infty, 0) = 0.$$

Now using the assumption that $([\mathfrak{b}^{-1}\mathfrak{g} : \mathfrak{b}'^{-1}\mathfrak{g}'], pN) = 1$ we may assume that the λ's are chosen in such a way that the μ_λ's are coprime to pN. So in particular, one has that $\mu_\lambda \mathcal{O}_{K_p}^\times = \mathcal{O}_{K_p}^\times$. Now make the crucial observation that η is again $\mathcal{O}$-good for the cone $\mu_\lambda^{-1}\mathcal{D}'$ (it uses $(\mu_\lambda, N) = 1$). We have

$$\fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}, 1, \mu_\lambda^{-1}\mathcal{D}')(\mu_\lambda^{-1}x) = \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1}\mathcal{D}')(x)$$

$$(10.10) \qquad\qquad\qquad = \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mathcal{D}')(x),$$

where the first equality follows from (10.9) and the second follows from the independence on the choice of the fundamental domain. From (10.10) and (10.6) we deduce that

$$(10.11) \qquad \prod_\lambda \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}, 1, \mu_\lambda^{-1}\mathcal{D}')(\mu_\lambda^{-1}x) = \prod_\lambda \pi_2(u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')^{h_\lambda}).$$

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On the other hand, we also have

$$\prod_\lambda \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1} \mathcal{D}')(\mu_\lambda^{-1} x) = \prod_{\lambda, \epsilon'} \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mu_\lambda \mathfrak{b}', \mathfrak{g}', 1, \mu_\lambda^{-1} \epsilon' \mathcal{D})(\mu_\lambda^{-1} x)$$

$$= \prod_{\epsilon'} \fint_{\mathcal{O}_{K_p}^\times} x \cdot d\nu_{\widetilde{\delta}}(\mathfrak{b}, \mathfrak{g}, 1, \epsilon' \mathcal{D})(x)$$

(10.12)
$$= \pi_2(u_D(\mathfrak{b}, \mathfrak{g})^e).$$

The first equality follows from (10.3). The second equality follows from (10.5). The third equality follows from the independence on the choice of the fundamental domain. Finally, combining (10.11) with (10.12) we find that

$$\pi_2(u_D(\mathfrak{b}, \mathfrak{g})^e) = \pi_2\left(\prod_\lambda u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}')^{h_\lambda} \right).$$

This concludes the proof of (10.1).

Finally, it remains to show that (10.1) is compatible to the adélic action of C_K. To fix the idea, assume that $\mathcal{O}' \subseteq \mathcal{O}$ (a similar argument works when this inclusion is reversed). Let $n = \mathrm{cond}(\mathcal{O})$ and let $g = [\mathcal{O}' : \mathfrak{g}']$. Let

$$\pi : C_K \to C_{\mathcal{O}}(ng),$$

be the natural map given by class field theory. We have natural surjective maps

$$p : C_{\mathcal{O}}(gn) \to C_{\mathcal{O}}(\mathfrak{g}) \quad \text{and} \quad p' : C_{\mathcal{O}}(gn) \to C_{\mathcal{O}'}(\mathfrak{g}').$$

Let $c \in C_K$, $p(c) = [\mathfrak{c}]$ and $p'(c) = [\mathfrak{c}']$. Without loss of generality we may assume that $\mathfrak{c}$ and $\mathfrak{c}'$ are chosen so that $(\mathfrak{c}, \mathfrak{b}) = \mathcal{O}$, $(\mathfrak{c}', \mathfrak{b}') = \mathcal{O}'$, $\mathfrak{c} \cap \mathcal{O}' = \mathfrak{c}'$, $(\mathfrak{c}, \mathfrak{g}) = \mathcal{O}$ and $(\mathfrak{c}', \mathfrak{g}') = \mathcal{O}'$. Let $\{\lambda\}$ be a complete set of representatives of $\mathfrak{b}^{-1}\mathfrak{g}/\mathfrak{b}'^{-1}\mathfrak{g}'$ such that $\lambda \in \mathfrak{c}^{-1}$ so that

(10.13)
$$\mathfrak{b}^{-1}\mathfrak{g} = \bigcup_\lambda (\mathfrak{b}'^{-1}\mathfrak{g}' + \lambda).$$

Since every $\lambda \in \mathfrak{b}^{-1}\mathfrak{g}$, we find, after intersecting both sides of the equality (10.13) with $\mathfrak{c}^{-1}$, that

$$(\mathfrak{b}\mathfrak{c})^{-1}\mathfrak{g} = \mathfrak{c}^{-1} \cap (\mathfrak{b}^{-1}\mathfrak{g}) = \bigcup_\lambda \mathfrak{c}^{-1} \cap (\mathfrak{b}'^{-1}\mathfrak{g} + \lambda) = \bigcup_\lambda (\mathfrak{c}^{-1} \cap (\mathfrak{b}'^{-1}\mathfrak{g}') + \lambda)$$

(10.14)
$$= \bigcup_\lambda ((\mathfrak{c}'\mathfrak{b}')^{-1}\mathfrak{g}' + \lambda).$$

The first equality follows from $\mathfrak{c} \cap \mathfrak{b} = \mathfrak{c}\mathfrak{b}$ and the second equality follows from (10.13). The third equality follows from the assumption that $\lambda \in \mathfrak{c}^{-1}$ and the last equality follows from the assumption that $\mathfrak{c} \cap \mathcal{O}' = \mathfrak{c}'$ and $(\mathfrak{c}', \mathfrak{b}') = \mathcal{O}'$. Now a direct computation shows that

$$c \star u_D(\mathfrak{b}, \mathfrak{g}) = u_D(\mathfrak{c}\mathfrak{b}, \mathfrak{g}) = \prod_\lambda u_D(\mu_\lambda \mathfrak{c}'\mathfrak{b}', \mathfrak{g}') = \prod_\lambda c \star u_D(\mu_\lambda \mathfrak{b}', \mathfrak{g}').$$

The first and the third equalities follow from the definition of $\star$ (see Definition 9.5). The second equality follows from (10.14) and the definition of u_D. This shows that the Shimura reciprocity law is compatible with the product (10.1). This concludes the proof. $\square$

Armed with Proposition 10.1 we may now find a non-trivial relationship between the p-adic invariants u_D and u_D^* where u_D^* is the "p-adic invariant u_D twisted by the the involution $*$" where $*$ is the involution given by $\tau \mapsto \frac{-1}{fN\tau}$.

Proposition 10.2 *Let* $\tau \in H^{\mathcal{O}}(\mathfrak{N})$, $\mathfrak{f} = f\mathcal{O}_\tau$ *and* $\mathfrak{f}^* = f\mathcal{O}_{\tau^*}$ *where* $\tau^* = \frac{-1}{fN\tau}$. *Set*

$$(10.15) \qquad (\mathfrak{b}^*)^{-1} = \Lambda_{\tau^*} \quad \text{and} \quad \mathfrak{b}^{-1} = \Lambda_{f\tau^*}.$$

Then

$$(10.16) \qquad u_D(\mathfrak{b}^*, \mathfrak{f}^*)^e = \prod_\lambda u_D(\mu_\lambda \mathfrak{b}, \mathfrak{f})^{h_\lambda},$$

where $\{\lambda\}$ *is a complete set of totally positive representatives of* $(\mathfrak{b}^*)^{-1}/\mathfrak{b}^{-1}$, $\mu_\lambda = \lambda f + 1$ *and the* h_λ's *and* e *are as in Proposition 10.1. Similarly, if one sets*

$$(10.17) \qquad \mathfrak{b}^{-1} = \Lambda_\tau \quad \text{and} \quad (\mathfrak{b}^*)^{-1} = \Lambda_{f\tau}.$$

one has that

$$(10.18) \qquad u_D(\mathfrak{b}, \mathfrak{f})^{e'} = \prod_{\lambda'} u_D(\mu_{\lambda'} \mathfrak{b}^*, \mathfrak{f}^*)^{h_{\lambda'}},$$

where $\{\lambda'\}$ *is a complete set of totally positive representatives of* $\mathfrak{b}^{-1}/(\mathfrak{b}^*)^{-1}$, $\mu_{\lambda'} = \lambda' f + 1$ *and the* $h_{\lambda'}$'s *and* e' *are as in Proposition 10.1.*

Proof First note that $\text{cond}(\mathcal{O}^*) = f \cdot \text{cond}(\mathcal{O})$, $\tau^* = \frac{-1}{fN\tau} \in H^{\mathcal{O}^*}(\mathfrak{N})$, $\frac{1}{N\tau} \in H^{\mathcal{O}}(\mathfrak{N})$, $\mathfrak{b}$ is $\mathfrak{f}$-int and $\mathfrak{b}^*$ is $\mathfrak{f}^*$-integral. We have $(\mathfrak{b}^*)^{-1} \supseteq \mathfrak{b}^{-1}$ and that $(\mathfrak{b}^*)^{-1}\mathfrak{f}^*/\mathfrak{b}^{-1}\mathfrak{f} \simeq (\mathfrak{b}^*)^{-1}/\mathfrak{b}^{-1} \simeq \mathbb{Z}/f\mathbb{Z}$. Now make the following crucial observation:

(1) The condition (†) is satisfied: namely, for all $x \in (\mathfrak{b}^*)^{-1}\mathfrak{f}^*$ $(x \neq -1)$ one has that $(1 + x)\mathfrak{b}$ is $\mathfrak{f}$-int and that $\Gamma_{(1+x)\mathfrak{b}}(\mathfrak{f}) \supseteq \Gamma_\mathfrak{b}(\mathfrak{f})$.

Now applying Proposition 10.1 we find that

$$(10.19) \qquad u_D(\mathfrak{b}^*, \mathfrak{f}^*)^e = \prod_\lambda u_D(\mu_\lambda \mathfrak{b}, \mathfrak{f})^{h_\lambda},$$

where $\mu_\lambda = 1 + f\lambda$ where $\{\lambda\}$ is a complete set of totally positive representatives of $(\mathfrak{b}^*)^{-1}/\mathfrak{b}^{-1}$, e and h_λ's are chosen as in Proposition 10.1. This proves (10.16). The proof of (10.18) follows from (10.16) by replacing τ by τ^*. $\square$

We may now give a precise version of Theorem 1.1 that was stated at the end of the introduction.

Theorem 10.1 *Let* $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}}(\mathfrak{N})$ *and set*

$$(\mathfrak{b}^*)^{-1} = \frac{1}{r}\Lambda_{(\tau^*)^\sigma} \quad \text{and} \quad \mathfrak{b}^{-1} = \frac{1}{r}\Lambda_{f(\tau^*)^\sigma},$$

where $\tau^* = \frac{-1}{fN\tau}$ and σ is the non-trivial automorphism of $\mathrm{Gal}(K/\mathbb{Q})$. Let $\mathcal{O}^* = \mathcal{O}_{\tau^*}$, $\mathfrak{f} = f\mathcal{O}$ and $\mathfrak{f}^* = f\mathcal{O}^*$. Then we have

$$(10.20) \qquad u_C(r,\tau)^{\nu(r,\tau)e} = u_D(\mathfrak{b}^*,\mathfrak{f}^*)^{12e} = \prod_\lambda u_D(\mu_\lambda\mathfrak{b},\mathfrak{f})^{12h_\lambda},$$

where $\mu_\lambda = 1 + f\lambda$, $\{\lambda\}$ being a complete set of totally positive representatives of $(\mathfrak{b}^*)^{-1}/\mathfrak{b}^{-1}$ and $\nu(r,\tau)$ is the integer which appears in Proposition 9.3 and the h_λ's and e are as in Proposition 10.1. Moreover, we have that

$$(10.21) \qquad u_D(\mathfrak{b},\mathfrak{f})^{12e'} = \prod_{\lambda'} u_D(\mu_{\lambda'}\mathfrak{b}^*,\mathfrak{f}^*)^{12e'} = \prod_{\lambda'} c_{\lambda'} \star \left(u_D(\mathfrak{b}^*,\mathfrak{f}^*)^{12h_{\lambda'}} \right)$$

$$(10.22) \qquad\qquad\qquad\qquad\qquad = \prod_{\lambda'} c_{\lambda'} \star \left(u_C(r,\tau^*)^{\nu(r,\tau^*)h_{\lambda'}} \right),$$

where $\mu_{\lambda'} = 1 + f\lambda'$ and $\{\lambda'\}$ is a complete set of representatives of $(f\mathfrak{b}^{-1})/(\mathfrak{b}^*)^{-1}$. Here $c_{\lambda'} \in C_K$ is chosen so that $\pi(c_{\lambda'}) \sim_{\mathfrak{f}^*} \lambda'\mathcal{O}^*$, where $\pi : C_K \to I_{\mathcal{O}^*}/P_{\mathcal{O}^*,1}(\mathfrak{f}^*)$ is the natural projection map and $\star$ is the action which appears in Definition 9.5.

We would like to emphasize here that the p-adic invariants $u_C(r,\tau)$ and $u_D(\mu_\lambda\mathfrak{b},\mathfrak{f})$'s which appear in (10.20) are relative to the order $\mathcal{O}$. Similarly, the p-adic invariants $u_D(\mu_\lambda\mathfrak{b},\mathfrak{f})$ and $u_C(r,\tau^*)$'s which appear in (10.22) are relative to the order $\mathcal{O}^*$.

Proof The first equality in (10.20) follows from Proposition 9.3. The second equality follows from (10.20) follows from (10.16). The first equality in (10.21) follows from (10.16) and the second equality follows from the Shimura reciprocity law applied to $u(\mathfrak{b}^*,\mathfrak{f}^*)$. Finally (10.22) follows from Proposition 9.3 and the compatibility of the Shimura reciprocity laws of u_C and u_D. This concludes the proof. $\square$

We have the following corollary which may be viewed as a precise version of Theorem 1.1.

Corollary 10.1 Let $\rho \in H^{\mathcal{O}_K}(\mathfrak{N}, f)$ and set $\tau = \rho$. Then one has that

$$(10.23) \qquad u_C(r,\tau)^{\nu(r,\tau)} = \prod_\lambda u_D(\mu_\lambda\mathfrak{b},\mathfrak{f})^{12h_\lambda}.$$

where the quantities $\mathfrak{b}, \mathfrak{f}, \mu_\lambda$ and h_λ are chosen as in (10.20) of Theorem 10.1. In a similar way, if one sets $\tau = \rho^* := \frac{-1}{fN\rho}$, one has that

$$(10.24) \qquad u_D(\mathfrak{b},\mathfrak{f})^{12} = \prod_{\lambda'} c_{\lambda'} \star u_C(r,\tau^*),$$

where the quantities $\mathfrak{b}, \mathfrak{f}, e'$ and $c_{\lambda'}$ are chosen as in (10.22) of Theorem 10.1.

Proof First let $Q_\rho(x,y) = Ax^2 + Bxy + Cy^2$ (note that $(A,f) = 1$). Let us treat the first case, i.e., when $\tau = \rho$. We have

$$(10.25) \qquad Q_{\tau^*}(x,y) = \mathrm{sign}(C) \left(Cf^2Nx^2 - Bfxy + \frac{A}{N}y^2 \right),$$

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so that

$$(10.26) \qquad \tau^* = \frac{B + f\sqrt{d_K}}{fCN}$$

From (10.20) of Theorem 10.1 we get

$$u_C(r,\tau)^{\nu(r,\tau)e} = \prod_\lambda u_D(\mu_\lambda \mathfrak{b}, \mathfrak{f})^{12h_\lambda}.$$

Using (10.26) one may check that $e = [\Gamma_{\mathfrak{b}^*}(\mathfrak{f}^*) : \Gamma_{\mathfrak{b}}(\mathfrak{f})] = 1$. This proves (10.23).

Now let us treat the second case, i.e., when $\tau = \frac{-1}{fN\rho} =: \rho^*$. Then in this case we have that $Q_\tau(x,y) = \text{sign}(C)\left(Cf^2Nx^2 - Bfxy + \frac{A}{N}y^2\right)$. Therefore,

$$(10.27) \qquad Q_{\tau^*}(x,y) = Ax^2 + Bxy + Cy^2.$$

so that

$$(10.28) \qquad \tau^* = \frac{-B + \sqrt{d_K}}{\frac{A}{N}}.$$

Applying (10.22) of Theorem 10.1 with the previous parameters we find that

$$u_D(\mathfrak{b}, \mathfrak{f})^{12e'} = \prod_{\lambda'} c_{\lambda'} \star \left(u_C(r,\tau^*)^{\nu(r,\tau^*)h_{\lambda'}}\right)$$

Using (10.28), a direct computation shows that for all λ' one has that $\nu(r,\tau^*)h_{\lambda'} = 1$. Moreover, one also have $e' = 1$. This concludes the proof of (10.24). $\quad\square$

Remark 10.2 In the special case where $f = 1$ and $\tau \in H^{\mathcal{O}}(\mathfrak{N})$ we may deduce from (10.20) that

$$(10.29) \qquad u_C(1,\tau) = u_D(I_{\tau^*}, \mathcal{O})^{12},$$

where $\tau^* = \frac{-1}{N\tau}$. Here, note that $I_{\tau^*}^{-1} = \Lambda_{(\tau^*)^\sigma}$ (for the definition of I_τ see Definition 3.1). Moreover, it follows from Remark 8.2 (or the computation carried in Appendix B) that

$$(10.30) \qquad u_C(1,\tau) = u_{DD}(\tau).$$

Combining (10.30) with (10.29) we deduce that

$$(10.31) \qquad u_D(I_{\tau^*}, \mathcal{O})^{12} = u_{DD}(\tau).$$

It thus follows from (10.31) that the p-adic invariant u_D (once his definition is extended to arbitrary orders as was done in Definition 9.4) may be viewed as a natural generalization of the Darmon-Dasgupta p-adic invariant u_{DD} with the subtlety that the appearance of τ in the argument of u_D is twisted by the Atkin-Lehner involution of level N namely $\tau \mapsto \frac{-1}{N\tau} = \tau^*$

For the rest of the section we assume that $f = 1$ and we let $\operatorname{cond}\mathcal{O} = n$ so that $\mathcal{O} = \mathcal{O}_n$ where $(n, N) = 1$. For every $\tau \in H^{\mathcal{O}}(\mathfrak{N})$ we may consider the p-adic invariant

$$u_D(I_{\tau^*}, \mathcal{O})^{12} = u_{DD}(\tau) = u(\alpha_\delta, \tau),$$

where the first equality follows from Remark 10.2. The p-adic invariant $u(\alpha_\delta, \tau)$ is the the Darmon-Dasgupta p-adic invariant constructed in [DD06]. Here $\delta \in \operatorname{Div}(N)$ is the fixed ambient divisor and α_δ is the associated modular unit to δ (see Appendix B). Conjecture 9.1 implies that $u_D(I_{\tau^*}, \mathcal{O})^{12} \in K(\mathcal{O}_n\infty)$. Moreover, for every $\mathcal{O}_K$-ideal $\mathfrak{a} \subseteq \mathcal{O}_K$ we have a p-adic invariant $u_D(\mathfrak{a}, \mathfrak{n})$ where $\mathfrak{n} = n\mathcal{O}_K$. In a similar way, Conjecture 9.2 predicts that $u_D(\mathfrak{a}, \mathfrak{n}) \subseteq K(\mathfrak{n}\infty)$. Since $K(\mathcal{O}\infty) \subseteq K(\mathfrak{n}\infty)$ it is natural to ask if a suitable product of p-adic invariants $u_D(\mathfrak{a}, \mathfrak{n})$'s (where $\mathfrak{a}$ here varies and $\mathfrak{n}$ is fixed) is equal to $u_D(I_{\tau^*}, \mathcal{O})^{12}$. The proposition below gives a positive answer. Before stating it, let us make a few observations.

Let $\tau \in H^{\mathcal{O}_K}(\mathfrak{N})$ so that $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ with $d_K = B^2 - 4AC$. Assume furthermore that $(C, n) = 1$ so that

$$Q_{\tau^*}(x, y) = \operatorname{sign}(C) \left(CNx^2 - Bxy + \frac{A}{N}y^2 \right),$$

and that

$$Q_{n\tau^*}(x, y) = \operatorname{sign}(C) \left(CNx^2 - Bnxy + \frac{n^2 A}{N}y^2 \right),$$

where $\tau^* = \frac{-1}{N\tau}$. We note that the condition $(C, n) = 1$ is equivalent to say that $\tau^* \in H^{\mathcal{O}_K}(N, n)$. If one sets

(10.32) $$\mathfrak{b}^{-1} = \Lambda_{n(\tau^*)^\sigma} \quad \text{and} \quad (\mathfrak{b}')^{-1} = \Lambda_{(\tau^*)^\sigma}.$$

one readily sees that $\operatorname{End}(\Lambda_{(\tau^*)^\sigma}) = \mathcal{O}_K$ and $\operatorname{End}(\Lambda_{n(\tau^*)^\sigma}) = \mathcal{O}_n$. We may now state the proposition.

Proposition 10.3 *Keeping the same notation as above we have the following identity*

(10.33) $$u_D(I_{\tau^*}, \mathcal{O}_n)^e = u_D(\mathfrak{b}, \mathcal{O}_n)^e = \prod_\lambda u_D(\mu_\lambda \mathfrak{b}', n\mathcal{O}_K)^{h_\lambda},$$

where $\mu_\lambda = 1 + \lambda$ and $\{\lambda\}$ is a complete set of totally positive representatives of $\mathfrak{b}^{-1}/n(\mathfrak{b}')^{-1}$,

$$h_\lambda = [\Gamma_{\mu_\lambda \mathfrak{b}'}(n\mathcal{O}_K) : \mathcal{O}_K(n\infty)^\times],$$

and $e = [\mathcal{O}_n(\infty)^\times : \mathcal{O}_K(n\infty)^\times]$, i.e., e is the order of the image of a generator ϵ_n of $\mathcal{O}_n(\infty)^\times$ inside the group $(\mathbb{Z}/n\mathbb{Z})^\times$.

Proof Apply Proposition 10.1 with the parameters $\mathcal{O} = \mathcal{O}_n$, $\mathcal{O}' = \mathcal{O}_K$, $\mathfrak{g} = \mathcal{O}_n$ and $\mathfrak{g}' = n\mathcal{O}_K$. $\square$

Remark 10.3 We keep the same notation as in (10.32). We would like to point out a mistake which appears in Theorem 8.3 of [Das08] which claims that

$$(10.34) \qquad u_{DD}(\mathfrak{b}, \mathcal{O}_n)^{\frac{1}{6}} = \prod_a u_D((a)\mathfrak{b}', n\mathcal{O}_K)^2,$$

where in the notation of [Das08] the p-adic invariant $u_{DD}(\mathfrak{b}, \mathcal{O}_n)^{\frac{1}{6}}$ corresponds to the p-adic invariant $u(\alpha, \tau)$. For an explanation concerning the appearance of this 6-th root see Remark 8.2. Here the product on the right hand side of (10.34) goes over a complete set of representatives of positive integers $\{a\}$ of $(\mathbb{Z}/n\mathbb{Z})^\times/\langle\epsilon_n\rangle$ which are coprime to pnN. When $n = 1$, the formula (10.34) holds true and is (up to a 6-th root extraction) equivalent to (10.33). However when $n > 1$ the formula (10.34) differs from the formula (10.33) since the set $\{a\}$ is strictly smaller than the set $\{\lambda\}$. A complete set of totally positive representatives of $\mathfrak{b}^{-1}/\mathfrak{n}(\mathfrak{b}')^{-1}$ is given for example by $\{r\}_{r=1}^n$. Note that $u_D(r\mathfrak{b}', \mathfrak{n}) = u_D(\frac{r}{\mathfrak{d}}\mathfrak{b}', \frac{\mathfrak{n}}{\mathfrak{d}})$ where $\mathfrak{n} = n\mathcal{O}_K$ and $(\mathfrak{n}, r\mathfrak{b}') = \mathfrak{d}$. Since $\mathfrak{d}$ is an $\mathcal{O}_K$-ideal ($\mathcal{O}_K$ being the maximal order) it is automatically invertible. Therefore $u_D(r\mathfrak{b}', \mathfrak{n})$ is a primitive p-adic invariant of conductor $\frac{\mathfrak{n}}{\mathfrak{d}}$. Here the word primitive is used in the sense of Definition 9.4. It is thus essential to consider p-adic invariants which are primitive of conductor $\mathfrak{d}$ for various divisors $\mathfrak{d}|\mathfrak{n}$. Let us note that the right hand side of equation (85) in [Das08] is indeed the (conjectural) Gross-Stark p-unit of $K(\mathcal{O}_n\infty)$. However, the left hand side of equation (85) in [Das08] is a product of powers of Gross-Stark p-units for ring class fields of various conductors $\mathfrak{d}|\mathfrak{n}$. The error is due to the discrepancy between the partial zeta-functions associated to ideal classes of the order $\mathcal{O}_n$, namely

$$(10.35) \qquad \zeta(\mathfrak{b}', \mathcal{O}_n, w_1, s) = \mathbf{N}(\mathfrak{b}')^{-s} \sum_{\mathcal{O}_n(\infty)^\times \backslash \{0 \neq \mu \in \mathfrak{b}'^{-1}\}} \frac{w_1(\mu)}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s}, \qquad \Re(s) > 1,$$

and the partial zeta-functions associated to the extension $K(\mathcal{O}_n\infty)/K$, namely

$$(10.36) \quad \zeta_R(\mathfrak{b}', \mathcal{O}_n, w_1, s) = \mathbf{N}(\mathfrak{b}')^{-s} \sum_{\mathcal{O}_n(\infty)^\times \backslash \{0 \neq \mu \in \mathfrak{b}'^{-1}, (\mu, R)=1\}} \frac{w_1(\mu)}{|\mathbf{N}_{K/\mathbb{Q}}(\mu)|^s}, \qquad \Re(s) > 1,$$

where $R = \{\nu \text{ is a place of } K: \nu|n\}$. When $n > 1$ the existence of invertible ideals of $\mathcal{O}_n$ that are not relatively prime to n, implies that (10.36) differs from (10.35). Since the Darmon-Dasgupta invariant $u(\alpha, \tau)$ was constructed from special values of (10.35) rather than (10.36) it explains the discrepancy between $u(\alpha, \tau) = u_{DD}(\mathfrak{b}, \mathcal{O}_n)^{\frac{1}{6}}$ and the p-adic Gross-Sark p-unit of $K(\mathcal{O}_n\infty)$.

A Special values of partial zeta functions and Dasgupta's refinement

Let F be a totally real number field and let S be a finite set of places of F which contains all the infinite places of F. Let $\mathfrak{f}$ be an integral ideal of F and let $M = F(\mathfrak{f}\infty)$ be the

narrow ray class field of F of conductor $\mathfrak{f}$. For an ideal $\mathfrak{a}$ of F we denote by $\sigma_{\mathfrak{a}} \in \mathrm{Gal}(M/F)$ the Frobenius at $\mathfrak{a}$. For any $\sigma \in \mathrm{Gal}(M/F)$ let

$$(A.1) \qquad \zeta_S(M/F, \sigma, s) := \sum_{\substack{(\mathfrak{a}, S) = 1 \\ \sigma_{\mathfrak{a}} = \sigma}} \frac{1}{\mathbf{N}(\mathfrak{a})^s}.$$

It was proved by Siegel and Klingen that the special values at negative integers of $\zeta_S(\sigma, s)$ are *rational numbers* (see [Kli62] and [Sie69]). One of the main tool which is used to study these rational values is the so-called *q-expansion principle* (see [DR80]) which says that for all integral ideals $\mathfrak{b}$ coprime to $\mathfrak{f}$ and all integers $k \geq 1$ one has

$$(A.2) \qquad \zeta_R(M/F, \sigma, 1-k) - \mathbf{N}(\mathfrak{b})^k \zeta_R(M/F, \sigma\sigma_{\mathfrak{b}}, 1-k) \in \mathbb{Z}\left[\frac{1}{\mathbf{N}(\mathfrak{b})}\right].$$

It follows from Theorem 2.4 of [Coa77] that the relation (A.2), when applied to all prime ideals $\mathfrak{b}$ of M coprime to $\mathfrak{f}$, implies that

$$(A.3) \qquad w_k(M/F)\zeta_R(M/F, \sigma, 1-k) \in \mathbb{Z},$$

where $w_k(M/F)$ is the largest integer, say e, for which the abelian group $\mathrm{Gal}(M(\mu_e)/F)$ has an exponent which divides k. Here μ_e denotes the group of roots of unity of order e. In particular, $w_1(M/F)$ is just the group of roots of unity of M. Thus the denominator of the rational number $\zeta_R(\sigma, 1-k)$ is a divisor of $w_k(M/F)$. It follows from (A.2) that if $\eta \in T$ then for all integers $k \geq 1$ one has

$$\zeta_{S,T}(M/F, \sigma, 1-k) \in \mathbb{Z}\left[\frac{1}{\mathbf{N}\eta}\right].$$

In particular, if T contains two primes of different residue characteristics then the special value $\zeta_{S,T}(M/F, \sigma, 1-k)$ is an integer. Another sufficient condition which guarantees the integrality of $\zeta_{S,T}(M/F, \sigma, 1-k)$ is to assume the existence of a prime $\eta \in T$ such that $(l, w_k(M/F)) = 1$ where $l = \mathbf{N}(\eta)$. This explains the raison d'être of Assumption 1.2.

There is another key property of the special values $\zeta_R(M/F, \sigma, 1-k)$ which plays a key role in the setting of p-adic Gross-Stark conjecture. Let τ be a complex conjugation of M/F and let $k \geq 1$ be an odd integer. Then

$$(A.4) \qquad \zeta_R(M/F, \sigma\tau, 1-k) = -\zeta_R(M/F, \sigma, 1-k).$$

For a proof of (A.4) see for example p. 12 of [Cha10]. The identity (A.4) forces additional restrictions on the type of global $\mathfrak{p}$-units which are predicted by the Strong Gross conjecture. Let us explain it. If $\sigma : M \hookrightarrow \mathbb{C}$ is a complex embedding and $u \in U_{\mathfrak{p}}$ one readily sees from the definition of $U_{\mathfrak{p}}$ that

$$(A.5) \qquad u^{\sigma\tau_{\infty}\sigma^{-1}} = u^{-1},$$

where τ_{∞} is the complex conjugation of $\mathbb{C}$. In particular, we see from (A.5) that every complex conjugation of M acts by -1 on the multiplicative group $U_{\mathfrak{p}}$. It thus follows that

$U_{\mathfrak{p}} \subseteq L_{CM}$ where L_{CM} stands for the largest CM subfield contained in M. Therefore if τ is a complex conjugation of M/F then $(u_T^\sigma)^\tau = (u_T^\sigma)^{-1}$ which is in harmony with the fact that

$$(A.6) \qquad \qquad \zeta_{R,T}(M/F, \sigma\tau, 0) = -\zeta_{R,T}(M/F, \sigma, 0),$$

where (A.6) follows from (A.4) and the definition of $\zeta_{R,T}(M/F, \sigma, s)$.

It is possible to use (1.7) of Conjecture 1.1 to get $\mathfrak{p}$-adic information about u_T. For example, if the set of finite places of S consists *exactly* of the primes which divide $\mathfrak{fp}$ then one may consider the tower of fields $L_n = F(\mathfrak{fp}^n\infty)$ with n increasing. In this case, class field theory provides an isomorphism

$$(A.7) \qquad \qquad rec_{\mathfrak{p}} : F_{\mathfrak{p}}^\times / \widehat{E}_{\mathfrak{p}}(\mathfrak{f}) \xrightarrow{\cong} \mathrm{Gal}(L_\infty/M),$$

where $L_\infty = \bigcup_n L_n$, $E_{\mathfrak{p}}(\mathfrak{f})$ denotes the group of totally positive $\mathfrak{p}$-units of F which are congruent to 1 modulo $\mathfrak{f}$ and $\widehat{E}_{\mathfrak{p}}(\mathfrak{f})$ denotes its closure in $F_{\mathfrak{p}}^\times$. Let

$$g_T := \varprojlim_n rec_{\mathfrak{p}}^{L_n}(u_T) \in \mathrm{Gal}(L_\infty/M),$$

where the transition maps are given by the restrictions. Then (1.7) predicts that

$$(A.8) \qquad \qquad rec_{\mathfrak{p}}^{-1}(g_T) \equiv u_T \pmod{\widehat{E}_{\mathfrak{p}}(\mathfrak{f})}.$$

In general the $\mathbb{Z}_p$-rank of $\widehat{E}_{\mathfrak{p}}(\mathfrak{f})$ will be larger than one and therefore (A.8) does not provide an exact formula for u_T. It is explained in [Das08] that by expanding the set S in an appropriate way one may gain more $\mathfrak{p}$-adic information about u_T. In Section 5.4 of [Das08], Dasgupta shows that by repeating this process indefinitely one can specify u_T in $F_{\mathfrak{p}}^\times$ to any specified degree of $\mathfrak{p}$-adic accuracy. However there is a lack of explicitness in this process which makes it not very suitable for numerical computations. Also, even from a theoretical point of view, this process is not completely satisfactory since it is rather indirect.

We would like to mention one more key result which follows from Lemma 5.17 of [Das08] and (A.6) which is not stated explicitly in [Das08].

Proposition A.1 *(Dasgupta) Assume that M is linearly disjoint from $F(\zeta_{p^m})$ for all $m \geq 1$. Let τ_∞ be a complex conjugation of M/F and suppose that $\mathfrak{c}$ is an $\mathcal{O}_K$-ideal of F such that $\sigma_{\mathfrak{c}} = \tau_\infty$. Then*

$$(A.9) \qquad \qquad u_D(\mathfrak{ac}, \mathfrak{f}) = u_D(\mathfrak{a}, \mathfrak{f})^{-1}.$$

Even when M is not linearly disjoint from $F(\zeta_{p^m})$ it is expected that (A.9) holds. The author does not know of a proof of (A.9) which avoids the artificial process of enlarging the set of places S. It would be quite interesting to find a more direct proof of (A.9).

B A norm formula from u_C to u_{DD}

We now discuss the compatibility of our p-adic invariant with the one constructed by Darmon and Dasgupta. One can relate the modular unit considered in [Cha09c] with the modular units used in [DD06]. We have for any positive integer N the identity

$$\text{(B.1)} \qquad \prod_{j=1}^{N-1} g_{(\frac{j}{N},0)}(N\tau)^{12} = \zeta \frac{\Delta(\tau)}{\Delta(N\tau)},$$

for some $\zeta \in \mu_N$ where

$$g_{(\frac{j}{N},0)}(\tau) = q_\tau^{\frac{1}{2}\widetilde{B}_2(\frac{j}{N})}(1 - q_{\frac{j\tau}{N}}) \prod_{n \geq 1}(1 - q_\tau^n q_{\frac{j\tau}{N}})(1 - q_\tau^n q_{\frac{-j\tau}{N}}),$$

and

$$\Delta(\tau) = q \prod_{n \geq 1}(1 - q^n)^{24},$$

where $q = e^{2\pi i \tau}$, $\tau \in \mathcal{H} = \{x + iy \in \mathbb{C} : y > 0\}$. Here $\widetilde{B}_2(x) = B_2(\{x\})$ where $B_2(x) = x^2 - x + \frac{1}{6}$ and $0 \leq \{x\} < 1$ denotes the fractional part of x. We note that

$$g_{0,0}(\tau)^{12} = \Delta(\tau).$$

We can thus rewrite (B.1) as

$$\text{(B.2)} \qquad \prod_{j=0}^{N-1} g_{(\frac{j}{N},0)}(N\tau)^{12} = \zeta \Delta(\tau),$$

for some $\zeta \in \mu_N$.

As in [DD06], choose a divisor $\delta = \sum_{d|N} n_d[d] \in Div(N)$ such that

$$\sum_{d|N} n_d d = 0 \text{ and } \sum_{d|N} n_d = 0.$$

To such a divisor Darmon and Dasgupta associate the modular unit

$$\text{(B.3)} \qquad \alpha_\delta(\tau) := \prod_{d|N} \Delta(d\tau)^{n_d},$$

which is $\Gamma_0(N)$-invariant. More generally, to a divisor $\delta = \sum_{d|N, r \in \mathbb{Z}/f\mathbb{Z}} n(d,r)[d,r] \in Div_f(N)$ we associate the modular unit

$$\text{(B.4)} \qquad \beta_\delta(z) := \prod_{d|N, r \in \mathbb{Z}/f\mathbb{Z}} g_{(\frac{r}{f},0)}(fd\tau)^{12n(d,r)}.$$

If we set $\delta' = \sum_{d|N, r \in \mathbb{Z}/f\mathbb{Z}} n(d, r)[d, r] \in Div_f(N)$ with $n(d, r) = n_d$ for all $r \in \mathbb{Z}/f\mathbb{Z}$, then we readily see that δ' is a good divisor with respect to any prime number p. Using equation (B.1) with $N = f$ we find

$$\beta_{\delta'}(\tau) = \prod_{d|f, r \in \mathbb{Z}/f\mathbb{Z}} g_{(\frac{r}{f}, 0)}(fd\tau)^{12n(d,r)}$$

$$= \zeta \prod_d \Delta(d\tau)^{n_d}$$

(B.5)
$$= \zeta \alpha_\delta(\tau),$$

for some $\zeta \in \mu_f$.

Let p be a prime number inert in K. Assume that (p, f, N) satisfy the usual assumptions and let $\tau \in H^{\mathcal{O}_f}(\mathfrak{N}, f)$ where $\mathcal{O}_f$ stands for the unique order of conductor f of K. By definition of δ' and $u_{C,\delta'}(1, \tau)$ we find that

$$u_{C,\delta'}(1, \tau) = \prod_{r=0}^{f-1} u_{C,\delta}(r, \tau).$$

Using (B.5) we deduce directly from the definitions of the p-adic invariants u_C and u_{DD} (see [DD06] for the definition of u_{DD}) that

$$u_{C,\delta'}(1, \tau) = u_{DD,\delta}(\tau).$$

Therefore it follows that

(B.6)
$$\prod_{r=0}^{f-1} u_{C,\delta}(r, \tau) = u_{DD,\delta}(\tau).$$

Conjecture 2.14 of [DD06] predicts that $u_{DD,\delta}(\tau) \in K(\mathcal{O}_f\infty)$. Conjecture 9.1 predicts that for each $r \in \mathbb{Z}/f\mathbb{Z}$, $u_{C,\delta}(r, \tau)$ is a strong p-unit of $K(f\mathcal{O}_f\infty)$. Now let us make the following additional assumption:

Assumption B.1 Let $\mathcal{O}_K = \mathbb{Z} + \omega\mathbb{Z}$ and let ϵ be the generator of $\mathcal{O}_K(f\infty)^\times$ such that $\epsilon > 1$. Assume that $\epsilon = u + vf\omega$ with $u, v \in \mathbb{Z}$ and $(v, f) = 1$.

Note that $u \equiv 1 \pmod{f}$ and that $\epsilon \in \mathcal{O}_f^\times$. Then we have the following proposition:

Proposition B.1 *Conjecture 9.1 and Assumption B.1 imply that for each $d|f$ one has*

(B.7)
$$\prod_{\substack{0 \leq r \leq f-1 \\ (r,f)=d}} u_{C,\delta}(r, \tau) \in K(\mathcal{O}_f\infty),$$

where $K(\mathcal{O}_f\infty)$ stands for the narrow ring class field of K of conductor f. In particular, one has

(B.8)
$$\prod_{r=0}^{f-1} u_{C,\delta}(r, \tau) \in K(\mathcal{O}_f\infty).$$

Proof The generalized ideal class group $I_{\mathcal{O}_f}(1)/P_{\mathcal{O}_f}(\infty)$ corresponds from class field theory to the narrow ring class field extension $K(\mathcal{O}_f\infty)$. Here

$$P_{\mathcal{O}_f}(\infty) = \left\{ \frac{\alpha}{\beta}\mathcal{O}_f : \alpha, \beta \in \mathcal{O}_f, \frac{\alpha}{\beta} \gg 0 \right\},$$

and two ideals $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}_f}(1)$ are equivalent modulo $P_{\mathcal{O}_f}(\infty)$ if and on if there exists $\lambda\mathcal{O}_f \in P_{\mathcal{O}_f}(\infty)$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Similarly, one may define $I_{\mathcal{O}_f}(f)/P_{\mathcal{O}_f}(f\infty)$ where

$$P_{\mathcal{O}_f}(f\infty) = \left\{ \frac{\alpha}{\beta}\mathcal{O}_f : \alpha, \beta \in \mathcal{O}_f, (\alpha\mathcal{O}_f, f\mathcal{O}_f) = (\beta\mathcal{O}_f, f\mathcal{O}_f) = \mathcal{O}_f, \frac{\alpha}{\beta} \gg 0 \right\}.$$

One can show that the natural map

(B.9)
$$I_{\mathcal{O}_f}(f)/P_{\mathcal{O}_f}(f\infty) \xrightarrow{\simeq} I_{\mathcal{O}_f}(1)/P_{\mathcal{O}_f}(\infty)$$
$$[\mathfrak{a}] \mapsto [\mathfrak{a}],$$

is an isomorphism. In order to show that (B.7) lies in $K(\mathcal{O}_f\infty)$ it is enough to show that $ker(res)$ acts trivially on the left hand side of (B.7) where

$$res : \mathrm{Gal}(K(f\mathcal{O}_f\infty)/K) \to \mathrm{Gal}(K(\mathcal{O}_f\infty)/K).$$

Using class field theory and (B.9) one has that the restriction map above corresponds to the map

$$\pi : I_{\mathcal{O}_f}(f)/P_{\mathcal{O}_f,1}(f\infty) \to I_{\mathcal{O}_f}(f)/P_{\mathcal{O}_f}(f\infty)$$
$$[\mathfrak{a}] \mapsto [\mathfrak{a}].$$

Therefore the subgroup $ker(res)$ corresponds to $\left(P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty)\right)/P_{\mathcal{O}_f,1}(f\infty)$. Let $\Theta : I_{\mathcal{O}_f}(f) \to I_{\mathcal{O}_{f^2}}(f)$ be the map given by $\mathfrak{a} \mapsto \mathfrak{a} \cap \mathcal{O}_{f^2}$. Using formula (9.6) we see that in order to show that (B.7) is in $K(\mathcal{O}_f\infty)$ it is enough to show that the collection of ideal classes

(B.10)
$$\{\widetilde{r}I_{\tau^*} \pmod{\sim_f} : 1 \le r \le f \text{ and } (r, f) = d\}$$

is stable as a set under the action of $\Theta(P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty))$. A direct computation shows that

$$P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty) = \{\alpha\mathcal{O}_f : \alpha = a + bf\omega, a, b \in \mathbb{Z}, (a, f) = 1 \text{ and } \alpha \gg 0\}.$$

We note that if $\alpha = a + bf\omega$ and $b \equiv 0 \mod f$ then $a + bf\omega \in \mathcal{O}_{f^2}$. Moreover, one may check that $\Theta(\alpha\mathcal{O}_f) = \alpha\mathcal{O}_{f^2}$ and therefore $\Theta(\alpha\mathcal{O}_f)$ is again a principal ideal. Now let $\alpha\mathcal{O}_f \in P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty)$. Then using Assumption B.1 one sees that it is always possible to multiply α by a suitable power of ϵ so that $\epsilon^m\alpha := \alpha' = a' + b'f\omega$ with $b' \equiv 0 \pmod{f}$. It thus follows that

$$\Theta(P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty)) = \{\alpha\mathcal{O}_{f^2} : \alpha = a + bf^2\omega, a, b \in \mathbb{Z}, (a, f) = 1 \text{ and } \alpha \gg 0\}.$$

Now let ϵ be as in Assumption B.1 and note that $\epsilon' = \epsilon^f = u' + v'f^2\omega$ for $u', v' \in \mathbb{Z}$, $u' \equiv 1$ (mod f) and $(v', f) = 1$. Now let $\alpha\mathcal{O}_{f^2} \in \Theta(P_{\mathcal{O}_f}(f\infty) \cap I_{\mathcal{O}_f}(f\infty))$ where $\alpha = a + bf^2\omega$. In a similar way to what we did previously we see that we may multiply α by a suitable power of ϵ' so that $(\epsilon')^{m'}\alpha := \alpha' = a' + b'f^2\omega$ with $b' \equiv 0$ (mod f). Now note that the multiplication by α' only reshuffles the ideal classes in (B.10). Therefore the Shimura reciprocity law of Conjecture 9.1 predicts that $\prod_{\substack{1 \leq r \leq f \\ (r,f)=d}} u_{C,\delta}(r, \tau)$ lies in $K(\mathcal{O}_f\infty)$. This concludes the proof.
$\square$

Remark B.1 The author thinks that Conjecture 9.1 alone (i.e. without the help of Assumption B.1) should imply relation (B.8).

Proposition B.2 *The identity* (10.1) *in Proposition 10.1 implies B.6 with the following parameters:* $\tau \in H^{\mathcal{O}_f}(\mathfrak{N}, f)$, $\mathfrak{b}^{-1} = \Lambda_{\frac{1}{N\tau^\sigma}}$, $\mathfrak{g} = \mathcal{O}_f$, $(\mathfrak{b}')^{-1} = \Lambda_{(\tau^*)^\sigma}$, $\mathfrak{g}' = f\mathcal{O}_{(\tau^*)^\sigma}$ *where* $\tau^* = \frac{-1}{fN\tau}$.

Proof We suppress the divisor δ for the notation. First note that $\text{End}_K(\mathfrak{b}) = \mathcal{O}_f$ and that $\text{End}_K(\mathfrak{b}') = \mathcal{O}_{f^2}$. Moreover, if $Q_\tau(x, y) = Ax^2 + Bxy + Cy^2$ $((A, f) = 1)$ then we have $B^2 - 4AC = f^2 d_K$. A direct computation shows that

$$Q_{\tau^*}(x, y) = \text{sign}(C)\left(CNf^2x^2 - Bfxy + \frac{A}{N}y^2\right),$$

so that

(B.11) $$\tau^* = \frac{B + f\sqrt{d_K}}{fCN}.$$

On one hand we have

(B.12) $$u_D(\mathfrak{b}, \mathcal{O}_f)^{12} = u_D(I_{\frac{1}{N\tau}}, \mathcal{O}_f)^{12} = u_{DD}(\tau),$$

where the first equality follows from the identity $\mathfrak{b} = I_{\frac{1}{N\tau}}$. For the second equality see the discussion in Remark 10.2.

Now for every $r \in \mathbb{Z}/f\mathbb{Z}$ we have that

(B.13) $$u_D(\widetilde{r}\mathfrak{b}', \mathfrak{g}')^{12} = u_D(\widetilde{r}I_{\tau^*}, f\mathcal{O}_{\tau^*})^{12}.$$

Using (B.13) in (10.1) of Proposition 10.1 we find that

(B.14) $$\prod_{r=1}^f u_D(\widetilde{r}\mathfrak{b}', \mathfrak{g}')^{12h_r} = u_D(\mathfrak{b}, \mathcal{O}_f)^{12e},$$

where

$$e = [\Gamma_\mathfrak{b}(\mathfrak{g}) : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')] \quad \text{and} \quad h_r = [\Gamma_{\widetilde{r}\mathfrak{b}'}(\mathfrak{g}') : \Gamma_{\mathfrak{b}'}(\mathfrak{g}')].$$

Now note that $\{\widetilde{r}\}_{\widetilde{r}=1}^f$ is a complete set of totally positive representatives of $\mathfrak{b}^{-1}\mathfrak{g}/(\mathfrak{b}')^{-1}\mathfrak{g}'$. Using Proposition 9.3 we find that

(B.15)
$$u_D(\widetilde{r}I_{\tau^*}, f\mathcal{O}_{\tau^*})^{12\nu(r,\tau)^{-1}} = u_C(r,\tau),$$

$$\nu(r,\tau) = [\Gamma_{\widetilde{r}\mathfrak{b}'}(\mathfrak{g}') : \epsilon(\langle\eta_{\tau^*}\rangle)]^{-1} \in \mathbb{Q}_{>0}.$$

For the definitions of $\epsilon(\eta_\tau)$ and $\epsilon(\eta_{\tau^*})$ (see Definition 4.1). Using (B.11), direct computations show that

$$\mathcal{O}_{f^2}(\infty)^\times \cap \mathcal{O}_K(f\infty)^\times = \Gamma_{\mathfrak{b}'}(\mathfrak{g}') = \Gamma_\mathfrak{b}(\mathfrak{g}).$$

In particular, we have

(B.16)
$$\nu(r,\tau) = [\Gamma_{\mathfrak{b}'}(\mathfrak{g}') : \Gamma_{\widetilde{r}\mathfrak{b}'}(\mathfrak{g}')] = h_r^{-1} \quad \text{and} \quad e = 1.$$

Using (B.16) in (B.14) we find that

(B.17)
$$\prod_{r=1}^f u_D(\widetilde{r}\mathfrak{b}', \mathfrak{g}')^{12\nu(r,\tau)^{-1}} = u_D(\mathfrak{b}, \mathfrak{f})^{12}.$$

Finally, combining (B.17) with (B.15) and (B.12) we find that

$$\prod_{r=1}^f u_C(r,\tau) = u_{DD}(\tau).$$

This concludes the proof. $\quad\square$

C   Additional proofs

C.1   Proof of Proposition 3.1

Let $(r,\tau), (r',\tau') \in (\mathbb{Z}/f\mathbb{Z} \times H^\mathcal{O}(\mathfrak{N}))/\sim_f$ and let $I_\tau = A\Lambda_\tau$, $I_{\tau'} = A'\Lambda_{\tau'}$ where $Q_\tau(x,y) = Ax^2 + Bxy + Cy^2$ and $Q_{\tau'}(x,y) = A'x^2 + B'xy + C'y^2$. We have $\mathrm{End}_K(\Lambda_\tau) = \mathrm{End}_K(\Lambda_{\tau'}) = \mathcal{O}$ where $\mathrm{disc}(\mathcal{O}) = D = B^2 - 4AC = B'^2 - 4A'C'$.

We first show that the map $\widetilde{\psi}$ is well defined, i.e., if $(r,\tau) \sim_f (r',\tau')$ then $\psi(r,\tau) \sim_f \psi(r',\tau')$. So assume that there exists a matrix $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(fN)$ such that $\gamma\tau = \tau'$ and $d^{-1}r \equiv r' \pmod{f}$.

We want to show that $[\widetilde{r}A\Lambda_\tau, \widetilde{r}A\Lambda_{N\tau}] \sim_f [\widetilde{r}'A'\Lambda'_\tau, \widetilde{r}'A'\Lambda'_{N\tau}]$. We have

(C.1)
$$\Lambda_\tau = (c\tau + d)\Lambda_{\tau'},$$

and

(C.2)
$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}\begin{pmatrix} \tau & \tau^\sigma \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} c\tau + d & 0 \\ 0 & c\tau^\sigma + d \end{pmatrix}\begin{pmatrix} \tau' & (\tau')^\sigma \\ 1 & 1 \end{pmatrix}.$$

Taking the determinant of (C.2) we find

$$(C.3) \qquad \frac{(\tau - \tau^\sigma)}{(\tau' - \tau'^\sigma)} = (c\tau + d)(c\tau^\sigma + d).$$

Since $\tau = \frac{-B+\sqrt{D}}{2A}$ and $\tau' = \frac{-B'+\sqrt{D}}{2A'}$ we find $(\tau - \tau^\sigma) = \frac{\sqrt{D}}{A}$ and $(\tau' - \tau'^\sigma) = \frac{\sqrt{D}}{A'}$. Substituting in (C.3) we find

$$\frac{A'}{A}(c\tau + d)^{-1} = (c\tau^\sigma + d).$$

From (C.1) we may deduce that $\frac{A'}{A}(c\tau + d)^{-1}(A\Lambda_\tau) = A'\Lambda_{\tau'}$ and thus $\frac{\widetilde{r'}}{\widetilde{r}}(c\tau^\sigma + d)(\widetilde{r}A\Lambda_\tau) = \widetilde{r'}A'\Lambda_{\tau'}$. Since $c \equiv 0 \pmod{f}$ and $\frac{\widetilde{r'}}{\widetilde{r}}d \equiv 1 \pmod{f}$ we see that

$$\frac{\widetilde{r'}}{\widetilde{r}}(c\tau^\sigma + d) \in (\widetilde{r}A\Lambda_\tau)^{-1}f + 1 = \frac{1}{\widetilde{r}}\Lambda_{\tau^\sigma}f + 1.$$

Moreover, since $N|c$ we also have

$$\frac{\widetilde{r'}}{\widetilde{r}}(c\tau^\sigma + d)(\widetilde{r}A\Lambda_{N\tau}) = \widetilde{r'}A'\Lambda_{N\tau'}.$$

It thus follows that $\psi(r, \tau) \sim_f \psi(r', \tau')$.

Let us now show now that the map ψ is injective. Assume that $\psi(r, \tau) \sim_f \psi(r', \tau')$, i.e.,

$$[\widetilde{r}A\Lambda_\tau, \widetilde{r}A\Lambda_{N\tau}] \sim_f [\widetilde{r'}A'\Lambda_{\tau'}, \widetilde{r'}A'\Lambda_{N\tau'}].$$

Then there exists a $\lambda \in (\widetilde{r}A\Lambda_\tau)^{-1}f + 1 = \frac{1}{\widetilde{r}}\Lambda_{\tau^\sigma}f + 1$ such that

$$(C.4) \qquad \lambda\widetilde{r}A\Lambda_\tau = \widetilde{r'}A'\Lambda_{\tau'} \quad \text{and} \quad \lambda\widetilde{r}A\Lambda_{N\tau} = \widetilde{r'}A'\Lambda_{N\tau'}.$$

We note that A' is the smallest integer such that $A'\Lambda_{\tau'} \subseteq \mathcal{O}$ where $\mathcal{O} = \mathcal{O}_\tau = \mathcal{O}_{\tau'}$. Write $\lambda = \frac{1}{\widetilde{r}}(c\tau^\sigma + d)$ with $c, d \in \mathbb{Z}$, $c \equiv 0 \pmod{f}$ and $d \equiv \frac{\widetilde{r}}{\widetilde{r'}} \pmod{f}$ and let $(c, d) = N$. We claim that $N = 1$. If not then

$$\frac{1}{N}(c\tau^\sigma + d)A\Lambda_\tau \subseteq \mathcal{O}.$$

Therefore we would have $\frac{A'}{N}\Lambda_{\tau'} \subseteq \mathcal{O}$ which contradicts the minimality of A'. Since $(c, d) = 1$ there exists $a, b \in \mathbb{Z}$ such that $ad - bc = 1$. Note that $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$. Let $\tau'' = \gamma\tau$. Then we have

$$(C.5) \qquad (c\tau + d)\Lambda_{\tau''} = \Lambda_\tau.$$

We let $Q_{\tau''}(x, y) = A''x^2 + B''xy + C''y^2$. We have

$$\frac{A''}{A} = (c\tau^\sigma + d)(c\tau + d).$$

Therefore from (C.5) we deduce that

(C.6)
$$A''\Lambda_{\tau''} = (c\tau^\sigma + d)A\Lambda_\tau.$$

Combining (C.4) with (C.6) we get

$$A''\Lambda_{\tau''} = A'\Lambda_{\tau'}.$$

From this we deduce that $A'' = A'$ and $\tau'' = (\tau' + n)$ for some $n \in \mathbb{Z}$. From this it follows that

$$\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} \tau \\ 1 \end{pmatrix} = (c\tau + d) \begin{pmatrix} \tau' \\ 1 \end{pmatrix}.$$

Since $\begin{pmatrix} 1 & -n \\ 0 & 1 \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \Gamma_0(f)$. We thus have shown that $(r, \tau) \sim_f (r', \tau')$ if and only if $\psi(r, \tau) \sim_f \psi(r', \tau')$. From this it follows that the map ψ is well defined and injective. It remains to show that ψ is surjective.

Let $\mathfrak{a}$ be an arbitrary integral invertible $\mathcal{O}_n$-ideal. For a vector $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$ we let

$$\left\langle \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} \right\rangle = \mathbb{Z}v_1 + \mathbb{Z}v_2,$$

be the $\mathbb{Z}$-lattice generated by v_1 and v_2. We can always find $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in M_2(\mathbb{Z})$ such that

$$\left\langle \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} n\omega_1 \\ 1 \end{pmatrix} \right\rangle = \mathfrak{a}.$$

Let $(a, c) = m$ and $a' = a/m$, $c' = c/m$. We thus have $(a', c') = 1$. Let $u, v \in \mathbb{Z}$ be such that $-ua' - vc' = 1$. Then

$$\left\langle \begin{pmatrix} A & B \\ 0 & D \end{pmatrix} \begin{pmatrix} n\omega_1 \\ 1 \end{pmatrix} \right\rangle = \mathfrak{a}.$$

where

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix} = \begin{pmatrix} u & v \\ c' & -a' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

We thus have $\mathfrak{a} = \mathbb{Z}(An\omega + B) + \mathbb{Z}D$. So $\mathfrak{a} = D(\mathbb{Z}\frac{An\omega+B}{D} + \mathbb{Z})$. Let $(A, B, D) = r$ and let $\frac{A}{r} = A'$, $\frac{B}{r} = B'$ and $\frac{C}{r} = C'$. Without loss of generality we may assume that and $D', A' > 0$. Now set $\tau = \frac{A'n\omega+B'}{D'}$. Note that $\tau > \tau^\sigma$ so that $(r, \tau) \in \mathbb{Z}/f\mathbb{Z} \times H^{\mathcal{O}_\tau}(\mathfrak{N})$. By construction we have $\widetilde{\psi}(r, \tau) = [\mathfrak{a}]$. Since $\mathfrak{a}$ was arbitrary it follows that $\widetilde{\psi}$ is surjective. $\square$

C.2 Proof of Proposition 2.1

Recall that $\mathcal{O} = \mathcal{O}_m$ and that $\mathcal{O}' = \mathcal{O}_{mn}$, $\mathfrak{f} = f\mathcal{O}$ and $\mathfrak{f}' = f\mathcal{O}'$. Let $\mathfrak{a}, \mathfrak{a}' \in I_{\mathcal{O}}(\mathfrak{f}n)$ and write $\mathfrak{a}$ as $\mathfrak{a} = rA_\tau \Lambda_\tau$ with $r \in \mathbb{Z}_{\geq 1}$ and $\tau \in K \backslash \mathbb{Q}$. Since $(\mathfrak{a}, \mathfrak{f}) = \mathcal{O}$ this implies that $(rA_\tau, fn) = 1$. Because $(rA_\tau, n) = 1$ we have that $\Theta(\mathfrak{a}) = \mathfrak{a} \cap \mathcal{O}' = rA_\tau \Lambda_{n\tau}$. Now note that $\mathrm{End}_K(\Lambda_{n\tau}) = \mathcal{O}'$ and that $(\mathfrak{a} \cap \mathcal{O}', \mathfrak{f}') = \mathcal{O}'$, so that indeed, the image of Θ lies in $I_{\mathcal{O}'}(\mathfrak{f}')$. Let us show that Θ is multiplicative. Since $\Theta(\mathfrak{a}\mathfrak{a}') \supseteq \Theta(\mathfrak{a})\Theta(\mathfrak{a}')$ the multiplicativity of Θ is equivalent to

$$(\text{C.7}) \qquad\qquad [\mathcal{O}' : \Theta(\mathfrak{a})\Theta(\mathfrak{a}')] = [\mathcal{O}' : \Theta(\mathfrak{a}\mathfrak{a}')].$$

Since $\mathfrak{a}$ is $\mathcal{O}$-invertible we have

$$(\text{C.8}) \qquad\qquad [\mathcal{O} : \mathfrak{a}\mathfrak{a}'] = [\mathcal{O} : \mathfrak{a}][\mathcal{O} : \mathfrak{a}'].$$

For an arbitrary ideal $\mathfrak{b} \in I_{\mathcal{O}}(\mathfrak{f}n)$ one may check that

$$(\text{C.9}) \qquad\qquad [\mathcal{O} : \mathfrak{b}] = [\mathcal{O}' : \Theta(\mathfrak{b})].$$

Now (C.7) follows from (C.9) and (C.8). This shows (i).

Let us show (ii). Let us assume that $\mathfrak{a} \sim_{fn} \mathfrak{a}'$, i.e., $\lambda \mathfrak{a} = \mathfrak{a}'$ for $\lambda \in \mathfrak{a}^{-1} fn + 1$. Since $\mathfrak{a}^{-1} = \frac{1}{r}\Lambda_{\tau^\sigma}$ this implies that $\lambda \in \frac{fn}{r}\Lambda_{\tau^\sigma} + 1$. Since $(rA_\tau, fn) = 1$ we can always find a positive integer b such that $b\lambda \in \mathcal{O}$ and $b \equiv 1 \pmod{fn}$ so that $\mathfrak{a}' \sim_{fn} b\mathfrak{a}'$. Therefore, without loss of generality we may assume that $\lambda \in \mathcal{O} \cap \left(\frac{fn}{r}\Lambda_{\tau^\sigma} + 1\right) = (1 + fn\mathcal{O})$. A direct computation shows that $\Theta(\lambda\mathcal{O}) = \lambda\mathcal{O}'$ and $\Theta(\mathfrak{a}') = \Theta(\lambda\mathcal{O})\Theta(\mathfrak{a})$. From this we find that $\lambda r A_\tau \Lambda_{n\tau} = \Theta(\mathfrak{a}')$. Finally, since

$$\lambda \in 1 + fn\mathcal{O} \subseteq \frac{f}{r}\Lambda_{n\tau} + 1,$$

we find that $\Theta(\mathfrak{a}) \sim_f \Theta(\mathfrak{a}')$. We thus have proved the implication $\mathfrak{a} \sim_{fn} \mathfrak{a}' \Longrightarrow \Theta(\mathfrak{a}) \sim_f \Theta(\mathfrak{a}')$. Therefore the map Θ gives rise to a map $\widetilde{\Theta} : C_{\mathcal{O}}(n\mathfrak{f}) \to C_{\mathcal{O}'}(\mathfrak{f}')$. It remains to show that the map $\widetilde{\Theta}$ is onto. So let $[\mathfrak{b}] \in C_{\mathcal{O}'}(\mathfrak{f}')$. We may write $\mathfrak{b}$ as $\mathfrak{b} = rA_\tau \Lambda_\tau$ with $r, A_\tau \in \mathbb{Z}_{\geq 1}$, $(rA_\tau, f) = 1$ and $\tau \in K\backslash\mathbb{Q}$ such that $\mathcal{O}_\tau = \mathcal{O}'$. We claim that we can always find a matrix $\gamma \in \Gamma_1(f)$ such that $\gamma\tau = \tau'$ and $(A_{\tau'}, fn) = 1$. Assuming this, let $Q_{\tau'}(x, y) = A'x^2 + B'xy + C'y^2$ so that $\tau' = \frac{-B' + mn\sqrt{d_K}}{2A'}$. Set $\rho = \frac{-B' + m\sqrt{d_K}}{2A'}$ and note that $\mathcal{O}_\rho = \mathcal{O}_m = \mathcal{O}$ and that $rA_{\tau'}\Lambda_\rho$ is integral. Since $rA_\tau\Lambda_\tau \sim_f rA_{\tau'}\Lambda_{\tau'}$ and $\Theta\left(rA_{\tau'}\Lambda_\rho\right) = rA_{\tau'}\Lambda_{\tau'}$, we see that Θ is onto. We leave the proof of the existence of the matrix γ to the reader. $\square$

C.3 Proof of Corollary 8.3

First note that for $x \in \mathbb{R}$ one has

$$\widetilde{B}_1(x) - B_1^*(x) = \begin{cases} 0 & \text{if} \quad 0 < x < 1 \\ -\frac{1}{2} & \text{if} \quad x \in \mathbb{Z}. \end{cases}$$

Moreover, for $x \in \mathbb{R}$ one has that

(C.10)
$$\widetilde{B}_1(-x) = -\widetilde{B}_1(x)$$

Let

$$y_1(h) = \frac{a}{\frac{c}{fd}}\left(h + \frac{j}{f} + \frac{v}{p^n}\right) - \frac{fdu}{p^n}, \quad y_2(h) = \frac{1}{\frac{c}{fd}}\left(h + \frac{j}{f} + \frac{v}{p^n}\right), \quad y_2'(h) = ly_2(h),$$

be the arguments appearing the $\widetilde{B}_1$'s of (8.2).

Comparing (8.2) with (8.3) we readily see that the two expressions are the same except that in (8.2) one evaluates $\widetilde{B}_1$ instead of B_1^*. First note that if $f > 1$ and $j \neq 0$ then for all $1 \le h \le \frac{c}{fd}$, one has $y_1(h), y_2(h), y_2'(h) \in \mathbb{Q}\backslash\mathbb{Z}$ and therefore $\xi(U_{u,v,n}) = 0$. So it is enough to prove (8.12) when $j = 0$. Note that since $U_{u,v,n}$ is a ball contained in $\mathbb{X}$ one has that $n \ge 1$ and $(u,v) \not\equiv (0,0) \pmod{p}$. Note in particular that for a fixed value of h one cannot have both $y_1(h)$ and $y_2(h)$ in $\mathbb{Z}$. We also note that if $y_2(h) \in \mathbb{Z}$ then $y_2'(h) \in \mathbb{Z}$. Let

$$R^*(y_1(h), y_2(h), y_2'(h), u, v) := B_1^*(y_1(h))\left[\left(\frac{N}{d}\right)B_1^*(y_2(h)) - \left(\frac{N}{d'}\right)B_1^*(y_2'(h))\right],$$

and similarly define $\widetilde{R}(y_1(h), y_2(h), y_2'(h), u, v)$ by replacing B_1^* by $\widetilde{B}_1$ in the expression above. In order to show that $\xi(U_{u,v,n}) + \xi(U_{-u,-v,n}) = 0$ it is enough to show that

(C.11)
$$\sum_{\substack{1 \le h \le \frac{c}{fd} \\ y_1(h) \in \mathbb{Z} \lor y_2(h) \in \mathbb{Z} \lor y_2'(h) \in \mathbb{Z}}} [R^*(y_1(h), y_2(h), y_2'(h), u, v) + R^*(y_1(h), y_2(h), y_2'(h), -u, -v)],$$

is equal to zero and similarly if one replaces R^* by $\widetilde{R}$. We will treat the following two cases:

(1) (1st case $v \equiv 0 \pmod{p^n}$). Note that in this case for all h one has $y_1(h) \notin \mathbb{Z}$. Note that the two following 5-tuples can be paired:

$$(y_1(h), y_2(h), y_2'(h), u, 0) \longleftrightarrow (-y_1(h), y_2(h), y_2'(h), -u, 0).$$

Therefore using (C.10) we see that (C.11) equals zero.

(2) (2nd case $v \not\equiv 0 \pmod{p^n}$) In this case $y_2(h) \notin \mathbb{Z}$ and $y_2'(h) \notin \mathbb{Z}$. Therefore we may assume that $y_1(h) \in \mathbb{Z}$. Note that the following two 5-tuples can be paired:

$$(y_1(h), y_2(h), y_2'(h), u, v) \longleftrightarrow (-y_1(h), -y_2(h), -y_2'(h), -u, -v).$$

Therefore using (C.10) one obtains that (C.11) is equal to zero.

A similar argument applies if one replaces R^* by $\widetilde{R}$. This concludes the proof of (8.12). Finally, the fact that $\frac{1}{3}\xi(U_{u,v,n})$ is an integer comes from the observation that

$$4R^*(y_1(h), y_2(h), y_2'(h), u, v) \in \mathbb{Z}$$

and similarly if one replaces R^* by $\widetilde{R}$. $\quad\square$

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