Some arithmetic properties of partial zeta functions weighted by a sign character

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Abstract

We introduce two types of zeta functions (Ψ-type and ζ-type) of one complex variable associated to an arbitrary number field K. We prove various arithmetic identities which involve both of them. We also study their special values at integral arguments.

Contents

1 Introduction

We introduce two types of zeta functions of a complex variable s which depend on the choice of a fixed order $\mathcal{O}$ of a number field K and some additional data. The first type may be viewed as a zeta function whose general term is weighted by the product of a sign character of $K^\times$ with a certain additive character of K. The second type may be viewed as a partial zeta function of K whose general term is weighted by a sign character of $K^\times$. Special cases of these two types of zeta functions appear in the work of Hecke (see [Hec59]) and Siegel (see [Sie68] and [Sie70]). Some of their arithmetic properties were known to Siegel (at least in the case where $\mathcal{O}$ is the maximal order of K) and probably to the experts. But to the author's best knowledge, no systematic treatment of them can be found in the literature. This paper concentrates on the main arithmetic properties on these two types of zeta functions.

We now briefly describe the content of this paper. In Section 2 we define functions of Ψ-type and ζ-type and we prove some key identities for both of these. In Section 3 we recall a functional equation (proved in [Cha09]) which relates these two types under the change of variables $s \mapsto 1-s$. We also show that a function of ζ-type can be written as a certain linear combination of functions of Ψ-type and vice versa. In Section 4 we set some notation about Hecke characters and Gauss sums. This allows us to give a precise relation between zeta functions of Ψ-type and L-functions associated to a primitive Hecke character. In Section 5 we study their special values at integral arguments which we relate to special values of classical partial zeta functions.

In a forthcoming paper, the author would like to construct certain Eisenstein series such that the constant term of their q-expansion is a special value at some negative integer of a function of Ψ-type or ζ-type. Once this is done one can then apply the so-called q-expansion principle to these Eisenstein series to study p-divisibility properties of these special values.

Notation

Let K be a number field of degree n over $\mathbb{Q}$ and let $\mathcal{O}_K$ be its maximal $\mathbb{Z}$-order. Let $\mathcal{O}$ be a $\mathbb{Z}$-order of K. A discrete $\mathcal{O}$-module $\Lambda \subseteq K$ will be called an $\mathcal{O}$-ideal. By an invertible $\mathcal{O}$-ideal (or $\mathcal{O}$-invertible ideal) we mean an $\mathcal{O}$-ideal Λ such that $End_K(\Lambda) = \{\lambda \in K : \lambda\Lambda \subseteq \Lambda\} = \mathcal{O}$. Note that every $\mathbb{Z}$-lattice $\Lambda \subseteq K$ (of maximal rank) is an invertible $\mathcal{O}$-ideal for $\mathcal{O} = End_K(\Lambda)$. A $\mathbb{Z}$-lattice $\Lambda \subseteq K$ (of maximal rank) will be called *integral* if $\Lambda \subseteq End_K(\Lambda)$. We note that the notion of invertibility is not well behaved under the intersection or the sum of ideals. If $\mathfrak{a}$ and $\mathfrak{b}$ are invertible $\mathcal{O}$-ideals then it is not necessarily true that $\mathfrak{a} \cap \mathfrak{b}$ and $(\mathfrak{a}, \mathfrak{b}) := \mathfrak{a} + \mathfrak{b}$ are invertible $\mathcal{O}$-ideals.

We recall some facts about invertible $\mathcal{O}$-ideals and regular prime ideals of $\mathcal{O}$. It is well known that an $\mathcal{O}$-ideal $\mathfrak{a}$ is invertible if and only if for every prime ideal $\mathfrak{p} \neq 0$ of $\mathcal{O}$ one has that $\mathfrak{a}_\mathfrak{p}$ is a principal $\mathcal{O}_\mathfrak{p}$-ideal (here $\mathfrak{a}_\mathfrak{p}$ denotes the localization of $\mathfrak{a}$ at $\mathfrak{p}$). For a proof of this criterion see Proposition 12.4 of [Neu99]. Let

$$cond(\mathcal{O}) = \{x \in \mathcal{O}_K : x\mathcal{O}_K \subseteq \mathcal{O}\}.$$

Note that $cond(\mathcal{O})$ is an integral $\mathcal{O}_K$-ideal. A prime ideal $\mathfrak{p}$ of $\mathcal{O}$ is called *regular* if $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring. We have the following criterion for regular prime ideals of $\mathcal{O}$:

$$\mathfrak{p} \not\supseteq cond(\mathcal{O}) \Longleftrightarrow \mathfrak{p} \text{ is regular.}$$

For a proof of this fact see Proposition 12.10 of [Neu99]. We note that if $\mathfrak{q}$ is a regular prime ideal of $\mathcal{O}$ then it is necessarily $\mathcal{O}$-invertible. Indeed, if $\mathfrak{p} \subseteq \mathcal{O}$ is a prime ideal distinct from $\mathfrak{q}$ then $\mathfrak{q}\mathcal{O}_{\mathfrak{p}} = \mathcal{O}_{\mathfrak{p}}$ and so is principal. If $\mathfrak{p} = \mathfrak{q}$ then $\mathfrak{q}\mathcal{O}_{\mathfrak{p}}$ is again principal since in this case $\mathcal{O}_{\mathfrak{p}}$ is a discrete valuation ring. Since the localization of $\mathfrak{q}$ at each prime of $\mathcal{O}$ is principal it follows that $\mathfrak{q}$ is $\mathcal{O}$-invertible.

Given an integral $\mathcal{O}$-ideal $\mathfrak{f}$ we define the set

$$I_{\mathcal{O}}(\mathfrak{f}) := \{\mathfrak{b} \subseteq \mathcal{O} : \mathfrak{b} \text{ is an invertible integral } \mathcal{O}\text{-ideal coprime to } \mathfrak{f}, \text{ i.e., } (\mathfrak{f}, \mathfrak{b}) = \mathcal{O}\}.$$

Consider the monoid $I_{\mathcal{O}}(1) = I_{\mathcal{O}}(\mathcal{O})$ where the multiplication is given by the usual multiplication of two ideals. Let $\{\sigma_1, \dots, \sigma_{r_1}\}$ be the set of real embeddings of K into $\mathbb{R}$ where $r_1 + 2r_2 = n$. Let $\lambda \in K$. The notation $\lambda \gg 0$ will be taken to mean that $\sigma_i(\lambda) > 0$ for $i = 1, \dots, r_1$. For every integral $\mathcal{O}$-ideal $\mathfrak{f}$ we define an equivalence relation $\sim_{\mathfrak{f}}$ on the monoid $I_{\mathcal{O}}(1)$. Let $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}}(1)$. We say that $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists an element $\lambda \in 1 + \mathfrak{f}\mathfrak{a}^{-1}$, $\lambda \gg 0$, such that $\lambda\mathfrak{a} = \mathfrak{b}$. Note that if $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ then $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f})$. The set $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is a finite monoid. The set of invertible elements of $I_{\mathcal{O}}(1)/\sim_{\mathfrak{f}}$ is exactly $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. We let

$$\mathcal{O}(\infty)^{\times} := \{\lambda \in \mathcal{O}^{\times} : \lambda \gg 0\},$$

where $\infty = \prod_{i=1}^{r_1} \infty_i$ stands for the product of the distinct real places of K.

We let

(1) $P_{\mathcal{O}}(\mathfrak{f}\infty) = \left\{\frac{\alpha}{\beta}\mathcal{O} : \alpha, \beta \in \mathcal{O}, \beta \neq 0, (\alpha\mathcal{O}, \mathfrak{f}) = (\beta\mathcal{O}, \mathfrak{f}) = \mathcal{O}, \frac{\alpha}{\beta} \gg 0\right\}$,

(2) $P_{\mathcal{O},1}(\mathfrak{f}) = \left\{\frac{\alpha}{\beta}\mathcal{O} : \alpha, \beta \in \mathcal{O}, \beta \neq 0, (\alpha\mathcal{O}, \mathfrak{f}) = (\beta\mathcal{O}, \mathfrak{f}) = \mathcal{O}, \alpha \equiv \beta \pmod{\mathfrak{f}}\right\}$,

(3) $P_{\mathcal{O},1}(\mathfrak{f}\infty) = \left\{\frac{\alpha}{\beta}\mathcal{O} \in P_{\mathcal{O},1}(\mathfrak{f}) : \frac{\alpha}{\beta} \gg 0\right\}$.

It is easy to see that for $\mathfrak{a}, \mathfrak{b} \in I_{\mathcal{O}}(\mathfrak{f})$, $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ if and only if there exists a $\lambda\mathcal{O} \in P_{\mathcal{O},1}(\mathfrak{f}\infty)$ such that $\lambda\mathfrak{a} = \mathfrak{b}$. Thus the quotient $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$ makes sense (even though $P_{\mathcal{O},1}(\mathfrak{f}\infty) \not\subseteq I_{\mathcal{O}}(\mathfrak{f})$) and can be identified with $I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$.

We let $C_{\mathcal{O}}(\mathfrak{f}) := I_{\mathcal{O}}(\mathfrak{f})/\sim_{\mathfrak{f}}$. In the case where $\mathfrak{f}$ is $\mathcal{O}$-invertible we call $C_{\mathcal{O}}(\mathfrak{f})$ the narrow ideal class group of K of conductor $\mathfrak{f}$. Note that because $\mathfrak{f}$ is assumed to be invertible the order $\mathcal{O}$ is already encoded in $\mathfrak{f}$. By class field theory, the ideal class group $C_{\mathcal{O}}(\mathfrak{f})$ corresponds to an abelian extension of K which we denote by $K(\mathfrak{f}\infty)$. We call $K(\mathfrak{f}\infty)$ the narrow ray class field of K of conductor $\mathfrak{f}$. Note that in the case where $\mathfrak{f}$ is an integral $\mathcal{O}_K$-ideal then $K(\mathfrak{f}\infty)$ is just the usual narrow ray class field of conductor $\mathfrak{f}$.

2 Functions of Ψ-type and ζ-type

2.1 Functions of Ψ-type

Before defining zeta functions of Ψ-type we recall some facts about the dual and the inverse of a $\mathbb{Z}$-lattice contained in K. Let $M \subseteq K$ be a $\mathbb{Z}$-lattice of maximal rank. We define $\mathbf{N}(M)$ to be the absolute value of the determinant of a matrix that takes a $\mathbb{Z}$-basis of $\mathcal{O}_K$ to a $\mathbb{Z}$-basis of M. Note that $\mathbf{N}(M)$ is always a positive rational number. We define the dual lattice of M as

$$M^* := \{x \in K : \mathbf{Tr}_{K/\mathbb{Q}}(xy) \in \mathbb{Z} \text{ for all } y \in M\}.$$

One can check that $(M^*)^* = M$ and $End_K(M) = End_K(M^*)$. From this we may deduce that M is an invertible $\mathcal{O}$-ideal if and only if M^* is an invertible $\mathcal{O}$-ideal. If $\mathfrak{c}$ is an invertible $\mathcal{O}$-ideal then we define $\mathfrak{c}^{-1} = \{x \in K : x\mathfrak{c} \subseteq \mathcal{O}\}$. From now on we fix a $\mathbb{Z}$-order $\mathcal{O}$ and we let $\mathfrak{d}_{\mathcal{O}} = \mathfrak{d} := (\mathcal{O}^*)^{-1}$. If $\mathfrak{f}$ is an invertible $\mathcal{O}$-ideal then one readily checks that $\mathfrak{f}^* = \mathfrak{f}^{-1}\mathfrak{d}^{-1}$. Note also that for any $\lambda \in K^\times$ one has $\mathbf{N}(\lambda\mathcal{O}) = |\mathbf{N}_{K/\mathbb{Q}}(\lambda)|\mathbf{N}(\mathcal{O})$.

Let $\mathfrak{f}$ be an integral invertible $\mathcal{O}$-ideal. Let w be a sign character of K, i.e., a product over elements of a subset of the characters

$$sign \circ \sigma_i : K^\times \to \mathbb{R}^\times \to \{\pm 1\}.$$

Let $\mathfrak{c}$ be an integral invertible $\mathcal{O}$-ideal of K. Following Siegel (see equation (9) and (12) of [Sie68]), we define

$$\Psi(\mathfrak{c}, \mathfrak{f}, w, s) := \mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^s \sum_{\Gamma_{\mathfrak{c}}(\mathfrak{f})\backslash\{0\neq\mu\in\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}} w(\mu)\frac{e^{2\pi i \mathbf{Tr}(\mu)}}{|\mathbf{N}(\mu)|^s}, \quad \Re(s) > 1, \tag{2.1}$$

where $\mathbf{Tr}(\mu) := \mathbf{Tr}_{K/\mathbb{Q}}(\mu)$ and $\mathbf{N}(\mu) := \mathbf{N}_{K/\mathbb{Q}}(\mu)$ are the usual trace and norm functions from K to $\mathbb{Q}$ and where

$$\Gamma_{\mathfrak{c}}(\mathfrak{f}) := \mathcal{O}(\infty)^\times \cap (1 + \mathfrak{f}\mathfrak{c}^{-1}).$$

It is understood that the summation in (2.1) is taken over a complete set of representatives of $\{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}$ modulo $\Gamma_{\mathfrak{c}}(\mathfrak{f})$. Note that for any $\epsilon \in \Gamma_{\mathfrak{c}}(\mathfrak{f})$ and $\mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}$ we have $\mu - \epsilon\mu \in \mathfrak{d}^{-1}$, so $\mathbf{Tr}(\mu - \epsilon\mu) \in \mathbb{Z}$. Therefore the summation (2.1) does not depend on the choice of representatives. Note also that by definition we have $\Gamma_{\mathfrak{c}}(\mathfrak{f}) = \Gamma_{\mathfrak{c}\mathfrak{b}}(\mathfrak{f}\mathfrak{b})$ for all integral invertible $\mathcal{O}$-ideals $\mathfrak{b}$ and therefore

$$\Psi(\mathfrak{c}, \mathfrak{f}, w, s) = \Psi(\mathfrak{c}\mathfrak{b}, \mathfrak{f}\mathfrak{b}, w, s). \tag{2.2}$$

Definition 2.1 *A function of Ψ-type associated to the number field K will be a function in the complex variable s which is equal to $\Psi(\mathfrak{c}, \mathfrak{f}, w, s)$ for some invertible integral $\mathcal{O}$-ideals $\mathfrak{c}, \mathfrak{f}$ and sign character w.*

Proposition 2.1 *The function $\Psi(\mathfrak{c}, \mathfrak{f}, w, s)$ depends on the first entry only modulo $\sim_{\mathfrak{f}}$.*

In order to prove this we will prove certain identities and a lemma. For a subgroup $\Gamma' \leq \Gamma_{\mathfrak{c}}(\mathfrak{f})$ of finite index, a straightforward computation shows that

$$[\Gamma_{\mathfrak{c}}(\mathfrak{f}) : \Gamma'] \cdot \Psi(\mathfrak{c}, \mathfrak{f}, w, s) = \mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^s \sum_{\Gamma' \backslash \{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}} w(\mu) \frac{e^{2\pi i \mathbf{Tr}(\mu)}}{|\mathbf{N}(\mu)|^s}, \qquad \Re(s) > 1. \tag{2.3}$$

Let $\rho \in K^\times$ be such that $\rho\mathfrak{c} \subseteq \mathcal{O}$. Using the inclusion $\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) \cap \Gamma_{\mathfrak{c}}(\mathfrak{f}) \subseteq \Gamma_{\mathfrak{c}}(\mathfrak{f})$ with (2.3) we may deduce that

(2.4)

$$[\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f}) \cap \Gamma_{\rho\mathfrak{c}}(\mathfrak{f})] \cdot \Psi(\rho\mathfrak{c}, \mathfrak{f}, w, s) = w(\rho)\mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^s \sum_{\Gamma_{\mathfrak{c}}(\mathfrak{f}) \cap \Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) \backslash \{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\}} w(\mu) \frac{e^{2\pi i \mathbf{Tr}(\rho\mu)}}{|\mathbf{N}(\mu)|^s}.$$

Lemma 2.1 *Let $\mathfrak{a}$ and $\mathfrak{b}$ be integral invertible $\mathcal{O}$-ideals such that $\mathfrak{b} = \lambda\mathfrak{a}$ for some $\lambda \in K^\times$. Then if $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f})$ we have $\Gamma_{\mathfrak{a}}(\mathfrak{f}) = \Gamma_{\mathfrak{b}}(\mathfrak{f})$.*

Proof By assumption we have $(1, \mathfrak{f}\mathfrak{a}^{-1}) = (\lambda, \mathfrak{a}^{-1}\mathfrak{f})$. Therefore $1 = \alpha\lambda + \beta$ for some $\alpha \in \mathcal{O}$ and $\beta \in \mathfrak{a}^{-1}\mathfrak{f}$. Let $\epsilon \in \Gamma_{\mathfrak{b}}(\mathfrak{f})$. Then we have

$$(\epsilon - 1) = (\epsilon - 1)(1 - \alpha\lambda) + (\epsilon - 1)\alpha\lambda \in \mathcal{O}(\mathfrak{a}^{-1}\mathfrak{f}) + \mathfrak{f}\mathfrak{b}^{-1}(\mathfrak{a}^{-1}\mathfrak{b}) \subseteq \mathfrak{a}^{-1}\mathfrak{f}.$$

Therefore $\Gamma_{\mathfrak{b}}(\mathfrak{f}) \subseteq \Gamma_{\mathfrak{a}}(\mathfrak{f})$. By symmetry we have $\Gamma_{\mathfrak{a}}(\mathfrak{f}) \subseteq \Gamma_{\mathfrak{b}}(\mathfrak{f})$. $\square$

Corollary 2.1 *Assume either $\mathfrak{a} \sim_{\mathfrak{f}} \mathfrak{b}$ or $\mathfrak{a} = \lambda\mathfrak{b}$ for some $\lambda \in K^\times$ where $\lambda = \frac{\alpha}{\beta}$ for some $\alpha, \beta \in \mathcal{O}$ and $(\alpha\beta\mathcal{O}, \mathfrak{f}) = \mathcal{O}$. Then $\Gamma_{\mathfrak{a}}(\mathfrak{f}) = \Gamma_{\mathfrak{b}}(\mathfrak{f})$.*

Proof In both cases one has $(\mathfrak{a}, \mathfrak{f}) = (\mathfrak{b}, \mathfrak{f})$. Therefore by the previous lemma one has $\Gamma_{\mathfrak{a}}(\mathfrak{f}) = \Gamma_{\mathfrak{b}}(\mathfrak{f})$. $\square$

Proof of Proposition 2.1 Use Lemma 2.1 and the identity (2.4). $\square$

Remark 2.1 Note that if there exists a $\rho \in \Gamma_{\mathfrak{c}}(\mathfrak{f})$ such that $w(\rho) = -1$, then we find using (2.4) that $\Psi(\mathfrak{c}, \mathfrak{f}, w, s) = 0$. The existence of such units should be avoided in order to have a non-zero function of Ψ-type.

The next lemma will be used in Section 3. In the case where $(\mathfrak{f}, cond(\mathcal{O})) = \mathcal{O}$ the following lemma gives a more explicit description of $\Gamma_{\mathfrak{a}}(\mathfrak{f})$.

Lemma 2.2 *Assume that $(\mathfrak{f}, cond(\mathcal{O})) = \mathcal{O}$. Let $\mathfrak{a}$ be an invertible integral $\mathcal{O}$-ideal and $(\mathfrak{a}, \mathfrak{f}) = \mathfrak{f}'$. Then*

$$\Gamma_{\mathfrak{a}}(\mathfrak{f}) = \mathcal{O}(\infty)^\times \cap (1 + \mathfrak{f}(\mathfrak{f}')^{-1}).$$

Proof First note that $\mathfrak{f}'$ is an invertible $\mathcal{O}$-ideal divisible only by regular primes of $\mathcal{O}$. We can thus write $\mathfrak{a} = \mathfrak{f}'\mathfrak{a}_0$ where $\mathfrak{a}_0$ is an integral invertible $\mathcal{O}$-ideal coprime to $\mathfrak{f}(\mathfrak{f}')^{-1}$. Let $\epsilon \in \Gamma_{\mathfrak{a}}(\mathfrak{f})$. Then $(\epsilon - 1) \in \mathfrak{f}(\mathfrak{f}')^{-1}\mathfrak{a}_0^{-1} \cap \mathcal{O} = \mathfrak{f}(\mathfrak{f}')^{-1}$. Therefore $\epsilon \in 1 + \mathfrak{f}(\mathfrak{f}')^{-1}$. Conversely, if $\epsilon \in \mathcal{O}(\infty)^\times$ and $\epsilon \in 1 + \mathfrak{f}(\mathfrak{f}')^{-1}$ then $\epsilon \in (1 + \mathfrak{f}\mathfrak{a}^{-1}) \cap \mathcal{O}(\infty)^\times = \Gamma_{\mathfrak{a}}(\mathfrak{f})$. $\square$

2.2 Functions of ζ-type

Let $\mathcal{O}$ be a $\mathbb{Z}$-order of K. Let $w : K^\times \to \{\pm 1\}$ be a sign character. Fix an integral invertible $\mathcal{O}$-ideal $\mathfrak{f}$. Given an invertible integral $\mathcal{O}$-ideal $\mathfrak{a}$ (so $1 \in \mathfrak{a}^{-1}$) we define

$$\zeta(\mathfrak{a}, \mathfrak{f}, w, s) := \mathbf{N}(\mathfrak{a})^{-s} \sum_{\Gamma_\mathfrak{a}(\mathfrak{f}) \backslash \{\mu \in \mathfrak{a}^{-1}, \mu \equiv 1 \pmod{\mathfrak{f}\mathfrak{a}^{-1}}\}} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s} \quad \Re(s) > 1, \tag{2.5}$$

where $\Gamma_\mathfrak{a}(\mathfrak{f}) = \mathcal{O}(\infty)^\times \cap (1 + \mathfrak{f}\mathfrak{a}^{-1})$. The expression on the right hand side of (2.5) may be viewed as a partial zeta function weighted by the sign character w.

Definition 2.2 *A function of ζ-type associated to the number field K will be a function in the complex variable s which is equal to $\zeta(\mathfrak{c}, \mathfrak{f}, w, s)$ for some invertible integral $\mathcal{O}$-ideals $\mathfrak{c}, \mathfrak{f}$ and sign character w.*

Let $\mathfrak{b}$ be an invertible integral $\mathcal{O}$-ideal. Then a direct computation reveals that

$$\zeta(\mathfrak{a}, \mathfrak{f}, w, s) = \zeta(\mathfrak{a}\mathfrak{b}, \mathfrak{f}\mathfrak{b}, w, s).$$

As for the functions of Ψ-type, the function $\zeta(\mathfrak{a}, \mathfrak{f}, w, s)$ depends on the first entry only modulo $\sim_\mathfrak{f}$. Let us prove it. Let $\Gamma' \leq \Gamma_\mathfrak{a}(\mathfrak{f})$ be a subgroup of finite index. Then

$$[\Gamma_\mathfrak{a}(\mathfrak{f}) : \Gamma'] \cdot \zeta(\mathfrak{a}, \mathfrak{f}, w, s) = \mathbf{N}(\mathfrak{a})^{-s} \sum_{\Gamma' \backslash \{0 \neq \mu \in \mathfrak{a}^{-1},\ \mu \equiv 1 \pmod{\mathfrak{f}\mathfrak{a}^{-1}}\}} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}. \tag{2.6}$$

Let $\lambda \in K^\times$ be such that $\mathfrak{a}$ and $\lambda\mathfrak{a}$ are integral invertible $\mathcal{O}$-ideals. Then applying (2.6) to the inclusion $\Gamma_\mathfrak{b}(\mathfrak{f}) \cap \Gamma_\mathfrak{a}(\mathfrak{f}) \subseteq \Gamma_\mathfrak{b}(\mathfrak{f})$ we deduce that

$$[\Gamma_{\lambda\mathfrak{a}}(\mathfrak{f}) : \Gamma_{\lambda\mathfrak{a}}(\mathfrak{f}) \cap \Gamma_\mathfrak{a}(\mathfrak{f})] \cdot \zeta(\lambda\mathfrak{a}, \mathfrak{f}, w, s) = w(\lambda)\mathbf{N}(\mathfrak{a})^{-s} \sum_{\Gamma_{\lambda\mathfrak{a}}(\mathfrak{f}) \cap \Gamma_\mathfrak{a}(\mathfrak{f}) \backslash \{0 \neq \mu \in \mathfrak{a}^{-1},\ \mu \equiv \lambda \pmod{\mathfrak{f}\mathfrak{a}^{-1}}\}} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}. \tag{2.7}$$

From Corollary 2.1 and (2.7) we deduce that $\zeta(\mathfrak{a}, \mathfrak{f}, w, s)$ depends on the first entry only modulo $\sim_\mathfrak{f}$.

3 A functional equation

Let $\{a_i\}_{i=1}^{r_1}$ be the signature of w, i.e., $w = \prod_{i=1}^{r_1} (sign \circ \sigma_i)^{a_i}$ where $a_i \in \{0, 1\}$. Then we define

$$F_w(s) := |d_K|^{s/2} \pi^{-ns/2} 2^{r_2(1-s)} \Gamma(s)^{r_2} \prod_{i=1}^{r_1} \Gamma\left(\frac{s + a_i}{2}\right)$$

where $d_K = \mathbf{N}(\mathcal{O}_K)$ is the discriminant of K and $\Gamma(s)$ is the Gamma function evaluated at s.

Theorem 3.1 *Let $\mathfrak{a}$ and $\mathfrak{f}$ be integral invertible $\mathcal{O}$-ideals. Then the functions $\Psi(\mathfrak{a},\mathfrak{f},w,s)$ and $\zeta(\mathfrak{a},\mathfrak{f},w,s)$ admit a meromorphic continuation to all of $\mathbb{C}$. Moreover they are related by the following functional equation:*

$$F_w(s)\Psi(\mathfrak{a},\mathfrak{f},w,s) = i^{\mathbf{Tr}(w)}F_w(1-s)\mathbf{N}(\mathfrak{f})^{1-s}\zeta(\mathfrak{a},\mathfrak{f},w,1-s) \tag{3.1}$$

where $\mathbf{Tr}(w) := \sum_i a_i$.

Proof The functional equation (3.1) appears (without any proof) as a special case of equation (19) of [Sie70]. For a detailed proof of (3.1), see Theorem 1.1 of [Cha09], where a functional equation is proved for a slightly more general class of zeta functions. The proof follows Hecke's classical method and relies on the functional equation of a multivariable theta function which is a direct consequence of the Poisson summation formula. $\square$

3.1 Identities between functions of Ψ-type and ζ-type

In this section we want to show that a certain linear combination of functions of Ψ-type is equal to a function of ζ-type and and vice versa. Before doing this, it is convenient to define a certain involution of the set of invertible $\mathcal{O}$-ideals of K.

Definition 3.1 *Let $\mathfrak{a}$ be an invertible $\mathcal{O}$-ideal of K. We define $\mathfrak{a}^{\iota} = \mathfrak{f}\mathfrak{d}\mathfrak{a}^{-1}$. Obviously we have $(\mathfrak{a}^{\iota})^{\iota} = \mathfrak{a}$.*

The next lemma gives a precise relation between functions of Ψ-type and functions of ζ-type.

Lemma 3.1 *Let $\mathfrak{f}$ and $\mathfrak{c}$ be invertible integral $\mathcal{O}$-ideals such that $(\mathfrak{c},\mathfrak{f}) = \mathcal{O}$. Assume also that $(\mathfrak{f}, cond(\mathcal{O})) = \mathcal{O}$. Then we have the following formulas:*

(1) $$\Psi(\mathfrak{c},\mathfrak{f},w,s) = \sum_{\lambda\in\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}} \pmod{\mathfrak{c}\mathfrak{d}^{-1}}} e^{2\pi i\mathbf{Tr}(\lambda)}w(\lambda)[\Gamma_{\lambda\mathfrak{c}^{\iota}}(\mathfrak{f}):\Gamma_{\mathfrak{c}}(\mathfrak{f})]\zeta(\lambda\mathfrak{c}^{\iota},\mathfrak{f},w,s).$$

(2) *Let $0\neq\lambda\in\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}$. Then*
$$\frac{[\Gamma_{\lambda\mathfrak{c}^{\iota}}(\mathfrak{f}):\Gamma_{\mathfrak{c}}(\mathfrak{f})]}{\mathbf{N}(\mathfrak{f})}w(\lambda)\sum_{\rho\in\mathcal{O} \pmod{\mathfrak{f}}}[\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}):\Gamma_{\mathfrak{c}}(\mathfrak{f})]e^{-2\pi i\mathbf{Tr}(\rho\lambda)}w(\rho)\Psi(\rho\mathfrak{c},\mathfrak{f},w,s) = \zeta(\lambda\mathfrak{c}^{\iota},\mathfrak{f},w,s).$$

In all the summations it will always be assumed that the representative of the zero congruence class is a non-zero element of K.

Proof Let us first show (1). A direct computation shows that

$$\Psi(\mathfrak{c},\mathfrak{f},w,s) = \mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^{s}\sum_{\lambda\in\mathfrak{c}\mathfrak{f}^{-1}\mathfrak{d}^{-1} \pmod{\mathfrak{c}\mathfrak{d}^{-1}}} e^{2\pi i\mathbf{Tr}(\lambda)}\sum_{\substack{\{0\neq\mu\in\mathfrak{c}\mathfrak{f}^{-1}\mathfrak{d}^{-1},\\ \mu\equiv\lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}\}/\Gamma_{\mathfrak{c}}(\mathfrak{f})}}\frac{w(\mu)}{|\mathbf{N}(\mu)|^{s}}. \tag{3.2}$$

One can verify that the congruence $\mu \equiv \lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}$ is preserved under the action of $\Gamma_{\mathfrak{c}}(\mathfrak{f})$ so that the second summation makes sense. The last equality can be rewritten as

(3.3)

$$\Psi(\mathfrak{c}, \mathfrak{f}, w, s) = \sum_{\lambda \in \mathfrak{c}\mathfrak{f}^{-1}\mathfrak{d}^{-1} \pmod{\mathfrak{c}\mathfrak{d}^{-1}}} e^{2\pi i \mathbf{Tr}(\lambda)} w(\lambda) \mathbf{N}\left(\frac{\mathfrak{c}}{\lambda\mathfrak{d}\mathfrak{f}}\right)^s \sum_{\left\{\substack{0 \neq \mu \in \lambda^{-1}\mathfrak{c}\mathfrak{f}^{-1}\mathfrak{d}^{-1}, \\ \mu \equiv 1 \pmod{\lambda^{-1}\mathfrak{c}\mathfrak{d}^{-1}}}\right\}/\Gamma_{\mathfrak{c}}(\mathfrak{f})} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}.$$

Now using the fact that $(\mathfrak{f}, cond(\mathcal{O})) = \mathcal{O}$ and applying Lemma 2.2 we find that

$$\Gamma_{\lambda\mathfrak{c}^{\iota}}(\mathfrak{f}) = \mathcal{O}(\infty)^{\times} \cap (1 + \mathfrak{f}(\mathfrak{f}')^{-1}),$$

where $\mathfrak{f}' = (\lambda\mathfrak{f}\mathfrak{d}, \mathfrak{f})$. In particular we deduce from this that $\Gamma_{\mathfrak{c}}(\mathfrak{f}) \subseteq \Gamma_{\lambda\mathfrak{c}^{\iota}}(\mathfrak{f})$. Using (2.6) we can rewrite (3.3) as

$$\Psi(\mathfrak{c}, \mathfrak{f}, w, s) = \sum_{\lambda \in \mathfrak{c}\mathfrak{f}^{-1}\mathfrak{d}^{-1} \pmod{\mathfrak{c}\mathfrak{d}^{-1}}} e^{2\pi i \mathbf{Tr}(\lambda)} w(\lambda) [\Gamma_{\lambda\mathfrak{c}^{\iota}}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f})] \zeta(\lambda\mathfrak{c}^{\iota}, \mathfrak{f}, w, s).$$

This proves (1).

Let us prove (2). Let $0 \neq \lambda \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}$. From (2.4) a direct computation shows that

$$\sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} [\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f})] e^{-2\pi i \mathbf{Tr}(\rho\lambda)} w(\rho) \Psi(\rho\mathfrak{c}, \mathfrak{f}, w, s) \tag{3.4}$$
$$= \mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^s \sum_{\left\{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{d}\mathfrak{f}}\right\}/\Gamma_{\mathfrak{c}}(\mathfrak{f})} \sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} \left(e^{2\pi i \mathbf{Tr}(\rho(\mu - \lambda))}\right) \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}.$$

Using the fact that

$$\sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} \left(e^{2\pi i \mathbf{Tr}(\rho(\mu - \lambda))}\right) = \begin{cases} \mathbf{N}(\mathfrak{f}) & \text{if } \mu \equiv \lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}, \\ 0 & \text{if } \mu \not\equiv \lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}, \end{cases}$$

we can rewrite (3.4) as

$$\sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} [\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f})] e^{-2\pi i \mathbf{Tr}(\rho\lambda)} w(\rho) \Psi(\rho\mathfrak{c}, \mathfrak{f}, w, s)$$
$$= \mathbf{N}(\mathfrak{f}) \mathbf{N}\left(\frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}}\right)^s \sum_{\left\{\substack{0 \neq \mu \in \frac{\mathfrak{c}}{\mathfrak{f}\mathfrak{d}} \\ \mu \equiv \lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}}\right\}/\Gamma_{\mathfrak{c}}(\mathfrak{f})} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}$$
$$= \mathbf{N}(\mathfrak{f}) \mathbf{N}\left(\lambda\mathfrak{c}^{\iota}\right)^{-s} w(\lambda) \sum_{\left\{\substack{0 \neq \mu \in (\lambda\mathfrak{c}^{\iota})^{-1} \\ \mu \equiv 1 \pmod{\mathfrak{f}(\lambda\mathfrak{c}^{\iota})^{-1}}}\right\}/\Gamma_{\mathfrak{c}}(\mathfrak{f})} \frac{w(\mu)}{|\mathbf{N}(\mu)|^s}.$$

Note that the summation on the right hand side of the first equality makes sense since for any $\epsilon \in \Gamma_{\mathfrak{c}}(\mathfrak{f}) = \mathcal{O}(\infty)^\times \cap (1+\mathfrak{f})$ we have $\epsilon\mu \equiv \mu \equiv \lambda \pmod{\mathfrak{c}\mathfrak{d}^{-1}}$. Finally using the inclusion $\Gamma_{\mathfrak{c}}(\mathfrak{f}) \subseteq \Gamma_{\lambda\mathfrak{c}^\iota}(\mathfrak{f})$ in (2.6) we deduce from the last equality that

$$\frac{[\Gamma_{\lambda\mathfrak{c}^\iota}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f})]}{\mathbf{N}(\mathfrak{f})} w(\lambda) \sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} [\Gamma_{\rho\mathfrak{c}}(\mathfrak{f}) : \Gamma_{\mathfrak{c}}(\mathfrak{f})] e^{-2\pi i \mathbf{Tr}(\rho\lambda)} w(\rho) \Psi(\rho\mathfrak{c}, \mathfrak{f}, w, s) = \zeta(\lambda\mathfrak{c}^\iota, \mathfrak{f}, w, s).$$

This proves (2). $\square$

4 Finite order Hecke characters

In this section we set some notation about characters of the narrow ideal class group of conductor $\mathfrak{f}$, i.e., characters of $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f})$. We have the following short exact sequence:

$$1 \to (\mathcal{O}/\mathfrak{f})^\times / (\mathcal{O}(\infty)^\times \pmod{\mathfrak{f}}) \xrightarrow{\theta} I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to I_{\mathcal{O}}(1)/P_{\mathcal{O}}(\infty) \to 1,$$

where $\theta(a \pmod{\mathfrak{f}}) = [\widetilde{a}\mathcal{O}]$ where $\widetilde{a} \in \mathcal{O}$ is chosen so that $\widetilde{a} \equiv a \pmod{\mathfrak{f}}$ and $\widetilde{a} \gg 0$. From this short exact sequence we see that every character $\chi : I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to \mathbb{C}^\times$ can be pulled back to a character

$$\chi_f := \chi \circ \theta : (\mathcal{O}/\mathfrak{f})^\times / (\mathcal{O}(\infty)^\times \pmod{\mathfrak{f}}) \to \mathbb{C}^\times,$$

where the subscript f of χ_f stands for finite. It will also be convenient to view χ_f as a function on the set of elements of

$$Q_{\mathcal{O}}(\mathfrak{f}) = \{\alpha\mathcal{O} : \alpha \in \mathcal{O}, (\alpha\mathcal{O}, \mathfrak{f}) = \mathcal{O}\}.$$

This makes sense since the latter set admits a natural map to $(\mathcal{O}/\mathfrak{f})^\times$. For an element $\alpha \in Q_{\mathcal{O}}(\mathfrak{f})$ we define

$$\chi_\infty(\alpha) := \chi([\alpha\mathcal{O}])/\chi_f(\alpha).$$

Let $\alpha \in Q_{\mathcal{O}}(\mathfrak{f})$. Choose an element $\beta \in Q_{\mathcal{O}}(\mathfrak{f})$ in such a way that $\alpha \equiv \beta \pmod{\mathfrak{f}}$ and $\beta \gg 0$. Then one finds that $\chi_f(\alpha) = \chi([\beta\mathcal{O}])$. Therefore $\chi_\infty(\alpha) = \frac{\chi([\alpha\mathcal{O}])}{\chi([\beta\mathcal{O}])}$. Since $\left(\frac{\alpha}{\beta}\right)^2 \gg 0$ and $\left(\frac{\alpha}{\beta}\right)^2 \equiv 1 \pmod{\mathfrak{f}}$, it follows that $\chi_\infty(\alpha) \in \{\pm 1\}$. The map $\chi_\infty : Q_{\mathcal{O}}(\mathfrak{f}) \to \{\pm 1\}$ can be written as $\prod_{i=1}^{r_1} (sign \circ \sigma_i)^{a_i}$ for a unique set of integers $\{a_i\}_{i=1}^{r_1}$ where $a_i \in \{0,1\}$. Therefore χ_∞ can be viewed (after extension to $K^\times$) as a sign character. Thus every character

$$\chi : I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to \mathbb{C}^\times,$$

when restricted to principal $\mathcal{O}$-ideals $\alpha\mathcal{O}$ coprime to $\mathfrak{f}$, can be written uniquely as $\chi = \chi_\infty \chi_f$ where

$$\chi_\infty : K^\times \to \{\pm 1\} \quad \text{and} \quad \chi_f : (\mathcal{O}/\mathfrak{f})^\times / (\mathcal{O}^\times(\infty) \pmod{\mathfrak{f}}) \to \mathbb{C}^\times.$$

If we view χ_f as a character of $(\mathcal{O}/\mathfrak{f})^\times$ and χ_∞ as a sign character of $K^\times$, one may deduce the following identity:

$$\chi_f(\epsilon)\chi_\infty(\epsilon) = 1 \ \ \forall \epsilon \in \mathcal{O}^\times. \tag{4.1}$$

Conversely, for every pair of characters $(w, \eta) \in (\widehat{(\mathbb{R} \otimes K)^\times}, \widehat{(\mathcal{O}/\mathfrak{f})^\times})$ satisfying (4.1), there exists a character $\chi : I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to \mathbb{C}^\times$ which lifts the pair (w, η), i.e, such that $\chi_f = \eta$ and $\chi_\infty = w$. Note that the number of such lifts is exactly h_K^+, where h_K^+ denotes the order of the narrow ideal class group of K, i.e., $I_{\mathcal{O}}(1)/P_{\mathcal{O}}(\infty)$.

In order to simplify the notation we set $G_{\mathfrak{f}\infty} = I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$ and $G_{\mathfrak{f}} = I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f})$. We have the following short exact sequence:

$$1 \to P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to G_{\mathfrak{f}\infty} \to G_{\mathfrak{f}} \to 1. \tag{4.2}$$

Let us assume that $P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \simeq (\mathbb{Z}/2\mathbb{Z})^r$. This means precisely that the index of the wide ray class field of conductor $\mathfrak{f}$ in the narrow ray class field of conductor $\mathfrak{f}$ is 2^r. Taking the Pontryagin dual of (4.2) we get

$$1 \to \widehat{G_{\mathfrak{f}}} \xrightarrow{\pi} \widehat{G}_{\mathfrak{f}\infty} \to \widehat{P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}}(\mathfrak{f}\infty) \to 1.$$

Let $\{\eta_j\}_{j=1}^r$ be a set of generators of $\widehat{P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}}(\mathfrak{f}\infty)$. For every $1 \le j \le r$, take an arbitrary lift of η_j to $\widehat{G}_{\mathfrak{f}\infty}$ and denote it again by η_j. By construction, the group generated by the η_j's is a complete set of representatives of $\widehat{G}_{\mathfrak{f}\infty}$ modulo $\widehat{G}_{\mathfrak{f}}$. Let $\chi \in G_{\mathfrak{f}\infty}$. Then there exists a unique set of integers $\{b_j\}_{j=1}^r$, $b_j \in \{0, 1\}$, such that $\chi_\infty = \prod_{j=1}^r \eta_j^{b_j}$. From this it follows that $\widehat{G}_{\mathfrak{f}}$ corresponds precisely to the set of characters $\chi \in \widehat{G}_{\mathfrak{f}\infty}$ such that $\chi_\infty = 1$.

4.1 Gauss sums

Let $\chi \in \widehat{G}_{\mathfrak{f}\infty}$ be a finite order Hecke character and $\alpha \in K^\times$ be such that $\gamma\mathcal{O} = \frac{\mathfrak{a}}{\mathfrak{d}\mathfrak{f}}$ where $(\mathfrak{a}, \mathfrak{f}) = \mathcal{O}$. For an element $\xi \in \mathcal{O}$ we define

$$g_\gamma(\chi, \xi) := \chi_\infty(\gamma)\overline{\chi(\mathfrak{a})} \sum_{\rho \in \mathcal{O} \pmod{\mathfrak{f}}} \overline{\chi_f}(\rho) e^{2\pi i \mathbf{Tr}(\rho\gamma\xi)}.$$

We set $\chi_f(\rho) = 0$ if $(\rho, \mathfrak{f}) \neq \mathcal{O}$. It is easy to see that if $\gamma' \in K^\times$ is another element such $\gamma'\mathcal{O} = \frac{\mathfrak{a}'}{\mathfrak{d}\mathfrak{f}}$ with $(\mathfrak{a}', \mathfrak{f}) = \mathcal{O}$ then $g_{\gamma'}(\chi, \xi) = g_\gamma(\chi, \xi)$. So from now on we omit the subscript γ keeping in mind that the modulus $\mathfrak{f}$ is fixed. When ξ is coprime to $\mathfrak{f}$ we have

$$g(\chi, \xi) = \chi_f(\xi) g(\chi, 1). \tag{4.3}$$

Furthermore, when χ is primitive, (4.3) remains valid for ξ not coprime to $\mathfrak{f}$ since in this case $g(\chi, \xi) = 0$.

4.2 Relation between $\Psi(\mathfrak{c}, \mathfrak{f}, \chi_\infty, s)$ and $L(\chi, s)$

In this subsection we would like to give a precise formula which relates functions of Ψ-type to Artin L-functions. Let χ be a primitive character of $G_{\mathfrak{f}\infty}$. Roughly speaking, this formula says that a certain linear combination of the functions $\Psi(\mathfrak{a}, \mathfrak{f}, \chi_\infty, s)$ (where $\mathfrak{a}$ is allowed to vary) is equal $L(\chi, s)$, up to a Gauss sum which depends on χ.

Definition 4.1 *Let $\mathcal{O}$ be an arbitrary $\mathbb{Z}$-order of K and let $\mathfrak{f}$ be an integral invertible $\mathcal{O}$-ideal. For a character $\chi : I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to \mathbb{C}^\times$ we define*

$$L(\chi, s) = \sum_{\mathfrak{a}\in I_{\mathcal{O}}(\mathfrak{f})} \frac{\chi(\mathfrak{a})}{\mathbf{N}(\mathfrak{a})^s}, \qquad \Re(s) > 1.$$

Proposition 4.1 *Let $\mathcal{O}$ be an arbitrary $\mathbb{Z}$-order of K and let $\chi : I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty) \to \mathbb{C}^\times$ be a primitive Hecke character. Then*

$$\sum_{c\in I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f})} \bar{\chi}(\mathfrak{a}_c)\Psi\left(\mathfrak{a}_c, \mathfrak{f}, \chi_\infty, s\right) = g(\chi, 1)L(\chi, s) \tag{4.4}$$

where $\mathfrak{a}_c \in c$ is an arbitrary chosen integral invertible $\mathcal{O}$-ideal in the class of c.

Proof See the calculation on p. 27 of [Sie68]. This calculation was done in the special case where $\mathcal{O} = \mathcal{O}_K$ but one readily checks that it generalizes to an arbitrary order. □

Remark 4.1 One can also deduce a similar formula when χ is not primitive but one needs to remove certain Euler factors from $L(\chi, s)$. This can be accounted by the fact that for a non primitive character χ, the identity (4.3) does not necessarily hold when ξ and $\mathfrak{f}$ are not coprime. Note that the summation of (4.4) goes over wide ideal classes modulo $\mathfrak{f}$, i.e., elements of $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f})$ and not over narrow ideal classes modulo $\mathfrak{f}$, i.e., elements of $I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$. Therefore there is no factor 2^r which appears on the right hand side of (4.4) as it is the case on p. 27 of [Sie68].

As a corollary of this, one may deduce the well known functional equation of $L(\chi, s)$ proved by Hecke in the special case where $\mathcal{O} = \mathcal{O}_K$.

Corollary 4.1 *We have*

$$\begin{aligned} F_{\chi_\infty}(1-s)g(\chi,1)L(\chi,1-s) &= (i)^{\mathbf{Tr}(\chi_\infty)}F_{\chi_\infty}(s)\mathbf{N}(\mathfrak{f})^s \sum_{c\in I_{\mathcal{O}}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f})} \zeta(\mathfrak{a}_c, \mathfrak{f}, \chi_\infty, s)\bar{\chi}(\mathfrak{a}_c) \\ &= (i)^{\mathbf{Tr}(\chi_\infty)}F_{\chi_\infty}(s)\mathbf{N}(\mathfrak{f})^s L(\overline{\chi}, s). \end{aligned}$$

Proof The proof follows from Proposition 4.1 and Theorem 3.1. □

5 Special values of zeta functions of Ψ-type and ζ-type

In this section we concentrate on special values of functions of Ψ-type and ζ-type evaluated at integers. First let us determine the set of trivial zeros of functions of ζ-type when restricted to the set of integers $m \geq 1$. Let $w : K^\times \to \{\pm 1\}$ be a sign character and let $\{a_i\}_{i=1}^{r_1}$ be the signature of w. For even integers $m \geq 2$ the quantity

$$\frac{F_w(m)}{F_w(1-m)} \tag{5.1}$$

is equal to 0 unless $r_2 = 0$ and $a_j = 0$ for all j. Similarly, for odd integers $m \geq 1$, the quantity

$$\frac{F_w(m)}{F_w(1-m)} \tag{5.2}$$

is 0 unless $r_2 = 0$ and $a_j = 1$ for all j. In order to avoid trivial zeros we make the following assumption.

Assumption: The number field K is totally real, i.e., $r_2 = 0$.

We let $w_0 = 1$ be the trivial sign character and $w_1 = sign \circ \mathbf{N}_{K/\mathbb{Q}}$. Let $\mathfrak{f}$ and $\mathfrak{a}$ be integral invertible $\mathcal{O}$-ideals. Using Theorem 3.1 with (5.1) and (5.2), we see that for integers $m \geq 1$ the quantity $\zeta(\mathfrak{a}, \mathfrak{f}, w, 1-m)$ can be different from 0 only when $w = w_0$ and m is even or when $w = w_1$ and m is odd.

Remark 5.1 Let $Q_{\mathcal{O},1}(\mathfrak{f}) = \{\alpha \in \mathcal{O} : \alpha \equiv 1 \pmod{\mathfrak{f}}\}$ and let $Q_{\mathcal{O},1}(\mathfrak{f}\infty) = \{\alpha \in Q_{\mathcal{O},1}(\mathfrak{f}) : \alpha \gg 0\}$. We have a natural map $Q_{\mathcal{O},1}(\mathfrak{f}) \to P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$. Therefore every sign character $\widetilde{w}$ of $P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$ pulls back to a sign character of $Q_{\mathcal{O},1}(\mathfrak{f})$ and thus, after extension to $K^\times$, to a sign character w of $K^\times$. Let w be a sign character of $K^\times$ which is not induced from a sign character of $P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$. Then using (2.7), it is not to hard to see that $\zeta(\mathfrak{a}, \mathfrak{f}, w, s)$ is identically equal to zero. Therefore the only sign characters which are interesting are the ones which are induced from the sign characters of $P_{\mathcal{O},1}(\mathfrak{f})/P_{\mathcal{O},1}(\mathfrak{f}\infty)$.

Definition 5.1 *We define*

$$\zeta(\mathfrak{a}, \mathfrak{f}\infty, s) := \mathbf{N}(\mathfrak{a})^{-s} \sum_{\Gamma_{\mathfrak{a}} \backslash \{\lambda \in \mathfrak{a}^{-1}, \lambda \equiv 1 \pmod{\mathfrak{f}\mathfrak{a}^{-1}}, \lambda \gg 0\}} \frac{1}{|\mathbf{N}(\lambda)|^s} = \sum_{\substack{\mathfrak{b} \subseteq \mathcal{O} \\ \mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{a}}} \frac{1}{\mathbf{N}(\mathfrak{b})^s}, \tag{5.3}$$

where $\mathfrak{b} \sim_{\mathfrak{f}} \mathfrak{c}$ means that $\mathfrak{b}$ and $\mathfrak{c}$ lie in the same narrow ray ideal class modulo $\mathfrak{f}$. We call $\zeta(\mathfrak{a}, \mathfrak{f}\infty, s)$ a partial zeta function.

Let $\lambda \in 1 + \mathfrak{f}$. Using orthogonality relations of characters we find

$$\sum_{w \text{ sign character}} \zeta(\lambda\mathfrak{a}, \mathfrak{f}, w, s) = 2^n \zeta(\lambda\mathfrak{a}, \mathfrak{f}\infty, s), \tag{5.4}$$

where $n = [K : \mathbb{Q}]$. Setting $\lambda = 1$ in (5.4) and combining it with (5.1) and (5.2), we see that for all even integers $m \geq 2$,

$$\zeta(\mathfrak{a}, \mathfrak{f}, w_0, 1-m) = 2^n \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m), \tag{5.5}$$

Similarly, for all odd integers $m \geq 1$,

$$\zeta(\mathfrak{a}, \mathfrak{f}, w_1, 1-m) = 2^n \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m). \tag{5.6}$$

Conversely, using (5.4) and orthogonality relations of characters, we may deduce that for any sign character η

$$\sum_{\lambda \in Q_{\mathcal{O},1}(\mathfrak{f})/Q_{\mathcal{O},1}(\mathfrak{f}\infty)} |\mathbf{N}(\lambda)|^{-s} \eta(\lambda) \zeta(\mathfrak{a}\lambda, \mathfrak{f}\infty, s) = \zeta(\mathfrak{a}, \mathfrak{f}, \eta, s).$$

Remark 5.2 Let $\rho \in 1 + \mathfrak{f}$. Then using the identity (2.7) with Lemma 2.1 we may deduce that

$$\zeta(\rho\mathfrak{a}, \mathfrak{f}, w, s) = w(\rho)\zeta(\mathfrak{a}, \mathfrak{f}, w, s). \tag{5.7}$$

Setting $w = w_1$ in (5.7) and using (5.5) we get that for all odd integers $m \geq 1$

$$\zeta(\rho\mathfrak{a}, \mathfrak{f}\infty, 1-m) = w_1(\rho)\zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m). \tag{5.8}$$

Similarly for all even integers $m \geq 2$ one has

$$\zeta(\rho\mathfrak{a}, \mathfrak{f}\infty, 1-m) = \zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-m). \tag{5.9}$$

The identity (5.8) at $m = 1$ plays a key role in the context of Gross-Stark conjectures.

One of the key results about special values at negative integers of partial zeta functions associated to totally real number fields is the following:

Theorem 5.1 (**Siegel, Klingen, Shintani**) *For any integer $k \geq 1$ the value $\zeta(\mathfrak{a}, \mathfrak{f}\infty, 1-k)$ is a rational number.*

Proof See [Sie69] and [Shi76]. □

Corollary 5.1 *Let $j \in \{0, 1\}$. Then for any integer $k \geq 1$ such that $k \equiv j \pmod 2$ we have that*

(1) $\dfrac{F_{w_j}(k)}{F_{w_j}(1-k)}(i)^{\mathbf{Tr}(w_j)}\Psi(\mathfrak{c}, \mathfrak{f}, w_j, k)$ *is a rational number,*

(2) $\Psi(\mathfrak{c}, \mathfrak{f}, w_j, 1-k) \in \mathbb{Q}(\zeta_f)$ *where* $\mathbf{N}(\mathfrak{f}) = f$ *and* $\zeta_f = e^{2\pi i/f}$.

(3) $\zeta(\mathfrak{c}, \mathfrak{f}, w_j, k) \in |d_K|^{-1/2}\pi^{nk} \cdot \mathbb{Q}(\zeta_f)^+$ *where* $\mathbb{Q}(\zeta_f)^+$ *is the maximal real subfield of* $\mathbb{Q}(\zeta_f)$.

Proof Part (1) follows from Theorem 5.1, Theorem 3.1 and the identities (5.5) and (5.6). Part (2) follows from the identity (1) of Proposition 3.1 and Theorem 5.1. Part (3) follows from part (2) and Theorem 3.1. □

Remark 5.3 It is easy to describe the Galois action of $Gal(\mathbb{Q}(\zeta_f)/\mathbb{Q})$ on the special values appearing in (2). Let $j \in \{0, 1\}$ and $k \geq 1$ be such that $k \equiv j \pmod 2$. Let a be a positive integer coprime to f and denote by τ_a the automorphism of $\mathbb{Q}(\zeta_f)$ defined by $\zeta_f^{\tau_a} = \zeta_f^a$. Then a straightforward computation which uses (1) of Proposition 3.1 shows that $\Psi(\mathfrak{c}, \mathfrak{f}, w_j, 1-k)^{\tau_a} = \Psi(a\mathfrak{c}, \mathfrak{f}, w_j, 1-k)$. In particular, if we apply the complex conjugation $\tau_\infty := \tau_{f-1}$, we find with the help of (2.4), that $\Psi(\mathfrak{c}, \mathfrak{f}, w_j, 1-k)^{\tau_\infty} = w_j(-1)\Psi(\mathfrak{c}, \mathfrak{f}, w_j, 1-k)$.

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